XLV National Mathematics Olympiad: 4th round, May 1997

Problem 1. Consider the polynomial

$$P_n(x) = \binom{n}{2} + \binom{n}{5}x + \binom{n}{8}x^2 + \cdots + \binom{n}{3k+2}x^k,$$

where $n \geq 2$ is a natural number and $k = \left[\dfrac{n-2}{3}\right]$.

(a) Prove that $P_{n+3}(x) = 3P_{n+2}(x) - 3P_{n+1}(x) + (x+1)P_n(x)$;

(b) Find all integer numbers a such that $P_n(a^3)$ is divisible by $3^{[\frac{n-1}{2}]}$ for all $n \geq 2$.

Solution: (a) Compare the coefficients in front of x^m, $0 \leq m \leq \left[\dfrac{n+1}{3}\right]$. It suffices to show that

$$\binom{n+3}{3m+2} = 3\binom{n+2}{3m+2} - 3\binom{n+1}{3m+2} + \binom{n}{3m+2} + \binom{n}{3m-1}.$$

Using the identity $\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1}$ we get that

$$\begin{aligned}
& \left(\binom{n+3}{3m+2} - \binom{n+2}{3m+2}\right) - 2\left(\binom{n+2}{3m+2} - \binom{n+1}{3m+2}\right) + \\
& + \left(\binom{n+1}{3m+2} - \binom{n}{3m+2}\right) - \binom{n}{3m-1} = \\
= \ & \binom{n+2}{3m+1} - 2\binom{n+1}{3m+1} + \binom{n}{3m+1} - \binom{n}{3m-1} = \\
= \ & \left(\binom{n+2}{3m+1} - \binom{n+1}{3m+1}\right) + \\
& - \left(\binom{n+1}{3m+1} - \binom{n}{3m+1}\right) - \binom{n}{3m-1} = \\
= \ & \binom{n+1}{3m} - \binom{n}{3m} - \binom{n}{3m-1} = \\
= \ & 0.
\end{aligned}$$

(b) Suppose a satisfies the condition of (b). Then $P_5(a^3) = 10 + a^3$ is divisible by 9 and so $a \equiv -1 \pmod 3$. On the contrary, let $a \equiv -1 \pmod 3$. Now $a^3 + 1 \equiv 0 \pmod 9$. Since $P_2(a^3) = 1, P_3(a^3) = 3$ and $P_4(a^3) = 6$ it follows by induction from (a) that $P_n(a^3)$ is divisible by $3^{\left[\frac{n-1}{2}\right]}$ for any n. Therefore the required values of a are all integer numbers congruent to 2 modulo 3.

Problem 2. Let M be the centroid of $\triangle ABC$. Prove the inequality

$$\sin \angle CAM + \sin \angle CBM \leq \frac{2}{\sqrt{3}}$$

(a) if the circumscribed circle of $\triangle AMC$ is tangent to the line AB;

(b) for any $\triangle ABC$.

Solution: We use the standard notation for the elements of $\triangle ABC$. Let G be the midpoint of AB.

(a) It follows from the conditions of the problem that

$$\left(\frac{c}{2}\right)^2 = GA^2 = GM \cdot GC = \frac{1}{3}m_c^2 = \frac{1}{12}(2a^2 + 2b^2 - c^2),$$

and therefore $a^2 + b^2 = 2c^2$. Using the median formula we get $m_a = \frac{\sqrt{3}}{2}b$ and $m_b = \frac{\sqrt{3}}{2}a$. Further

$$A = \sin\angle CAM + \sin\angle CBM = S\left(\frac{1}{bm_a} + \frac{1}{am_b}\right) = \frac{(a^2+b^2)\sin\gamma}{\sqrt{3}ab}.$$

From the Cosine Law $a^2 + b^2 - 2ab\cos\gamma = c^2 = \dfrac{a^2+b^2}{2}$, so $a^2 + b^2 = 4ab\cos\gamma$. Therefore $A = \dfrac{2}{\sqrt{3}}\sin 2\gamma \leq \dfrac{2}{\sqrt{3}}$.

(b) There are two circles through C and M tangent to the line AB. Denote the contact points by A_1 and B_1 and let $A_1 \in GA^{\rightarrow}$, $B_1 \in GB^{\rightarrow}$. Since G is the midpoint of A_1B_1 and $CM : MG = 2:1$, M must be the centroid of $\triangle A_1B_1C$. Furthermore it is clear that $\angle CAM \leq \angle CA_1M$ and $\angle CBM \leq \angle CB_1M$. Suppose $\angle CA_1M \leq 90^\circ$ and $\angle CB_1M \leq 90^\circ$. Now $\sin\angle CAM + \sin\angle CBM \leq \sin\angle CA_1M + \sin\angle CB_1M \leq \dfrac{2}{\sqrt{3}}$.

