XLVIII National Mathematics Olympiad: 3rd round, 17–18 April 1999

Problem 1. Find all triples (x, y, z) of natural numbers such that y is a prime number, y and 3 do not divide z, and $x^3 - y^3 = z^2$.

Nikolay Nikolov

Solution: Since $(x-y)((x-y)^2+3xy)) = x^3 - y^3 = z^2$, it follows from the conditions of the problem that $x - y$ and $(x-y)^2 + 3xy$ are relatively prime. Therefore $x - y = u^2$ and $x^2 + xy + y^2 = v^2$. Thus $3y^2 = (2v - 2x - y)(2v + 2x + y)$ and since y is a prime number, there are three cases to consider:

1. $2v - 2x - y = y, 2v + 2x + y = 3y$. Now $x = 0$, which is impossible.

2. $2v - 2x - y = 1, 2v + 2x + y = 3y^2$. Now $3y^2 - 1 = 2(2x + y) = 2(2u^2 + 3y)$ and it follows that 3 divides $u^2 + 1$, which is impossible.

3. $2v-2x-y=3, 2v+2x+y=y^2$. Now $y^2-3=2(2x+y)=2(2u^2+3y)$ and it follows that $(y-3)^2-(2u)^2=12$. Therefore $y=7, u=1, x=8, z=13$. Direct verification shows that $(8,7,13)$ is a solution of the problem.

Problem 2. A convex quadrilateral of area S is inscribed in a circle whose centre is a point interior to the quadrilateral. Prove that the area of the quadrilateral whose vertices are the projections of the point of intersection of the diagonals on the sides does not exceed $\frac{S}{2}$. *Christo Lesov*

Solution: Let $ABCD$ be a quadrilateral inscribed in a circle with centre O and radius R. Denote by E the point of intersection of AC and BD. Denote further by M, N, P, Q and F the projections of E on AB, BC, CD, DA and MN, respectively. We know that $MN = BE\sin\angle ABC = \frac{BE\cdot AC}{2R}$. Also,

$$EF = EM\sin\angle EMN = \frac{AE\cdot BE\sin\angle AEB}{AB}\sin\angle CBE.$$

Since $BE\sin\angle CBE = CE\sin\angle BCE = CE\frac{AB}{2R}$ and $AE\cdot CE = R^2-OE^2$, it follows that $EF = \frac{R^2-OE^2}{2R}\sin\angle AEB$. Therefore

$$S_{\triangle MEN} = \frac{MN\cdot EF}{2} = \frac{AC\cdot BE\sin\angle AEB(R^2-OE^2)}{8R^2}.$$

Similar equalities hold for $S_{\triangle NEP}, S_{\triangle PEQ}$ and $S_{\triangle QEM}$. By combining the above we obtain

$$S_{MNPQ} = \frac{AC \cdot BD \sin \angle AEB(R^2 - OE^2)}{4R^2} \leq \frac{S_{ABCD}}{2},$$

Q. E. D.

Problem 3. In a competition 8 judges marked the contestants by *yes* or *no*. It is known that for any two contestants, two judges gave both a *yes*; two judges gave the first one a *yes* and the second one a *no*; two judges gave the first one a *no* and the second one a *yes*, and finally, two judges gave both a *no*. What is the greatest possible number of contestants? *Emil Kolev*

Solution: Denote the number of contestants by n. Consider a table with 8 rows and n columns such that the cell in the ith row and jth column contains 0 (1) if the ith judge gave the jth contestant a *no* (a *yes*). The conditions of the problem now imply that the table formed by any two columns contains among its rows each of the pairs 00, 01, 10 and 11 twice. We shall prove that 8 columns having this property do not exist. Assume the opposite. It is easily seen that if in any column all 0s are replaced by 1s and *vice versa*, the above property is retained. Therefore without loss of generality suppose that the first row consists of 0s. Denote the number of 0s in the ith row by a_i. It is clear that the total number of 0s is $8 \cdot 4 = 32$. Further, the number of occurrences of 00 is $\binom{8}{2} \cdot 2 = 56$. On the other hand the same number is $\sum_{i=1}^{8} \binom{a_i}{2}$. Since $a_1 = 8$, it follows that $\sum_{i=2}^{8} a_i = 24$. It is easy to prove now that $\sum_{i=2}^{8} \binom{a_i}{2} \geq 30$. Therefore $56 = \sum_{i=1}^{8} \binom{a_i}{2} \geq 58$, which is false.

The diagram on the right shows that it is possible to have exactly 7 contestants:

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}$$

Problem 4. Find all pairs (x, y) of integer numbers such that $x^3 = y^3 + 2y^2 + 1$.

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Solution: It is obvious that $x > y$. On the other hand $x < y + 1 \iff (y+1)^3 > y^3 + 2y^2 + 1 \iff y(y+3) > 0$. Therefore if $y > 0$ or $y < -3$ the problem has no solution. Direct verification yields all pairs (x, y) which satisfy the equality $x^3 = y^3 + 2y^2 + 1$, namely $(-2, -3)$, $(1, -2)$ and $(1, 0)$.

Problem 5. Let B_1 and C_1 be points on the sides AC and AB of $\triangle ABC$. The straight lines BB_1 and CC_1 intersect at point D. Prove that the quadrilateral AB_1DC_1 is circumscribed if and only if the incircles of $\triangle ABD$ and $\triangle ACD$ are tangent.

Rumen Kozarev and Nikolay Nikolov

Solution: Note that the incircles of $\triangle ABD$ and $\triangle ACD$ are tangent if and only if $AB+AD-BD = AC+AD-CD$, so $AB+CD = AC+BD$.

Suppose AB_1DC_1 is circumscribed and the incircle touches AB_1, B_1D, DC_1, C_1A in the points M, N, P, Q, respectively. Therefore $AB+CD = AQ+BQ+CP-DP = AM+BN+CM-DN = AC+BD$.

Conversely, let the incircles of $\triangle ABD$ and $\triangle ACD$ be tangent. Denote the point of intersection of the tangent through C (different from CA) with the incircle of $\triangle ABB_1$ by D'. It follows from the above that $BD'-CD' = AB-AC = BD-CD$, so $DD' = |CD-CD'|$. Therefore $D' \equiv D$, which completes the proof.

Problem 6. Each interior point of an equilateral triangle of side 1 lies in one of six circles of the same radius r. Prove that $r \geq \dfrac{\sqrt{3}}{10}$.

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Solution: Divide each side of the triangle into five equal parts and draw lines parallel to the sides through these points. Thus the triangle is divided into 25 equilateral triangles of side $\dfrac{1}{5}$. The total number of vertices is $21 > 6 \cdot 3$. Therefore there exist 4 points which are interior to one and the same circle. It is easy to see now that $r \geq \dfrac{\sqrt{3}}{10}$, which solves the problem.