SPRING MATHEMATICAL COMPETITION

1995

Grade 8.

Problem 1. Find all values of a, for which the system

$$\left|\begin{array}{rcl} x+4|y| & = & |x| \\ |y|+|x-a| & = & 1 \end{array}\right.$$

has exactly two solutions.

Solution. Let (x, y) be a solution with $x \geq 0$. Then $y = 0$ and $|x-a| = 1$, i.e. $x = a \pm 1$. It is obvious that when $a \geq 1$ the system has two solutions with $x \geq 0$, namely $(a-1, 0)$, $(a+1, 0)$. When $-1 \leq a < 1$ the system has only one solution $(a+1, 0)$ with $x \geq 0$. Let $x < 0$. Then $|y| = -\dfrac{x}{2}$ and consequently $|x-a| = 1 + \dfrac{x}{2}$. Since $|x-a| \geq 0$, then $1 + \dfrac{x}{2} \geq 0$ or $x \geq -2$. We have $x - a = 1 + \dfrac{x}{2}$ and $x = 2(a+1)$, $x - a = -1 - \dfrac{x}{2}$. From here $x = \dfrac{2}{3}(a-1)$. From $-2 \leq 2(a+1) < 0$ we get$-2 \leq a < -1$ and from $-2 \leq \dfrac{2}{3}(a-1) < 0$ we get $-2 \leq a < 1$. Obviously if (x, y) is a solution of the system with $x < 0$, then $y \neq 0$ and therefore $(x, -y)$ is a solution too. Thus, when $a < -2$ the system has no solution. When $a = -2$ we get $x = 2(a+1) = -2$ and $x = \dfrac{2}{3}(a-1) = -2$, i.e. the system has exactly two solutions $(-2, \pm 1)$. If $1 > a > -2$, then $2(a+1) \neq \dfrac{2}{3}(a-1)$ and the two values of x give solutions, thus we get four solutions with $x < 0$.

Therefore the system has two solutions only when $a \geq 1$, namely $(a-1, 0)$ and $(a+1, 0)$ and when $a = -2$, namely $(-2, \pm 1)$.

Problem 2. Let M be the midpoint of the side BC of the parallelogram $ABCD$, N be the common point of AM and BD, while P be the common point of AD and CN. Prove that

a) $AP = AD$;

b) $CP = BD$ iff $AB = AC$.

Solution. a) Consider $\triangle ABC$. The lines AM and BD are medians and thus N is the center of gravity. If Q is the intersection point of CP and AB, then Q is the midpoint of AB. It's easy to see that $\triangle APQ \cong \triangle BCQ$ (Figure. 1), from where $AP = BC = AD$.

b) Let $AB = AC$. Then $\triangle ABC$ is isosceles and AM is a median in it. It is easy to see that $\triangle NBC$ is isosceles and $BN = CN$. Analogously NA is a median and an altitude in $\triangle PND$, thus $PN = DN$, i.e. $PC = PN + CN = DN + BN = BD$ (Figure 2). Let $CP = BD$. Trough B we draw a line BF, parallel to CP. Obviously $PFBC$ is a parallelogram and therefore $\angle CPD = \angle BFP$. Since $CP = BF$, then $\triangle DBF$ is isosceles. Consequently

$\angle NDA = \angle BFP = \angle CPD$ and $DN = NP$. A is the midpoint of PD and NA is perpendicular to AD and BC. We get that AM is perpendicular to BC and we deduce from here that $\triangle ABC$ is isosceles, i.e. $AB = AC$.

Figure 1.

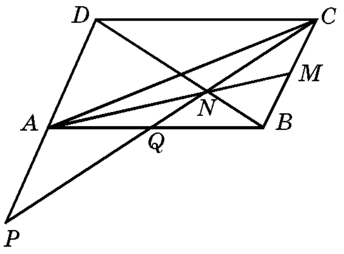
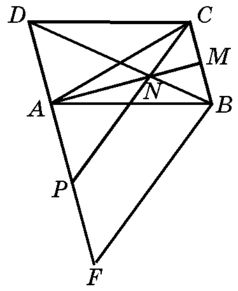


Figure 2.



Problem 3. A convex polygon with n sides, $n \geq 4$, is given. No four vertices of it lie on one and the same circle.

a) Prove that there exists a circle through 3 vertices of the polygon which contains the remaining vertices in its interior.

b) Prove that there exists a circle through 3 consecutive vertices of the polygon which contains the remaining vertices in its interior.

Solution. a) Let AB be a side of the polygon. All the vertices lie in one of the half-planes with respect to AB. The segment AB is seen under different angles from the vertices which are different from A and B. Let C be the vertex from which AB is seen under the smallest angle. The circle we are looking for is defined by A, B and C.

b) We shall use the following two lemmas.

