SPRING MATHEMATICAL COMPETITION

1996

Grade 8

Problem 1. Prove that for all real $a \in (1,2)$ the area of the figure encountered by the graphs of the functions $y = 1 - |x-1|$ and $y = |2x - a|$ is less than $\frac{1}{3}$.

Solution. Firstly, we shall find the common points of the given functions. For this purpose we solve the equation

$$|2x - a| = 1 - |1 - x|. \tag{1}$$

Since $1 < a < 2$, then $\frac{a}{2} < 1$, and we shall consider the cases: $x \le \frac{a}{2}$, $\frac{a}{2} < x < 1$ and $x \ge 1$. We have:

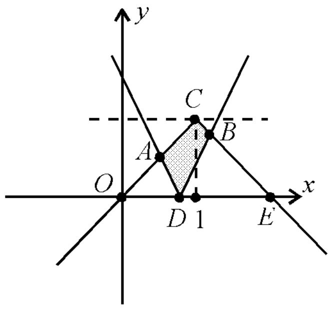
1. When $x \le \frac{a}{2}$ the equation (1) takes the form $a - 2x = x$. Then $x = \frac{a}{3}$, which satisfies (1), because $\frac{a}{3} < \frac{a}{2}$.

2. When $\frac{a}{2} < x < 1$ the equation (1) takes the form $2x - a = x$, i.e. $x = a$, which does not satisfy (1), because $a > 1$.

3. When $x \ge 1$ the equation (1) takes the form $2x - a = 2 - x$. Then $x = \frac{a+2}{3}$, which satisfies (1), because $\frac{a+2}{3} > 1$ when $a > 1$.

Thus, the graphs of the two functions have two common points (Figure 1), the first of which (denoted by A) has coordinates $x_A = \frac{a}{3}$, $y_A = \frac{a}{3}$, while the second one (denoted by B) has coordinates $x_B = \frac{a+2}{3}$, $y_B = 2 - \frac{a+2}{3} = \frac{4-a}{3}$. Denote by C, D and E the points, with coordinates $x_C = 1$, $y_C = 1$; $x_D = \frac{a}{2}$, $y_D = 0$ and $x_E = 2$, $y_E = 0$, respectively.

Figure 1.

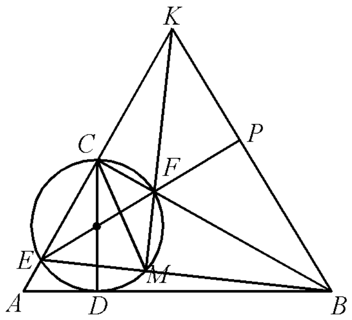


Then, the figure encountered by the two graphs is the quadrilateral $ACBD$ and its area S is obtained by subtracting the areas of the triangles ODA and BDE from the area of the triangle OEC. Therefore,

$$\begin{aligned} S &= S_{OEC} - S_{ODA} - S_{BDE} \\ &= 1 - \frac{1}{2} \cdot OD \cdot y_A - \frac{1}{12} \cdot DE \cdot y_B \\ &= 1 - \frac{1}{2} \cdot \frac{a}{2} \cdot \frac{a}{3} - \frac{1}{2} \cdot \left(2 - \frac{a}{2}\right) \cdot \frac{4-a}{3} \end{aligned}$$

$$= 1 - \frac{1}{12} \cdot a^2 - \frac{1}{12} \cdot (4-a)^2$$
$$= \frac{1}{6} \cdot \left(-a^2 + 4a - 2\right)$$
$$= \frac{1}{6} \cdot \left(2 - (a-2)^2\right) < \frac{1}{3}.$$

Figure 2.



Problem 2. The altitude CD of the rectangle triangle ABC ($\angle ACB = 90°$) is a diameter of the circle k, which meets the sides AC and BC in E and F, respectively. The intersection point of the line BE and the circle k, which is different from E, is denoted by M. Let the intersection point of the lines AC and MF be K, and the intersection point of the lines EF and BK be P.

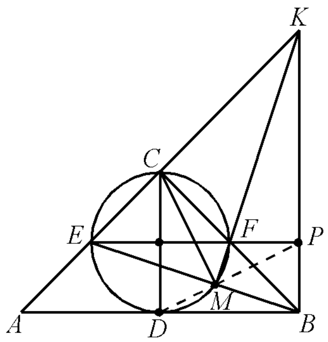
a) Prove that the points B, F, M and P are concyclic;

b) If the points D, M and P are colinear, find the angles A and B of the triangle ABC.

Solution. a) Since $\angle ECF = 90°$, then EF is a diameter of the circle k and consequently $\angle EMF = 90° = \angle BMK$. Then BC and KM are altitudes in $\triangle BEK$ (Figure 2), which means that the point F is the altitude center of this triangle. It follows from here that $EP \perp BK$, i.e. $\angle BPF = 90°$.

Thus, $\angle BPF = \angle BMF = 90°$ and consequently the points B, F, M and P lie on a circle with diameter BF.