It remains to consider the case $\angle CA_1M > 90°$, $\angle CB_1M \leq 90°$ (the above angles could not both be obtuse). It follows from $\triangle CA_1M$ that $CM^2 > CA_1^2 + A_1M^2$, so

$$\frac{1}{9}(2b_1^2 + 2a_1^2 - c_1^2) > b_1^2 + \frac{1}{9}(2b_1^2 + 2c_1^2 - a_1^2)$$

(a_1, b_1, c_1 are sides of $\triangle A_1B_1C$). We know from (a) that $a_1^2 + b_1^2 = 2c_1^2$ and the above inequality becomes $a_1^2 > 7b_1^2$. Again from (a) we obtain

$$\sin \angle CB_1M = \frac{b_1 \sin \gamma_1}{a_1\sqrt{3}} = \frac{b_1}{a_1\sqrt{3}}\sqrt{1 - \left(\frac{a_1^2 + b_1^2}{4a_1b_1}\right)^2}.$$

Substituting $\dfrac{b_1^2}{a_1^2} = x$ we get that

$$\sin \angle CB_1M = \frac{1}{4\sqrt{3}}\sqrt{14x - x^2 - 1} < \frac{1}{4\sqrt{3}}\sqrt{2 - \frac{1}{49} - 1} = \frac{1}{7}$$

since $x < \dfrac{1}{7}$. Therefore

$$\sin \angle CAM + \sin \angle CBM < 1 + \sin \angle CB_1M < 1 + \frac{1}{7} < \frac{2}{\sqrt{3}}.$$

Note: The inequality holds only for $\triangle ABC$ with angles $\alpha = 22.5°$, $\beta = 112.5°$, $\gamma = 45°$ or $\alpha = 112.5°$, $\beta = 22.5°$, $\gamma = 45°$.

Problem 3. Let n and m be natural numbers such that $m + i = a_ib_i^2$ for $i = 1, 2, \ldots, n$, where a_i and b_i are natural numbers and a_i is not divisible by a square of a prime number. Find all n for which there exists an m such that $a_1 + a_2 + \cdots + a_n = 12$.

Solution: It is clear that $n \le 12$. Since $a_i = 1$ if and only if $m+i$ is a perfect square, at most three of the numbers a_i equal 1 (prove it!). It follows now from $a_1 + a_2 + \cdots + a_n = 12$ that $n \le 7$.

We show now that the numbers a_i are pairwise distinct. Assume the contrary and let $m + i = ab_i^2$ and $m + j = ab_j^2$ for some $1 \le i < j \le n$. Therefore $6 \ge n - 1 \ge (m + j) - (m + i) = a(b_j^2 - b_i^2)$. It is easy to see that the former is true only if $(b_i, b_j, a) = (1, 2, 2)$ or $(2, 3, 1)$ and in either case $a_1 + a_2 + \cdots + a_n > 12$.

All possible values of a_i are 1, 2, 3, 5, 6, 7, 10 and 11. There are three possibilities for n: $n = 2$ and $\{a_1, a_2\} = \{1, 11\}, \{2, 10\}, \{5, 7\}$; $n = 3$ and $\{a_1, a_2, a_3\} = \{1, 5, 6\}, \{2, 3, 7\}$; $n = 4$ and $\{a_1, a_2, a_3, a_4\} = \{1, 2, 3, 6\}$. Suppose $n = 4$ and $\{a_1, a_2, a_3, a_4\} = \{1, 2, 3, 6\}$. Now $(6b_1b_2b_3b_4)^2 = (m + 1)(m + 2)(m + 3)(m + 4) = (m^2 + 5m + 5)^2 - 1$, which is impossible. Therefore $n = 2$ or $n = 3$.

If $n = 3$ and $(a_1, a_2, a_3) = (1, 5, 6)$, then $m = 3$ has the required property, and if $n = 2$ and $(a_1, a_2) = (11, 1)$, then $m = 98$ has the required property. (It is not difficult to see that the remaining cases are not feasible.)