Lemma 1. Let the segment AB be seen from the point X under the angle α and from the point Y under the angle β, where $0 < \alpha < \beta < 90°$. Then the radius of the circumcircle of $\triangle ABX$ is greater than the radius of the circumcircle of $\triangle ABY$.

Lemma 2. For each convex polygon there exist a side and a vertex from which this side is seen under an acute angle.

Proof. Let A_i, A_{i+1} and A_{i+2} be three consecutive vertices of the polygon (Figure 3). At least one of the angles $\alpha A_{i+1}A_iA_{i+2}$ and $\alpha A_iA_{i+2}A_{i+1}$ is acute. This is enough for the proof.

Figure 3.

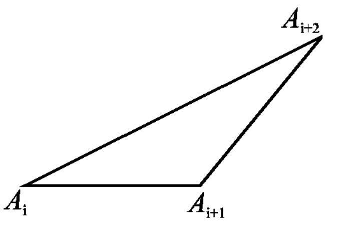
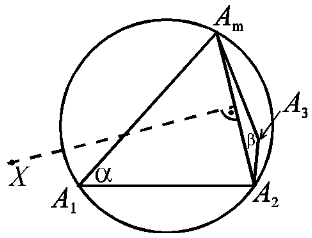


Figure 4.



Let now M be the set of all pairs, that are consisted of a side and a vertex from which this side is seen under an acute angle. Let (AB, C) be an element of M. Consider the circumcircle of $\triangle ABC$ and let N be the set of all such circles. We denote by k the circle in N with the biggest radius. Let k be the circumcircle of $\triangle A_1A_2A_m$, where A_1A_2 is a side of the polygon (Figure

4). We shall show that k is the circle we are looking for. Assume that there exists a vertex A_p which is outside k. Then $\angle A_1A_pA_2 < \angle A_1A_mA_2 < 90°$ and the circumcircle of $\triangle A_1A_2A_p$ is from N and its radius is greater than the radius of k (Lemma 1). We get a contradiction.

Consider one of the acute angles in $\triangle A_1A_2A_m$. Let $\alpha = \angle A_2A_1A_m$ be acute. We shall prove that $A_m = A_3$, which means that k passes through 3 consecutive vertices of the polygon. Assume that $A_m \neq A_3$. Then A_3 is situated in the way which is shown in the Figure 4. If $\beta = \angle A_2A_3A_m$, then $\alpha + \beta > 180°$, i.e. $180° - \beta < \alpha < 90°$. Thus $\angle A_2A_mA_3 < 90°$, which means that (A_2A_3, A_m) is an element of M. Let c be the circumcircle of $\triangle A_2A_3A_m$ and X be the intersection point of c and the segment bisector of A_2A_m. Since $\angle A_2XA_m = 180° - \beta < \alpha = \angle A_2A_1A_m$, then the radius of c is greater than the radius of k (Lemma 1). This is a contradiction.

Grade 9.

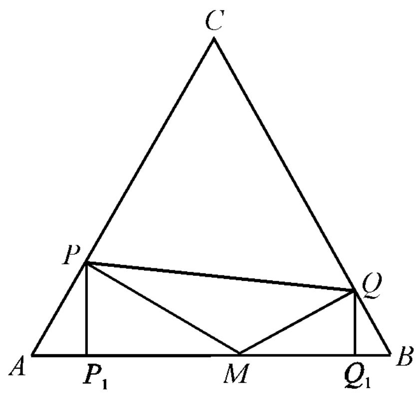
Problem 1. Let M be an arbitrary point on the side $AB = 1$ of the equilateral triangle ABC. The points P and Q are orthogonal projections of M on AC and BC, while P_1 and Q_1 are orthogonal projections of P and Q on AB.

a) Prove that $P_1Q_1 = \dfrac{3}{4}$.

b) Find the position of M for which the segment PQ is with the smallest length.

Solution. a) We have $S_{ABC} = S_{ACM} + S_{BCM} = \dfrac{1}{2}(AC.MP + BC.MQ) = \dfrac{1}{2}(MP + MQ)$ (Figure 5). On the other hand $S_{ABC} = \dfrac{\sqrt{3}}{4}$. Thus $MP + MQ = \dfrac{\sqrt{3}}{2}$. Now from the rectangular triangles P_1MP and MQ_1Q we evaluate $P_1M = \dfrac{\sqrt{3}}{2}MP$ and $MQ_1 = \dfrac{\sqrt{3}}{2}MQ$. From here $P_1Q_1 = P_1M + MQ_1 = \dfrac{\sqrt{3}}{2}(MP + MQ) = \dfrac{3}{4}$.