Figure 3.



b) We have:

$$\begin{aligned}\angle BDM &= 90° - \angle MDC = 90° - \angle MEC \\ &= \angle CBE = \angle FBM = \angle FPM.\end{aligned}$$

If the points D, M and P are colinear (Figure 3), then the equality between the angles $\angle BDM$ and $\angle FPM$ implies that the lines AB and EF are parallel, i.e. EF and CD are diameters of the circle k, which are perpendicular to each other. This means that CD is the bisector of $\angle ACB$. Consequently, $AC = BC$ and $\angle BAC = \angle ABC = 45°$.

Problem 3. In a state every town is connected with the nearest town by a straight way. The distances between the pairs of towns are pairwise different. Prove that

a) no two ways have common points;

b) every town is connected by ways with at most 5 other towns;

c) there is no closed piecewise line, consisted of ways.

Solution. a) Suppose that the ways AC and BD meet each other (Figure 4) and let C be the nearest town to A, while D be the nearest town to B. Then $AC < AD$ and $BD < BC$, from where $AC + BD < AD + BC$.

On the other hand, if O is the common point of AC and BD, then $AO + OD > AD$ and $BO + OC > BC$, from where $AC + BD > AD + BC$. This is a contradiction.

b) Let the town X be connected by ways with the towns A and B (Figure 5). Then AB is the longest side of $\triangle XAB$. Indeed, if we assume that for example AX is the longest side, then

A should not be the nearest town to X, and X should not be the nearest town to A as well. Consequently, the way AX should not exist.

Figure 4.

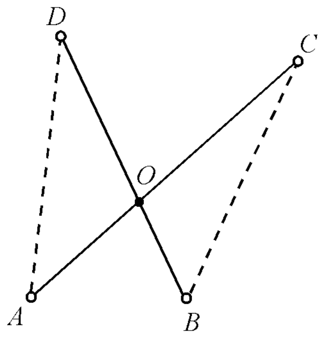


Figure 5.

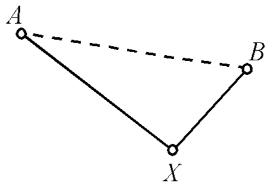
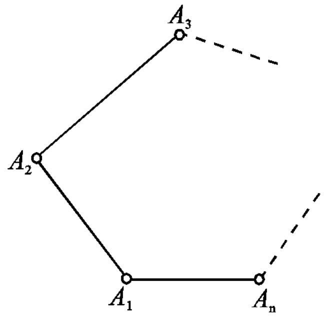


Figure 6.



Therefore $\angle AXB$ is the biggest angle in $\triangle XAB$, from where $\angle AXB > 60°$. Now, if we assume that the town X is connected with at least 6 towns, then the sum of the angles at X would be greater than $6 \cdot 60° = 360°$, which is impossible. Thus, every town is connected with at most 5 other towns.

c) Assume that there exists a closed piecewise line $A_1A_2\ldots A_n$, consisted of ways (Figure 6). The distances A_1A_n and A_1A_2 are different. Let $A_1A_n < A_1A_2$. Then A_2 is not the nearest town to A_1 and consequently (because the way A_1A_2 exists) A_1 is the nearest town to A_2. It follows from here that $A_1A_2 < A_2A_3$.

Proceeding in this way we obtain the chain: $A_1A_n < A_1A_2 < A_2A_3 < \ldots < A_nA_1$, which leads to contradiction.

Grade 9

Problem 1. Find the values of the real parameter b, for which the difference between the maximal and the minimal values of the function $f(x) = x^2 - 2bx + 1$ in the interval $[0, 1]$ is equal to 4.

Figure 7.

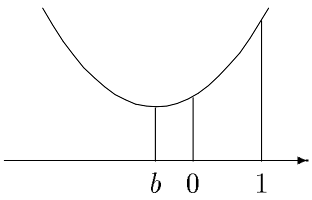


Figure 8.

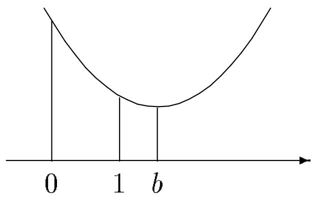
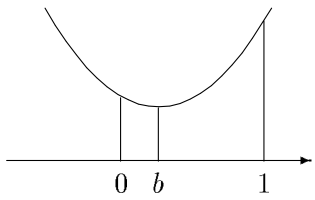


Figure 9.



Solution. It is clear that the minimal value of the quadratic function $f(x)$ is obtained when $x = b$. We shall consider the following three cases:

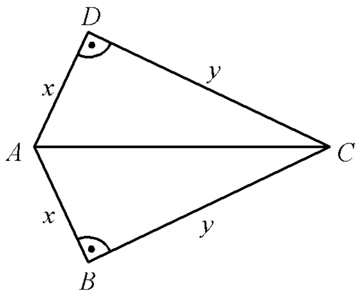
1. Let $b < 0$. In this case the function $f(x)$ is increasing in the interval $[0, 1]$ (Figure 7) and the maximal value is $f(1) = 2 - 2b$, while the smallest one is $f(0) = 1$. From the condition $f(1) - f(0) = 1 - 2b = 4$ we find $b = -\frac{3}{2}$, which is a solution of the problem.

