XLV National Mathematics Olympiad: 3rd round, April 1997

Problem 1. Find all natural numbers a, b and c such that the roots of the equation

$$\begin{aligned} x^2 - 2ax + b &= 0 \\ x^2 - 2bx + c &= 0 \\ x^2 - 2cx + a &= 0 \end{aligned}$$

are natural numbers.

Solution: Let $\{x_1, x_2\}$, $\{x_3, x_4\}$ and $\{x_5, x_6\}$ be the roots of the first, second and third equation respectively, and let all of them be natural numbers.

Assume that $x_i \geq 2$ for all $i = 1, 2, \ldots, 6$. Then $2a = x_1 + x_2 \leq x_1x_2 = b$, $2b = x_3 + x_4 \leq x_3x_4 = c$ and $2c = x_5 + x_6 \leq x_5x_6 = a$. Thus $2(a + b + c) \leq a + b + c$, which is impossible, since a, b, c are natural numbers.

Therefore at least one of the numbers x_i equals 1. Without loss of generality suppose $x_1 = 1$, so $1 - 2a + b = 0$.

If $x_i \geq 2$ for $i = 3, 4, 5, 6$, then

$$2(b + c) = (x_3 + x_4) + (x_5 + x_6) \leq x_3 x_4 + x_5 x_6 = c + a,$$

whence $2(2a - 1 + c) \leq c + a \Rightarrow c \leq 2 - 3a$, which is impossible when a, b, c are natural numbers.

So at least one of x_3, x_4, x_5, x_6 equals 1. Let $x_3 = 1$. Now $1 - 2b + c = 0$. Assuming that $x_5 \geq 2$ and $x_6 \geq 2$, we get $2c = x_5 + x_6 \leq a$, so $2(2b - 1) \leq \dfrac{b+1}{2}$. Thus $7b \leq 5$, a contradiction.

Therefore at least one of the numbers x_5, x_6 is 1 and it follows that $1 - 2c + a = 0$. Further

$$0 = (1 - 2a + b) + (1 - 2b + c) + (1 - 2c + a) = 3 - (a + b + c)$$

and since a, b, c are natural numbers, it follows that $a = b = c = 1$.

Direct verification shows that $a = b = c = 1$ satisfy the conditions of the problem.

Problem 2. Given a convex quadrilateral $ABCD$ which can be inscribed in a circle. Let F be the intersecting point of diagonals AC and BD and E be the intersecting point of the lines AD and BC. If M and N are the midpoints of AB and CD, prove that

$$\frac{MN}{EF} = \frac{1}{2} \cdot \left| \frac{AB}{CD} - \frac{CD}{AB} \right|.$$

Solution: Let $\angle AEB = \gamma$, $EC = c$, $ED = d$, $\overrightarrow{i} = \frac{1}{c} \cdot \overrightarrow{EC}$ and $\overrightarrow{j} = \frac{1}{d} \cdot \overrightarrow{ED}$. Since $ABCD$ is an inscribed quadrilateral, $\frac{AB}{CD} = \frac{AE}{CE} = \frac{BE}{DE} = k$. Therefore $\overrightarrow{EA} = kc\overrightarrow{j}$ and $\overrightarrow{EB} = kd\overrightarrow{i}$. Since $F \in AC$ and $F \in BD$ there exist x and y such that

$$\overrightarrow{EF} = x\overrightarrow{EA} + (1-x)\overrightarrow{EC} = xkc\overrightarrow{j} + (1-x)c\overrightarrow{i}$$

and

$$\overrightarrow{EF} = y\overrightarrow{EB} + (1-y)\overrightarrow{ED} = ykd\overrightarrow{i} + (1-y)d\overrightarrow{j}.$$

Comparing the coefficients of $\overrightarrow{i}$ and $\overrightarrow{j}$ in these equalities gives $xkc = (1-y)d$ and $ykd = (1-x)c$. This implies $x = \frac{kd-c}{(k^2-1)c}$. Therefore

$$\overrightarrow{EF} = \frac{k}{k^2-1}\left((kd-c)\overrightarrow{j} + (kc-d)\overrightarrow{i}\right)$$

and thus

$$EF^2 = \left(\frac{k}{k^2-1}\right)^2 \left((kd-c)^2 + (kc-d)^2 + 2(kd-c)(kc-d)\cos\gamma\right).$$

On the other hand

$$\begin{aligned}\overrightarrow{MN} = \frac{1}{2}\cdot\left(\overrightarrow{AD} + \overrightarrow{BC}\right) &= \frac{1}{2}\cdot\left(\overrightarrow{ED} - \overrightarrow{EA} + \overrightarrow{EC} - \overrightarrow{EB}\right)\\ &= \frac{1}{2}\cdot\left((d-kc)\overrightarrow{j} + (c-kd)\overrightarrow{i}\right)\end{aligned}$$

and it follows that

$$MN^2 = \frac{1}{4}\cdot\left((d-kc)^2 + (c-kd)^2 + 2(d-kc)(c-kd)\cos\gamma\right).$$

Therefore $\dfrac{MN^2}{EF^2} = \dfrac{1}{4}\left(\dfrac{k^2-1}{k}\right)^2 = \dfrac{1}{4}\left(k - \dfrac{1}{k}\right)^2$ and so

$$\frac{MN}{EF} = \frac{1}{2} \cdot \left|\frac{AB}{CD} - \frac{CD}{AB}\right|$$

Problem 3. Prove that the equation

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0$$

has no solution in rational numbers.