Figure 5.



b) The orthogonal projection of the segment PQ on AB is the segment P_1Q_1 and thus $PQ \geq P_1Q_1$. Therefore PQ is with minimal length when it is parallel to P_1Q_1. The last is true exactly when $AP = BQ$. We get that $\triangle AMP \cong \triangle BMQ$ and hence $AM = BM$. So, PQ is minimal when M is the midpoint of AB.

Problem 2. The quadratic function $f(x) = -x^2 + 4px - p + 1$ is given. Let S be the area of the triangle with vertices at the intersection points of the parabola $y = f(x)$ with the x-axis and the vertex of the same parabola. Find all rational p, for which S is an integer.

Solution. The discriminant of $f(x)$ is $D = 4(4p^2 - p + 1)$ and $D > 0$ for all real p. Consequently $f(x)$ has two real roots x_1 and x_2, i.e. $f(x)$ intersects the x-axis in two different points — A and B. The vertex C of the parabola has coordinates $2p$ and $h = f(2p) = 4p^2 - p + 1 > 0$. We have

$$AB = |x_1 - x_2| = \sqrt{(x_1 - x_2)^2} = \sqrt{(x_1 + x_2)^2 - 4x_1x_2} = 2\sqrt{4p^2 - p + 1}.$$

Now we find $S = S_{ABC} = \dfrac{AB.h}{2} = (4p^2 - p + 1)^{\frac{3}{2}}$. Denote $q = 4p^2 - p + 1$. Since q is rational and $q^3 = S^2$ is an integer, then q is an integer too. Then $\dfrac{S}{q}$ is rational and $\left(\dfrac{S}{q}\right)^2 = q$ is integer, thus $\dfrac{S}{q}$ is integer too. Therefore $q=n^2$, where n is a positive integer, i.e. $4p^2-p+1-n^2=0$. The quadratic equation (with respect to p) has a rational root exactly when its discriminant $16n^2-15$ is a square of a rational number. Consequently $16n^2-15=m^2$, and we can consider m to be a positive integer. From the equality $(4n-m)(4n+m)=15$ we get $4n-m=1$, $4n+m=15$ or $4n-m=3$, $4n+m=5$. From here $n=2$, $m=7$ or $n=1$, $m=1$. The rational numbers we are looking for are $0, 1, \dfrac{1}{4}, -\dfrac{3}{4}$.

Problem 3. Let n be a positive integer and X be a set with n elements. Prove that

a) The number of all subsets of X (X and $\emptyset$ included) is equal to 2^n.

b) There exist 2^{n-1} subsets of X each pair of which is with common element.

c) There do not exist $2^{n-1}+1$ subsets of X, each pair of which is with common element.

Solution. a) We use induction with respect to n. The base of the induction is obvious. Assume that the assertion is true for a set with $n-1$ elements and let X be with n elements. We can assume that $X=\{1,2,\dots,n\}$. Let $Y=\{1,2,\dots,n-1\}$. All subsets of X are divided into two groups: I group — those which do not contain n and II group — those, which contain n. Both groups have one and the same number of elements because each set of the II group is obtained from exactly one set of the I group by annexing n. Thus the number of the elements of X is twice greater than the number of the subsets of the I group. But the subsets of the I group are exactly the subsets of Y and according to the inductive assumption their number is 2^{n-1}. Thus the number of the subsets of X is 2^n.

b) According to a) the number of the subsets of X from the II group is 2^{n-1} and each pair of them has a common element — the number n.

c) If $A\subseteq X$, let $\overline{A}=X\setminus A$. All subsets of X are divided into pairs $\{A,\overline{A}\}$ and the number of these pairs is 2^{n-1}. Now if we have $2^{n-1}+1$ arbitrary subsets of X, according to the pigeonhole principle, it is not possible that they are in different pairs of the type $\{A,\overline{A}\}$. Consequently among the given 2^n+1 subsets there exist two pairs of the type $\{A,\overline{A}\}$, which obviously have no common element.

Grade 10.

Problem 1. Find all values of the real parameters p and q, for which the roots of the equations $x^2-px-1=0$ and $x^2-qx-1=0$ form (in a suitable order) an arithmetic progression with four members.