Problem 4. Let a, b and c be positive numbers such that $abc = 1$. Prove the inequality

$$\frac{1}{1 + a + b} + \frac{1}{1 + b + c} + \frac{1}{1 + c + a} \le \frac{1}{2 + a} + \frac{1}{2 + b} + \frac{1}{2 + c}.$$

Solution: Let $x = a + b + c$ and $y = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = ab + bc + ca$ $(abc = 1)$. It follows from Cauchy's Inequality that $x \ge 3$ and $y \ge 3$. Since both sides of the given inequality are symmetric functions of

a, b and c, we transform the expression as a function of x, y and $abc = 1$. After simple calculations we get

$$\frac{3 + 4x + y + x^2}{2x + y + x^2 + xy} \leq \frac{12 + 4x + y}{9 + 4x + 2y},$$

which is equivalent to

$$3x^2y + xy^2 + 6xy - 5x^2 - y^2 - 24x - 3y - 27 \geq 0.$$

Write the last inequality in the form

$$\begin{aligned}(\frac{5}{3}x^2y - 5x^2) + (\frac{xy^2}{3} - y^2) + (\frac{xy^2}{3} - 3y) + (\frac{4}{3}x^2y - 12x) + \\ + (\frac{xy^2}{3} - 3x) + (3xy - 9x) + (3xy - 27) \quad \geq \quad 0.\end{aligned}$$

When $x \geq 3$, $y \geq 3$, all terms in the left hand side are non-negative and the inequality is true. Equality holds when $x = 3$, $y = 3$, which implies $a = b = c = 1$.

Problem 5. Given a $\triangle ABC$ with bisectors BM and CN ($M \in AC$, $N \in AB$). The ray $MN^{\rightarrow}$ intersects the circumcircle of $\triangle ABC$ at point D. Prove that

$$\frac{1}{BD} = \frac{1}{AD} + \frac{1}{CD}.$$

Solution: Let A_1, B_1 and C_1 be the orthogonal projections of D on the lines BC, CA and AB, respectively. It follows from $\triangle DAB_1$ and the Sine Law that $DB_1 = DA \cdot \sin \angle DAB_1 = DA \cdot \sin \angle DAC = \dfrac{DA \cdot DC}{2R}$ (R is the circumradius of $\triangle ABC$). Analogously $DA_1 =$

$\dfrac{DB \cdot DC}{2R}$ and $DC_1 = \dfrac{DA \cdot DB}{2R}$. Our equality is now equivalent to $AD \cdot CD = BD \cdot CD + AD \cdot BD$ and so it suffices to prove that

$$DB_1 = DA_1 + DC_1 \tag{1}$$

Denote by m the distance from M to AB and BC and by n the distance from N to AC and BC. Let $\dfrac{DM}{MN} = x$ $(x > 1)$. Further, $\dfrac{DB_1}{n} = x$, $\dfrac{DC_1}{m} = x - 1$ and $\dfrac{DA_1 - m}{n - m} = x$. Therefore $DB_1 = nx$, $DC_1 = m(x-1)$ and $DA_1 = nx - m(x-1) = DB_1 - DC_1$ and (1) holds.

Problem 6. Let X be a set of $n+1$ elements, $n \geq 2$. Ordered n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ formed from distinct elements of X are called 'disjoint' if distinct indices i and j exist such that $a_i = b_j$. Find the maximal number of n-tuples any two of which are 'disjoint'.

Solution: For $n \geq 2$ denote by $A(n+1)$ the maximum number of ordered n-tuples such that any two of them are 'disjoint'. Also let $S(X)$ be a set of such n-tuples for which $|S(X)| = A(n+1)$. It is clear that for any $\alpha \in X$ the following holds:

$$|\{(a_1, a_2, \ldots, a_n) \in S(X) | a_1 = \alpha\}| \leq A(n).$$

Thus $A(n+1) \leq (n+1)A(n)$. Therefore

$$A(n+1) \leq (n+1)n \ldots A(3).$$

Direct verification shows that $A(3) = 3$, so $A(n+1) \leq \dfrac{(n+1)!}{2}$.

We prove now that $A(n+1) = \dfrac{(n+1)!}{2}$ by constructing a set of $\dfrac{(n+1)!}{2}$ ordered n-tuples, any two of which are 'disjoint'. We may assume that $X = \{1, 2, \ldots, n+1\}$. Consider a set E of all even permutations of $1, 2, \ldots, n+1$. (A permutation $(a_1, a_2, \ldots, a_{n+1})$ is called even if the number of pairs (i, j) such that $i < j$ and $a_i > a_j$ is an even number.) The set

$$\{(a_1, a_2, \ldots, a_n) | (a_1, a_2, \ldots, a_n, a_{n+1}) \in E\}$$

has $\dfrac{(n+1)!}{2}$ ordered n-tuples, any two of which are 'disjoint'.