Solution: It is easy to see that the equation is equivalent to

$$(2x + 3)^2 + (2y + 3)^2 + (2z + 3)^2 = 7.$$

It has a solution in rational numbers if and only if there exist integer numbers a, b, c and a natural number m such that

$$(*) \qquad a^2 + b^2 + c^2 = 7m^2.$$

Suppose such numbers exist and give m its smallest possible value. There are two cases:

(I) $m = 2n$ is an even number. Now $a^2 + b^2 + c^2$ is divisible by 4. This implies that all numbers a, b, c are even ones, so $a = 2a_1$, $b = 2b_1$, $c = c_1$. It follows now that $a_1^2 + b_1^2 + c_1^2 = 7n^2$, which contradicts the way m have been chosen.

(II) $m = 2n + 1$ is an odd number. Now $m^2 \equiv 1 \pmod 8$ and therefore $a^2 + b^2 + c^2 \equiv 7 \pmod 8$, which is impossible when a, b, c are integer.

Problem 4. Find all continuous functions $f(x)$ defined in the set of real numbers and such that

$$f(x) = f\left(x^2 + \frac{1}{4}\right)$$

for all real x.

Solution: Let $f(x)$ be a function satisfying the conditions of the problem. Obviously $f(x)$ is an even function.

Let $x_0 \geq 0$. There are two cases:

(I) $0 \leq x_0 \leq \frac{1}{2}$. Consider the sequence

$$x_0, x_1, \ldots, x_n, \ldots \tag{1}$$

defined by the equalities $x_{n+1} = x_n^2 + \frac{1}{4}$.

It is easy to see by induction that $0 \leq x_n \leq \frac{1}{2}$ for all n. Moreover

$$x_{n+1} - x_n = x_n^2 - x_n + \frac{1}{4} = \left(x_n - \frac{1}{2}\right)^2 \geq 0,$$

which implies that (1) is a monotone increasing function. Since it is bounded, it follows that it is a convergent function. Let $\lim_{n\to\infty} x_n = \alpha$. Now $\alpha^2 - \alpha + \frac{1}{4} = 0$, so $\alpha = \frac{1}{2}$.

On the other hand, since $f(x)$ is a continuous function,

$$\lim_{n\to\infty} f(x_n) = f\left(\frac{1}{2}\right).$$

But

$$f(x_{n+1}) = f\left(x_n^2 + \frac{1}{4}\right) = f(x_n)$$

for all n. Thus $f(x_0) = f(x_1) = \cdots$ which means that $f(x_{0)} = f\left(\frac{1}{2}\right)$ for all $x_0 \in \left[0, \frac{1}{2}\right]$.

(II) $x_0 > \frac{1}{2}$. Consider the following sequence:

$$x_0, x_1, \ldots, x_n, \ldots \tag{2}$$

defined by $x_{n+1} = \sqrt{x_n - \frac{1}{4}}$.

As in the previous case, we show that (2) is a convergent function and $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$. Further $\lim\limits_{n\to\infty} f(x_n) = f\left(\frac{1}{2}\right)$ and since

$$f(x_{n+1}) = f\left(x_{n+1}^2 + \frac{1}{4}\right) = f(x_n)$$

for all n we get that $f(x_{0)} = f\left(\frac{1}{2}\right)$.

Therefore $f(x)$ is a constant function in the interval $[0, +\infty)$, and since it is even, it is a constant for all x.

Conversely, any constant function satisfies the conditions of the problem.

Problem 5. Two squares K_1 and K_2 with centres M and N and sides of length 1 are placed in the plane in a way that $MN = 4$, two of the sides of K_1 are parallel to MN and one of the diagonals of

K_2 lies on the line MN. Find the locus of midpoints of segments XY where X is an interior point for K_1 and Y is an interior point for K_2.

Solution: The locus is the interior of a regular hexagon centred at the midpoint of the segment MN and with a side length of $\frac{1}{2}$.

To prove this, proceed as follows.

Fix the point Y in the interior of K_2. When X varies in the interior of K_1 the locus of the midpoints of XY is a square K_1' which is homothetic to K_1 by homothecy with centre Y and coefficient $\frac{1}{2}$. Obviously the side of this square is $\frac{1}{2}$ and its centre is the midpoint of MY. When Y varies in the interior of K_2 then the locus of the midpoints of MY is a square K_2' which is homothetic to K_2 by homothecy with centre M and coefficient $\frac{1}{2}$. The side of this square equals $\frac{1}{2}$ and its centre Q is the midpoint of MN. Finally, when the centres of K_1' vary in the interior of K_2', the squares vary in the interior of a regular hexagon centred at Q and with a side length of $\frac{1}{2}$.

Problem 6. Find the number of non-empty sets of $S_n = \{1, 2, \ldots, n\}$ such that there are no two consecutive numbers in one and the same set.

Solution: Denote the required number by f_n. It is easy to see that $f_1 = 1,\ f_2 = 2,\ f_3 = 4$.

Divide the subsets of S_n having no two consecutive numbers into two groups—those that do not contain the element n and those that do. Obviously the number of subsets in the first group is f_{n-1}.

Let T be a set of the second group. Therefore either $T = \{n\}$ or $T = \{a_1, \ldots, a_{k-1}, n\}$, where $k > 1$. It is clear that $a_{k-1} \neq n - 1$, so $\{a_1, \ldots a_{k-1}\} \subset S_{n-2}$, whence the number of sets in the second group is $f_{n-2} + 1$. Therefore

$$f_n = f_{n-1} + f_{n-2} + 1.$$

After substituting $u_n = f_n + 1$ we get

$$u_1 = 2, u_2 = 3, \qquad u_n = u_{n-1} + u_{n-2}.$$

Therefore the sequence $\{u_n\}_{n=1}^{\infty}$ coincides with the Fibonacci sequence from its third number onwards. Thus we obtain

$$f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right) - 1.$$