Solution. Denote by x_1, x_2 the roots of the equation $x^2-px-1=0$ and by y_1, y_2 the roots of $x^2-qx-1=0$. It is clear that for all p and q the numbers x_1, x_2, y_1, y_2 are real and $x_1x_2=y_1y_2=-1$. Assume that $x_1<0<x_2$ and $y_1<0<y_2$. If four numbers a, b, c and d form an arithmetic progression then the numbers d,c,b,a form an arithmetic progression too. So, we can assume that x_1, x_2, y_1, y_2 in a suitable order form an increasing arithmetic progression. If $x_1<y_1$ (the case $y_1<x_1$ is analogous) then there are two possibilities:

I. The arithmetic progression is x_1, y_1, y_2, x_2. Then $x_1+x_2=y_1+y_2$, from where $p=q$, i.e. $x_1=y_1$, $x_2=y_2$ which is impossible.

II. The arithmetic progression is x_1, y_1, x_2, y_2. Then $x_2-x_1=y_2-y_1$, from where $\sqrt{p^2+4}=\sqrt{q^2+4}$, i.e. $p^2=q^2$ and since $p\neq q$, then $p=-q\neq 0$.

In the same way we have $y_1=\dfrac{x_1+x_2}{2}=\dfrac{p}{2}$ and therefore $\dfrac{p^2}{4}-\dfrac{pq}{2}-1=0$. Since $p=-q$, then $p^2=\dfrac{4}{3}$ and $p=\pm\dfrac{2}{\sqrt{3}}$. Hence $(p,q)=\left(\dfrac{2}{\sqrt{3}},-\dfrac{2}{\sqrt{3}}\right)$, $\left(-\dfrac{2}{\sqrt{3}},\dfrac{2}{\sqrt{3}}\right)$.

Problem 2. Triangle ABC with $AB = 22$, $BC = 19$, $CA = 13$ is given.

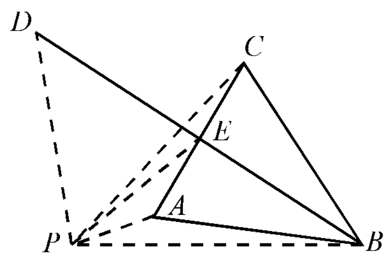
a) If M is the center of gravity of $\triangle ABC$, prove that $AM^2 + CM^2 = BM^2$.

b) Find the locus of points P from the plane of $\triangle ABC$, for which $AP^2 + CP^2 = BP^2$.

c) Find the minimal and maximal values of BP, if $AP^2 + CP^2 = BP^2$.

Solution. a) We have

Figure 6.



$$
\begin{aligned}
AM = \frac{2}{3}m_a &= \frac{2}{3}\cdot\frac{1}{2}\sqrt{2b^2+2c^2-a^2} = \sqrt{105},\\
CM = \frac{2}{3}m_c &= \frac{2}{3}\cdot\frac{1}{2}\sqrt{2a^2+2b^2-c^2} = 8,\\
BM = \frac{2}{3}m_b &= \frac{2}{3}\cdot\frac{1}{2}\sqrt{2a^2+2c^2-b^2} = 13.
\end{aligned}
$$

Hence $AM^2 + CM^2 = 105 + 64 = 169 = BM^2$.

b) Let E be the midpoint of AC and D be symmetric to B with respect to E (Figure 6). We shall prove that the locus we are looking for is a circle k with center D and radius 26. For an arbitrary point P we have $4PE^2 = 2PA^2 + 2PC^2 - AC^2$ and $4PE^2 = 2PB^2 + 2PD^2 - BD^2$, from where $2(PA^2 + PC^2 - PB^2) = 2PD^2 - (BD^2 - AC^2)$. But $BD = 2BE = 39$, i.e. $PA^2 + PC^2 - PB^2 = PD^2 - (26)^2$. It follows from here that the equality $PA^2 + PC^2 = PB^2$ is equivalent to $PD = 26$.

c) Let the circle k intersects the line BD at the points M and N. It follows from b) that BP is minimal when P coincides with M and then $BP = BM = 13$. BP is maximal when P coincides with N, which gives $BN = 65$.

Problem 3. Find the smallest positive integer n, for which there exist n different positive integers $a_1, a_2, \ldots, a_n$ satisfying the conditions:

a) the smallest common multiple of $a_1, a_2, \ldots, a_n$ is 1995;

b) for each $i, j \in \{1, 2, \ldots, n\}$ the numbers a_i and a_j have a common divisor $\neq 1$;

c) the product $a_1 a_2 \ldots a_n$ is a perfect square and is divisible by 243.

Find all n-ples $(a_1, a_2, \ldots, a_n)$, satisfying a), b) and c).

