XLVII National Mathematics Olympiad: 4th round, 16–17 May 1998

Problem 1. Let n be a natural number. Find the least natural number k for which there exist k sequences of 0's and 1's of length $2n+2$ with the following property: any sequence of 0's and 1's of length $2n+2$ coincides in at least $n+2$ positions with some of these k sequences.

Solution: We shall prove that $k = 4$. Assume that $k \le 3$ and let the respective sequences be $a_1^i, a_2^i, \ldots, a_{2n+2}^i$ for $i = 1, \ldots, k$. Since $k \le 3$ there is a sequence $b_1, b_2, \ldots, b_{2n+2}$ such that $(b_{2l+1}, b_{2l+2}) \ne (a_{2l+1}^i, a_{2l+2}^i)$ for $l = 0, 1 \ldots, n$ and $i = 1, \ldots, k$. This is a contradiction. For $k = 4$ it is easily seen that the sequences $000\ldots0$, $011\ldots1$, $100\ldots0, 111\ldots1$ have the required property.

Problem 2. The polynomials $P_n(x, y)$, $n = 1, 2, \ldots$ are defined by

$P_1(x, y) = 1, P_{n+1}(x, y) = (x+y-1)(y+1)P_n(x, y+2)+(y-y^2)P_n(x, y).$

Prove that $P_n(x,y) = P_n(y,x)$ for all x, y and n.

Solution: We know that $P_1(x,y) = 1$ and $P_2(x,y) = xy + x + y - 1$. Assume that $P_{n-1}(x,y)$ and $P_n(x,y)$, $(n \geq 2)$ are symmetric polynomials. Then

$$\begin{aligned} P_{n+1}(x,y) &= (x+y-1)(y+1)P_n(x,y+2) + (y-y^2)P_n(x,y) \\ &= (x+y-1)(y+1)P_n(y+2,x) + (y-y^2)P_n(y,x) \\ &= (x+y-1)(y+1)\left(\begin{array}{l} (x+y+1)(x+1)P_{n-1}(y+2,x+2) \\ +\,(x-x^2)P_{n-1}(y+2,x) \end{array}\right) \\ &\quad + (y-y^2)\left(\begin{array}{l} (y+x-1)(x+1)P_{n-1}(y,x+2) \\ +\,(x-x^2)P_{n-1}(y,x) \end{array}\right) \\ &= (x+y-1)(y+1)(x+y+1)(x+1)P_{n-1}(y+2,x+2) \\ &\quad + (y-y^2)(x-x^2)P_{n-1}(y,x) \\ &\quad + (x+y-1)(y+1)(x-x^2)P_{n-1}(y+2,x) \\ &\quad + (y-y^2)(x+y-1)(x+1)P_{n-1}(x+2,y). \end{aligned}$$

and by induction it follows that all the polynomials are symmetric.

Problem 3. On the sides of a non-obtuse triangle ABC a square, a regular n-gon and a regular m-gon $(n, m > 5)$ are constructed externally, so that their centres are vertices of a regular triangle. Prove that $m = n = 6$ and find the angles of ABC.

Solution: Let the square, the n-gon and the m-gon be constructed on the sides AB, BC and CA, respectively. Denote their centres by O_1, O_2 and O_3; denote by A_1, B_1 and C_1 the centres of the equilateral triangles constructed externally on BC, CA and AB.

The lines O_1C_1, O_2A_1 and O_3B_1 intersect at the circumcentre O of $\triangle ABC$. Since $\triangle A_1A_2A_3$ is equilateral, it follows straightforwardly that $\triangle O_1O_2O_3$ is equilateral if and only if $C_1A_1||O_1O_2$, $A_1B_1||O_2O_3$ and $B_1C_1||O_1O_3$. This is equivalent to

$$\frac{OC_1}{C_1O_1} = \frac{OA_1}{A_1O_2} = \frac{OB_1}{B_1O_3} = k.$$

On the other hand,

$$\begin{aligned} \frac{OC_1}{C_1O_1} &= \frac{\cot C + \tan 30^\circ}{\cot 45^\circ - \tan 30^\circ}, \\ \frac{OA_1}{A_1O_2} &= \frac{\cot A + \tan 30^\circ}{\cot \frac{180^\circ}{n} - \tan 30^\circ}, \\ \frac{OB_1}{B_1O_3} &= \frac{\cot B + \tan 30^\circ}{\cot \frac{180^\circ}{m} - \tan 30^\circ}. \end{aligned}$$

Set $\cot \dfrac{180^\circ}{n} = x$ and $\cot \dfrac{180^\circ}{m} = y$. The above identities imply that

$$\cot A = kx - \frac{k+1}{\sqrt{3}}, \cot B = ky - \frac{k+1}{\sqrt{3}}, \cot C = k - \frac{k+1}{\sqrt{3}}.$$

