# XLVII National Mathematics Olympiad: 3rd round, 25–26 April 1998

**Problem 1.** Find the least positive integer number $n$ $(n \geq 3)$ with the following property: for any colouring of $n$ different points $A_1, A_2, \ldots, A_n$ on a line and such that $A_1A_2 = A_2A_3 = \ldots = A_{n-1}A_n$ in two colours, there are three points $A_i, A_j, A_{2j-i} (1 \leq i < 2j - i \leq n)$ which have the same colour.

**Solution:** Assume the two colours are white and black. Consider 8 points coloured as follows: $A_1, A_2, A_5, A_6$ (white), $A_3, A_4, A_7, A_8$ (black). Obviously no three points $A_i, A_j, A_{2j-i}$ $(1 \leq i < 2j - i \leq n)$ have the same colour and therefore $n \geq 9$.

If we can show that $n = 9$ has the required property, we will be done. Suppose there are 9 points coloured black or white and no three points $A_i, A_j, A_{2j-i}$ $(1 \leq i < 2j - i \leq n)$ have the same colour.

First assume that for $i = 3$, $i = 4$ or $i = 5$ points $A_i$ and $A_{i+2}$ have the same colour (say white). Then the points $A_{i-2}, A_{i+1}, A_{i+4}$

should be black (note that $i-2 \geq 1$ and $i+4 \leq 9$), which is a contradiction.

Suppose now that for $i = 3, 4, 5$ the points $A_i$ and $A_{i+2}$ have different colours. Without loss of generality assume $A_5$ is a white point. Then $A_3$ and $A_7$ are black. Because of the symmetry we may suppose that $A_4$ is white and $A_6$ is black. Consiquently $A_8$ is white, $A_2$ is black $(2+8 = 2 \cdot 5)$ and $A_9$ is white $(7+9 = 2 \cdot 8)$. Therefore $A_1$ should be both white $(1+3 = 2 \cdot 2)$ and black $(1+9 = 2 \cdot 5)$, which is again a contradiction.

Consequently the assumption is not true and so $n = 9$.

**Problem 2.** Let $ABCD$ be a quadrilateral such that $AD = CD$ and $\angle DAB = \angle ABC < 90^\circ$. The line passing through $D$ and the midpoint of the segment $BC$ intersects the line $AB$ at the point $E$. Prove that $\angle BEC = \angle DAC$.

**Solution:** Let $M$ be the midpoint of $BC$ and let $AD$ and $BC$ meet at point $N$ and $AN$ and $EC$ meet at point $P$. It follows from Menelaus' Theorem aplied to $\triangle DMN$ and $\triangle DEN$ that $DP \cdot NC \cdot ME = PN \cdot CM \cdot ED$ and $DA \cdot NB \cdot ME = AN \cdot BM \cdot ED$. Combining the above equalities with $AN = BN$, $BE = CE$ and $AD = CD$, we get $DP \cdot NC = DC \cdot PN$. Therefore $CP$ is bisector of $\angle DCN$.

Consequently $\angle ACP = \angle ACD + \angle DCP = \frac{1}{2}(\angle NDC + \angle DCN) = \angle NAB$ so $\angle DCP = \angle CAB$ and we obtain $\angle BEC = \angle ABC - \angle BCE = \angle BAD - \angle DCP = \angle DAC$, *Q. E. D.*

**Note:** The assertion is also true if the condition $\angle ABC < 90°$ is left out.

**Problem 3.** Let $\mathbb{R}^+$ be the set of all positive real numbers. Prove that there is no function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(f(x))^2 \geq f(x+y)(f(x)+y)$$

for arbitrary positive real numbers $x$ and $y$.

**Solution:** Suppose there exists a function satisfying the conditions of the problem. Write the initial equality in the form

$$f(x) - f(x+y) \geq \frac{f(x)y}{f(x)+y}.$$

First we prove that $f(x) - f(x+1) \geq \frac{1}{2}$ for $x > 0$. Obviously $f$ is a (strictly) monotone non-increasing function. Fix $x > 0$ and choose a natural number $n$, such that $n \cdot F(x+1) \geq 1$. When $k = 0, 1, \ldots, n-1$, we obtain that

$$f(x + \frac{k}{n}) - f(x + \frac{k+1}{n}) \geq \frac{f(x+\frac{k}{n})\frac{1}{n}}{f(x+\frac{k}{n}) + \frac{1}{n}} \geq \frac{1}{2n}.$$

Adding the above inequalities gives $f(x) - f(x+1) \geq \frac{1}{2}$.

Let the natural number $m$ be such that $m \geq 2f(x)$. Therefore

$$f(x) - f(x+m) = \sum_{i=0}^{m-1} (f(x+i) - f(x+i+1)) \geq \frac{m}{2} \geq f(x),$$

and so $f(x+m) \leq 0$. But this contradicts the fact that $f$ is strictly positive.