Solution. Since $1995 = 3.5.7.19$ and $a_i/1995$, then for all i

$$a_i = 3^{\alpha_i} 5^{\beta_i} 7^{\gamma_i} 19^{\delta_i} \qquad (*)$$

where the numbers α_i, β_i, γ_i, δ_i are equal to 0 or 1. There is no a_i which is divisible by 9. We have $a_1 a_2 \ldots a_n = k^2$, where k is a positive integer and since $243 = 3^5$ divides k^2, then 3^6 divides k^2 and since among the numbers $(*)$ there is no one which is divisible by 9, then $n \geq 6$. The numbers $(*)$, which are divisible by 3 are

$$3;\quad 3.5;\quad 3.7;\quad 3.19;\quad 3.5.7;\quad 3.5.19;\quad 3.7.19;\quad 3.5.7.19 \qquad (**)$$

They are 8 in number. Let $n = 6$. Then the numbers $a_1, a_2, \ldots, a_n$ are among $(**)$. It is easy to see that the product of any 6 numbers from $(**)$ is not a perfect square. Thus $n \geq 7$. Let $n = 7$. It is not possible that all the numbers $a_1, a_2, \ldots, a_n$ are divisible by 3, because in such a case $3^7/k^2$ and $3^8 \not| \, k^2$. Therefore 6 numbers from $a_1, a_2, \ldots, a_n$ are divisible by 3, i.e. they are from $(**)$ and at least one of them (for example a_1) is not divisible by 3. It follows from a) that among $a_1, a_2, \ldots, a_7$ there is at least one which is divisible by 5, at least one which is divisible by 7 and at least one which is divisible by 19. Since each pair of these numbers has a common divisor, and $3 \not| \, a_1$, then a_1 must be divisible by 5.7.19, i.e. $a_1 = 5.7.19$. At last note that the product of all numbers (**) is equal to $3^8.5^4.7^4.19^4$. The only possibility is $a_1a_2\dots a_7 = 3^6.5^4.7^4.19^4$. From here $a_2a_3\dots a_7 = 3^6.5^3.7^3.19^3$ and the only possibility is

$$\{a_2, a_3, \dots, a_7\} = \{3\cdot 5,\ 3\cdot 7,\ 3\cdot 19,\ 3\cdot 5\cdot 7,\ 3\cdot 5\cdot 19,\ 3\cdot 7\cdot 19\}.$$

Therefore $n = 7$ and

$$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\} = \{3\cdot 5,\ 3\cdot 7,\ 3\cdot 19,\ 3\cdot 5\cdot 7,\ 3\cdot 5\cdot 19,\ 3\cdot 7\cdot 19\ 5\cdot 7\cdot 19\}.$$

Grade 11.

Problem 1. Let $a_n = \dfrac{n+1}{2^{n+1}}\left(\dfrac{2^1}{1} + \dfrac{2^2}{2} + \dots + \dfrac{2^n}{n}\right)$, $n = 1, 2, 3, \dots$. Prove that
a) $a_{n+1} \le a_n$ for all $n \ge 3$;
b) the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent and find its limit.
Solution. a) We have

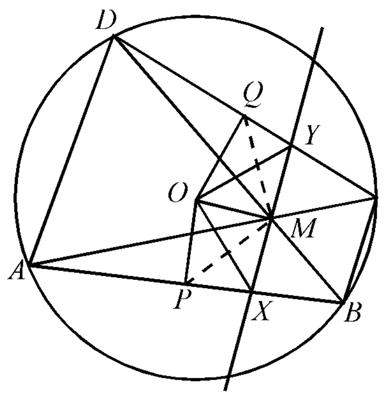
$$a_{n+1} = \frac{n+2}{2^{n+2}}\left(\frac{2^1}{1} + \frac{2^2}{2} + \dots + \frac{2^{n+1}}{n+1}\right) = \frac{n+2}{2(n+1)}(a_n + 1).$$

From here

$$a_{n+2} - a_{n+1} = \frac{(n+2)^2(a_{n+1} - a_n) - (a_n + 1)}{2(n+1)(n+2)}, \quad n = 1, 2, 3 \dots.$$

Since $a_n > 0$ for all n, then if $a_{n+1} - a_n \le 0$, we have $a_{n+2} - a_{n+1} \le 0$. But $a_3 = \dfrac{5}{3}$ and $a_4 = \dfrac{5}{3}$, i.e. $a_4 - a_3 = 0$. Hence $a_5 - a_4 \le 0$, $a_6 - a_5 \le 0$, …, $a_{n+1} - a_n \le 0$.

Figure 7.



b) The sequence $a_1, a_2, a_3, \dots$ is decreasing when $n \ge 3$ and it is bounded ($a_n > 0$ for all n). Therefore this sequence is convergent. Let $\lim_{n\to\infty} a_n = a$. From the equality $a_{n+1} = \dfrac{n+2}{2(n+1)}(a_n + 1)$ after passing to infinity we get $a = \dfrac{1}{2}(a+1)$, i.e. $a = 1$.