From the identity $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$ we get

$$\begin{aligned} k &= \frac{2(x+y+1-\sqrt{3})}{\sqrt{3}xy + (\sqrt{3}-2)(x+y) + \sqrt{3} - 2}, \\ \cot C &= \frac{x+y-xy+3-2\sqrt{3}}{\sqrt{3}xy + (\sqrt{3}-2)(x+y) + \sqrt{3} - 2}. \end{aligned}$$

Since $m, n \geq 6$ it follows that $x, y \geq \sqrt{3}$,i. e. $xy \geq \sqrt{3}(x+y) - 3$. The inequality $\cot C \geq 0$ implies $x + y - xy + 3 - 2\sqrt{3} \geq 0$. and therefore $x + y + 3 - 2\sqrt{3} \geq xy \geq \sqrt{3}(x+y) - 3$,i. e. $x + y \leq 2\sqrt{3}$. This shows that $x = y = \sqrt{3}$, i.e., $m = n = 6$. Hence $\cot C = 0, \cot A = \cot B = 1$, so $\angle C = 90^\circ, \angle A = \angle B = 45^\circ$.

Problem 4. Let $a_1, a_2, \ldots, a_n$ be real numbers, not all of them zero. Prove that the equation

$$\sqrt{1+a_1x} + \sqrt{1+a_2x} + \ldots + \sqrt{1+a_nx} = n$$

has at most one nonzero real root.

Solution: The given equation is equivalent to $x \sum_{i=1}^{n} \frac{a_i}{\sqrt{1+a_ix}} = n$. Since the function $\frac{a_i}{\sqrt{1+a_ix}}$ $(a_i \neq 0)$ is strictly decreasing, it follows that this equation has at most one nonzero root.

Problem 5. Let m and n be natural numbers such that $A = ((m+3)^n + 1)/3m$ is integer. Prove that A is an odd integer number.

Solution: Assume that A is an even integer number, i.e., $(m+3)^n + 1 = 6km$. Then m is an even integer number. Moreover $m|3^n + 1$, which shows that $m = 3t + 2$ and n is odd. Let $m = 2^\alpha m_1$, where $\alpha \geq 1$ and m_1 is odd. Then $2^\alpha | 3^n + 1$ and therefore $\alpha \leq 2$. Since $m_1 | 3^n + 1$, it follows that $m_1 | a^2 + 3$, where $a = 3^{\frac{n+1}{2}}$. It is well-known that in this case $m_1 = 6t_1 + 1$. Since $m = 2^\alpha(6t_1 + 1)$ has the form $3t + 2$ and $1 \leq \alpha \leq 2$ we see that $\alpha = 1$. Then $m = 12t_1 + 2$ and from $(m+3)^n + 1 = 6km$ it follows that $4|5^n + 1$, which is impossible.

Problem 6. The sides and the diagonals of a regular n-gon X are coloured in k colours so that:

(i) for each colour a and any two vertices A and B of X, the segment AB is coloured in colour a or there is a vertex C such that AC and BC are coloured in colour a;

(ii) the sides of any triangle with vertices among the vertices of X are coloured in at most two colours.

Prove that $k \leq 2$.

Solution: Assume that the colouring involves at least three different colours a, b, c. We shall construct an infinite subset of vertices of X, which will imply a contradiction.

Let $Z \in X$ and A_1 is a vertex such that the colour of A_1Z is a. From (i) it follows that there is a vertex B_1, such that the colour of B_1Z and B_1A_1 is b. Analogously there is a vertex C_1 such that the colour of C_1Z and C_1B_1 is c. Considering the triangles C_1A_1Z and $C_1A_1B_1$ we see (using condition (ii)) that the colour of C_1A_1 is c. Let $A_2 \in X$ be such that the colour of A_2C_1 and A_2Z is a. It is easily seen that $A_2 \neq A_1$ and the colour A_2A_1 and A_2B_1 is a. Now we shall proceed by induction. Let the vertices $A_2, B_2, C_2, \ldots, A_{k-1}, B_{k-1}, C_{k-1}$ be such that the colour of A_iA_j, A_iB_j, A_iC_j is a, the colour of B_iA_j, B_iB_j, B_iC_j is b and the colour C_iA_j, C_iB_j, C_iC_j is c, $(2 \leq j < i < k)$.

Take a vertex A_k such that the colour of A_kC_{k-1} and A_kZ is a. Then considering the triangles ZA_kB_j and $C_{k-1}A_kB_j$ $(j < k)$, ZA_kC_j and $B_{j+1}A_kC_j$ $(j + 1 < k)$, $A_kA_jB_j$ and $A_kA_jB_j$ we see that the colour of A_kB_j, A_kC_j and A_kC_j is a. The vertices B_k and C_k are constructed in a similar way.