**Problem 4.** Let $f(x)=x^3-3x+1$. Find the number of different real solutions of the equation $f(f(x))=0$.

**Solution:** Since $f'(x)=3(x-1)(x+1)$ it follows that $f$ is strictly monotone non-decreasing in the intervals $(-\infty,-1]$ and $[1,\infty)$ and strictly monotone non-increasing in the interval $[-1,1]$. Moreover $\lim_{x\to\pm\infty} f(x)=\pm\infty, f(-1)=3, f(1)=-1, f(3)=19>0$ and so the equation $f(x)=0$ has three distinct roots $x_1,x_2,x_3$ such that $x_1<-1<x_2<1<x_3<3$. Therefore $f(x)=x_1$ has only one real root (which is less than $-1$) and $f(x)=x_2$ and $f(x)=x_3$ have three distinct real roots each (one in each of the intervals $(-\infty,-1),(-1,1)$ and $(1,\infty)$). Since the roots of $f(f(x))=0$ are exactly the roots of these three equations, we conclude that it has seven distinct real roots.

**Problem 5.** The convex pentagon $ABCDE$ is inscribed in a circle with radius $R$. The inradii of the triangles $ABC$, $ABD$, $AEC$ and $AED$ are denoted by $r_{ABC}$, $r_{ABD}$, $r_{AEC}$ and $r_{AED}$. Prove that

a.) $\cos\angle CAB+\cos\angle ABC+\cos\angle BCA=1+\dfrac{r_{ABC}}{R}$;

b.) If $r_{ABC}=r_{AED}$ and $r_{ABD}=r_{AEC}$, then $\triangle ABC\cong\triangle AED$.

**Solution:** a.) Using the standard notation for the elements of $\triangle ABC$, we obtain that

$$\frac{r}{R}=\frac{S}{pR}=\frac{4S^2}{pabc}=\frac{4(p-a)(p-b)(p-c)}{abc}=$$

$$= \frac{1}{2abc}(a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2)-2abc) =$$
$$= \cos\angle CAB + \cos\angle ABC + \cos\angle BCA - 1.$$

b.) Applying continuously the equality from a.) and using the fact that $ABCDE$ is inscribed in a circle convex pentagon, it is easy to see that

$$r_{ABC} + r_{AEC} + r_{EDC} = r_{AED} + r_{ABD} + r_{BCD}.$$

The condition of the problem and the above equality imply $r_{BCD} = r_{EDC}$. Since $\triangle BCD$ and $\triangle EDC$ have a side in common and equal circumradiuses we get that $\triangle BCD \cong \triangle EDC$ (prove it using a.)). In particular $BC = ED$ and using $r_{ABC} = r_{AED}$ again, we get $\triangle ABC \cong \triangle AED$.

**Problem 6.** Show that the equation

$$x^2y^2 = z^2(z^2 - x^2 - y^2)$$

has no solution in positive integer numbers.

**Solution:** Suppose $x, y, z$ is a solution for which $\frac{xy}{z}$ (which is a natural number—why?) has its minimum value.

Write $x, y, z$ in the form $x = dx_1, y = dy_1, z = dz_1$, where $d = (x, y, z)$. Our equation is equivalent to $x_1^2y_1^2 = z_1^2(z_1^2 - x_1^2 - y_1^2)$. Let $u = (x_1, z_1), v = (y_1, z_1), x_1 = ut, y_1 = vw$. Since $z_1$ divides $x_1y_1$, we get $z_1 = uv$. A substitution in the last equality gives

$(u^2+w^2)(v^2+t^2) = 2u^2v^2$. Further, since $(x_1, y_1, z_1) = 1$, it follows that $(u, w) = 1, (v, t) = 1$. Therefore

$$u^2 + w^2 = v^2, v^2 + t^2 = 2u^2$$

or

$$u^2 + w^2 = 2v^2, v^2 + t^2 = u^2.$$

Without loss of generality we may assume that the first pair of equalities hold. It is easily seen that $v$ and $u$ are odd integer numbers. It follows now from $u^2 + w^2 = v^2$ that $u = m^2 - n^2, w = 2mn, v = m^2 + n^2$, where $m$ and $n$ are coprime natural numbers (of distinct parity). Substitution in $v^2 + t^2 = 2u^2$ shows that $t^2 + (2mn)^2 = (m^2 - n^2)^2$ and so $t = p^2 - q^2, mn = pq, m^2 - n^2 = p^2 + q^2$ for some natural numbers $p$ and $q$. Therefore

$$p^2q^2 = m^2(m^2 - p^2 - q^2),$$

which shows that $p$, $q$, $m$ is a solution of the original equation. It remains to be seen that $\frac{pq}{m} = n < d(p^2 - q^2)2mn = \frac{xy}{z}$, which contradicts the way we have chosen $x, y, z$.