Problem 2. The quadrilateral $ABCD$ is inscribed in a circle with center O. The diagonals AC and BD intersect each other in the point M, $M \ne O$. The line through M which is perpendicular to OM intersects the sides AB and CD of the quadrilateral $ABCD$ in the points X and Y, respectively. Prove that $AB = CD$ iff $BX = CY$.

Solution. If $AB = CD$, then $ABCD$ is isosceles trapezoid. Hence $OM \perp AD$ and $OM \perp BC$, from where $XY \| BC$ and $BX = CY$.

Let $BX = CY$. Denote by P and Q the midpoints of AB and CD, respectively. The quadrilaterals $OPXM$ and $OQYM$ are inscribed, thus $\angle OPM = \angle OXM$ and $\angle OQM = \angle OYM$. We shall prove that $\angle OPM = \angle OQM$. The triangles ABM and DCM are congruent and

MP and MQ are medians in them. Therefore $\triangle MPB \sim \triangle MQC$ and $\angle MPB = \angle MQC$. Then $\angle OPM = 90^\circ - \angle MPB = 90^\circ - \angle MQC = \angle OQM$. (Or $\angle OPM = \angle MPB - 90^\circ = \angle MQC - 90^\circ = \angle OQM$.)

Hence $\angle OXM = \angle OYM$ and $OX = OY$. Therefore $\triangle OXB \cong \triangle OYC$. From here $\angle OBA = \angle OCD$ and the isosceles triangles ABO and DCO are equal, i.e. $AB = CD$

Remark. If $X = P$ and $Y = Q$, then $PB = QC = \frac{1}{2}AB = \frac{1}{2}CD$.

Problem 3. Let n be a positive integer and let

$$f(x) = x^n + (k+1)x^{n-1} + (2k+1)x^{n-2} + \cdots + ((n-1)k+1)x + nk + 1.$$

a) Prove that $f(1-k) = n+1$.

b) Prove that if $n \geq 3$ and k is an integer ($k \neq 0$), then the equation $f(x) = 0$ has no integer solution.

Solution. a) Since

$$\begin{aligned} & x^{n-1} + 2x^{n-2} + \cdots + (n-1)x + n \\ = \; & (x^{n-1} + \cdots + x + 1) + (x^{n-2} + \cdots + x + 1) + \cdots + (x^2 + x + 1) + (x+1) + 1 \\ = \; & \frac{x^n - 1}{x-1} + \frac{x^{n-1}-1}{x-1} + \cdots + \frac{x^3-1}{x-1} + \frac{x^2-1}{x+1} + \frac{x-1}{x-1}, \end{aligned}$$

then

$$\begin{aligned} x^{n-1} + 2x^{n-2} + \cdots + (n-1)x + n &= \frac{1}{x-1}\left(\frac{x^{n+1}-1}{x-1} - (n+1)\right) \\ &= \frac{x^{n+1} - (n+1)x + n}{(x-1)^2} \end{aligned}$$

when $x \neq 1$. Thus

$$\begin{aligned} f(x) &= x^n + x^{n-1} + \cdots + x + 1 + k(x^{n-1} + 2x^{n-2} + \cdots + n) \\ &= \frac{x^{n+1}-1}{x-1} + k\frac{x^{n+1} - (n+1)x + n}{(x-1)^2}, \end{aligned}$$

i.e.

$$f(x) = \frac{x^{n+2} + (k-1)x^{n+1} - [k(n+1)+1]x + kn + 1}{(x-1)^2}$$

when $x \neq 1$.

From here

$$f(1-k) = \frac{-[k(n+1)+1](1-k) + kn + 1}{k^2} = n+1,$$

when $1 - k \neq 1$. If $1 - k = 1$, i.e. $k = 0$, then $f(1) = n+1$.

b) Let $n \geq 3$ and k be an integer ($k \neq 0$). Assume that $f(a) = 0$. Obviously $a \neq 0$. We have

$$a^n + a^{n-1} + \cdots + a + 1 = -k(a^{n-1} + 2a^{n-2} + \cdots + (n-1)a + n).$$

If $a = -1$, the left hand side of the equation is equal to 0 or ± 1, while the right hand side is neither 0 nor ± 1, because $|n - (n-1) + (n-2) - \cdots| > 1$ when $n \geq 3$ ($k \neq 0$). Thus $a \neq -1$. The equation $f(a) = 0$ can be written in the following way:

$$(-a - k + 1)(a^{n-1} + 2a^{n-2} + \cdots + (n-1)a + n) = n + 1.$$

Hence $r_n(a) = a^{n-1} + 2a^{n-2} + \cdots + (n-1)a + n$ divides $n+1$. If $a > 1$, then $r_n(a) > n+1$ when $n \geq 3$ and this is impossible. From $a \neq 0$ and $a \neq -1$ it follows that $a \leq -2$. We shall prove that the inequality $|r_n(t)| \geq n+2$ is satisfied for all integers $t \leq -2$ and for all $n \geq 3$ except $t = -2$, $n = 3$ and $t = -2$, $n = 4$. Since $r_n(-2) = \dfrac{(-2)^{n+1} + 3n + 2}{9}$, then $r_3(-2) = 3$, $r_4(-2) = -2$ and $r_5(-2) = 9$, i.e. $|r_5(-2)| \geq 7$. From $r_3(t) = t^2 + 2t + 3$ it follows that $|r_3(t)| \geq 5$ when $t \leq -3$.

Now we shall use induction. If $|r_n(t)| \geq n+2$ for $t \leq -2$ and $n \geq 3$, then $r_{n+1}(t) = t.r_n(t) + n + 1$ and $|r_{n+1}(t)| \geq |t|.|r_n(t)| - (n+1) \geq 2(n+2) - n - 1 = n+3$, i.e. $|r_{n+1}(t)| \geq n+3$. Hence $|r_n(t)| \geq n+2$ when $n \geq 3$ and $t \leq -2$ except the cases $n = 3, t = -2$ and $n = 4, t = -2$. Thus $r_n(a)$ does not divide $n+1$ when $n \geq 3$ and $a \leq -2$, because $r_3(-2) = 3$, $r_4(-2) = -2$. For all others $n \geq 3$ and $a \leq -2$ we have $|r_n(a)| \geq n+2$.

Grade 12.

Problem 1. The function $f(x) = \sqrt{1-x}$ $(x \leq 1)$ is given. Let $F(x) = f(f(x))$.

a) Solve the equations $f(x) = x$ and $F(x) = x$.

b) Solve the inequality $F(x) > x$.

c) If $a_0 \in (0,1)$, prove that the sequence $\{a_n\}_{n=0}^{\infty}$, determined by $a_n = f(a_{n-1})$ for $n = 1, 2, \ldots$, is convergent and find its limit.

Solution. The functions $f(x), F(x)$ and $F(x) - x$ are defined for all $x \in [0,1]$.

a) The equation $f(x) = x$, i.e. $\sqrt{1-x} = x$ has only one root $\alpha = \dfrac{-1+\sqrt{5}}{2}$. It is clear that $\alpha \in (0,1)$ and the roots of the equation $F(x) = x$ are $x_1 = 0$, $x_2 = 1$, $x_3 = \alpha$.

b) In $[0,\alpha]$ the function $F(x) - x$ has a constant sign. The contrary would imply that there is $\beta \in (0,\alpha)$ such that $F(\beta) = \beta$ and this contradicts the result from a). Analogously in $[\alpha, 1]$ the function $F(x) - x$ has a constant sign. On the other hand $\dfrac{1}{4} \in (0,\alpha)$, $\dfrac{3}{4} \in (\alpha, 1)$ and $F\left(\dfrac{1}{4}\right) - \dfrac{1}{4} > 0$, $F\left(\dfrac{3}{4}\right) - \dfrac{3}{4} < 0$. From here $F(x) > x$ iff $x \in \left(0, \dfrac{-1+\sqrt{5}}{2}\right)$.

c) Let $a_0 = \alpha$. It follows from a) that $a_n = \alpha$ for $n = 0, 1, 2, \ldots$ and hence the sequence is convergent and its limit is α. Let now $a_0 < \alpha$. Since $f'(x) = -\dfrac{1}{2\sqrt{1-x}} < 0$ for all $x \in [0,1)$, then $f(x)$ is decreasing. From here $frac(a_0) > frac(\alpha) = \alpha$, i.e. $a_1 > \alpha$. By induction $a_{2n} \in (0,\alpha)$ and $a_{2n+1} \in (\alpha, 1)$ for all $n = 0, 1, 2, \ldots$. On the other hand it follows from the result of b) that for all $x \in (0,\alpha)$ we have $F(x) > x$, while for $x \in (\alpha, 1)$ we have $F(x) < x$, respectively. Also $F'(x) = f'(f(x)).f'(x) > 0$, i.e. $F(x)$ is increasing and hence $F(x) \in (x, \alpha)$ if $x \in (0,\alpha)$ and $F(x) \in (\alpha, x)$ if $x \in (\alpha, 1)$. By induction we get

$$a_0 < a_2 < \cdots < a_{2n} < \cdots < \alpha,$$
$$a_1 > a_3 > \cdots > a_{2n+1} > \cdots > \alpha.$$

Both sequences are convergent and let their limits be α_1 and α_2, respectively. We have $F(a_{2n}) = a_{2n+2}$ and $F(a_{2n+1}) = a_{2n+3}$ for $n = 0, 1, 2, \ldots$. The function $F(x)$ is continuous and thus $F(\alpha_1) = \alpha_1$ and $F(\alpha_2) = \alpha_2$. We get $\alpha_1 = \alpha_2 = \alpha$ because the only solution of $F(x) = x$ in $(0,1)$ is α. Therefore the sequence $\{a_n\}_{n=0}^{\infty}$ is convergent and its limit is α. The case $\alpha < a_0$ is analogous.

Problem 2. The sides AC and BC of the triangle ABC are diameters of two circles, each of which touches internally a circle k, which is concentric to the incircle of $\triangle ABC$.

a) Prove that $AC = BC$.

b) If $\cos \angle BAC = \frac{3}{4}$, find the ratio of the radii of k and the incircle of $\triangle ABC$.

Solution. a) Let I be the center of the incircle of $\triangle ABC$, N be the common point of this circle with AC and M be the midpoint of AC. Let r_1 be the radius of k and r be the radius of the incircle of $\triangle ABC$. From the condition it follows that $IM = \frac{b}{2} - r_1$ and from the rectangular $\triangle INM$ we get $MN^2 = \left(\frac{b}{2} - r_1\right)^2 - r^2$. On the other hand $AN = \frac{b+c-a}{2}$, i.e. $MN = |AM - AN| = \frac{|c-a|}{2}$. Therefore

$$(c-a)^2 = (b-2r_1)^2 - 4r^2. \tag{1}$$

Analogously

$$(c-b)^2 = (a-2r_1)^2 - 4r^2. \tag{2}$$

Assume that $a \neq b$. From (1) and (2) we get $(b-a)(2c-a-b) = (b-a)(a+b-4r_1)$, i.e. $2c-a-b = a+b-4r_1$. From here $b - 2r_1 = c - a$ and from (1) it follows that $r = 0$, which is impossible. Thus $a = b$.

b) Let $\frac{r_1}{r} = t$ and $\angle BAC = \alpha$. Then $c = 2b\cos\alpha = \frac{3}{2}b$ and

$$r = \frac{c}{2}\tan\frac{\alpha}{2} = b\cos\alpha\tan\frac{\alpha}{2} = b\cos\alpha\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{3b}{4\sqrt{7}}.$$

Since $b > 2r_1$, from (1) it follows that $t = \dfrac{b - \sqrt{(c-b)^2 + 4r^2}}{2r}$ and substituting c and r we get $t = \frac{2}{3}(\sqrt{7} - 2)$.

Problem 3. n points ($n > 4$), no three of which are colinear are given in the plane. More than n triangles are constructed with vertices among these points. Prove that at least two triangles have exactly one common vertex.

Solution. Assume the contrary and let k be the smallest number for which the assertion is not true. This means that there are constructed at least $(k+1)$ triangles using k points. It follows from the pigeonhole principle that there exists a point A which is a vertex of at least 4 triangles. Let ABC be the first triangle. At least one of the points B and C is a vertex of the second triangle, which we denote by ABD. If ACX is the third triangle then $X = D$. Thus the forth triangle must contain B or C, which is impossible. Therefore if A and B are vertices of two triangles then they are vertices of all the four triangles. Let A be a vertex of t triangles, $t \geq 4$. These triangles are of the kind ABA_1, ABA_2, …, ABA_t, where all the points $A_1, A_2, \ldots, A_t$ are pairwise different. Obviously it is not possible to exist a triangle of the type BXY, where X and Y are points which are different from $A_1, A_2, \ldots, A_t$. Triangles BA_iA_j and $A_iA_jA_m$ do not exist too. Hence the points A, B, A_1, A_2, …, A_t are vertices only of the triangles ABA_1, ABA_2, …, ABA_t. In such a way we use $t+2$ points and get t triangles. It is not possible that $t + 2 = k$, because all triangles are $t < k$. The number of remaining points is $k_0 = k - t - 2$ and by them there are constructed at least $k + 1 - t > k_0$ triangles, such that no two of them have exactly one common vertex. The triangles are more than the points k_0, and thus $k_0 > 4$. We have found a number $k_0 < k$ for which the assertion is not true. This contradicts the choice of k.