2. Let $b > 1$. Now the function $f(x)$ is decreasing in the interval $[0, 1]$ (Figure 8) and the maximal value is $f(0) = 1$, while the minimal one is $f(1) = 2 - 2b$. From the condition $f(0) - f(1) = 2b - 1 = 4$ we find $b = \dfrac{5}{2}$, which is a solution of the problem.

3. Let $0 \le b \le 1$ (Figure 9). The minimal value of $f(x)$ in the interval $[0, 1]$ is $f(b) = 1 - b^2$, while the the maximal one is $f(0)$ or $f(1)$. From $f(0) - f(b) = b^2 = 4$ we find $b = \pm 2$, which are not solutions (because in this case $0 \le b \le 1$). From $f(1) - f(b) = (1 - b)^2 = 4$ we find $b = 3$ or $b = -1$, which are not solutions too.

Finally, the answers are: $b = -\dfrac{3}{2}$ and $b = \dfrac{5}{2}$.

Figure 10.



Problem 2. The quadrilateral $ABCD$ is inscribed in a circle with radius 1, a circle can be inscribed in it and $AB = AD$. Prove that:

a) the area of the quadrilateral $ABCD$ does not exceed 2;

b) the inradius of the quadrilateral $ABCD$ does not exceed $\dfrac{\sqrt{2}}{2}$.

Solution. We have (Figure 10) $AB + CD = AD + BC$ and from $AB = AD$ we get: $BC = CD$. Let $AB = AD = x$, $BC = CD = y$. The triangles ACB and ACD are equal (by **SSS**), from where $\angle B = \angle D$. But $ABCD$ is inscribed in a circle and consequently $\angle B + \angle D = 180°$. Therefore $\angle B = \angle D = 90°$. Then AC is diameter of the circumcircle of the quadrilateral and particularly $AC = 2$ and $x^2 + y^2 = AC^2 = 4$.

Let S, p and r be the area of the quadrilateral, its semiperimeter and inradius, respectively.

a) We have $S = S_{ABC} + S_{ACD} = 2S_{ACB} = xy$ and

$$xy = \sqrt{x^2y^2} \le \frac{x^2 + y^2}{2} = \frac{AC^2}{2} = 2$$

Thus, $S \le 2$ (and $S = 2$ only if $x = y$ and then the quadrilateral $ABCD$ is a square).

b) We shall make use of the formula $S = pr$, which is true for all polygons that can be inscribed in a circle. We have:

$$r^2 = \frac{S^2}{p^2} = \frac{S^2}{(x+y)^2} = fracS^2x^2 + y^2 + 2xy = \frac{S^2}{4 + 2S}.$$

Now

$$\begin{aligned} r \le \frac{\sqrt{2}}{2} \quad &\Longleftrightarrow \quad r^2 \le \frac{1}{2} \quad \Longleftrightarrow \quad \frac{S^2}{4 + 2S} \le \frac{1}{2} \\ &\Longleftrightarrow \quad S^2 - S - 2 \le 0 \quad \Longleftrightarrow \quad (S + 1)(S - 2) \le 0 \\ &\Longleftrightarrow \quad S \le 2 \end{aligned}$$

(because $S > 0$). The last inequality is true according to a) and consequently the inequality $r \le \dfrac{\sqrt{2}}{2}$ is true too.

Problem 3. This problem is the same as problem 3, grade 8.

Grade 10

Problem 1. Find in the plane the locus of points with coordinates (x, y), for which there exists exactly one real number z, satisfying the equality:

$$xz^4 + yz^3 - 2\left(x + |y|\right)z^2 + yz + x = 0.$$

Solution. Let (x, y) be the coordinates of a point from the locus, we are looking for. This means that there exists exactly one real number z, for which

$$xz^4 + yz^3 - 2\left(x + |y|\right)z^2 + yz + x = 0.$$

If $x = 0$, the above equality takes the form

$$z\left(yz^2 - 2|y|z + y\right) = 0.$$

This equality is satisfied for at least two different values of z (for example $z = 0$ and $z = 1$, if $y \geq 0$ and $z = 0$ and $z = -1$, if $y < 0$), which shows that the condition of the problem is not verified. Consequently, the points from the y-axis do not belong to the locus.

Let $x \neq 0$. Thus, if z satisfies the given equality, then $z \neq 0$. It is easy to see that in this case the number $\dfrac{1}{z}$ satisfies the same equality and consequently $z = \dfrac{1}{z}$. From here $z^2 = 1$, i.e. $z = \pm 1$.

Case 1. Let $z = 1$. Then $x + y - 2\left(x + |y|\right) + y + x = 0$, from where $y - |y| = 0$. This shows that $y \geq 0$. If $y = 0$ and then the given equality takes the form $x\left(z^2 - 1\right)^2 = 0$, and consequently it is satisfied also by $z = -1$. Thus, we can assume that $y > 0$. Then,

$$xz^4 + yz^3 - 2(x + y)z^2 + yz + x = 0,$$

from where

$$(z - 1)^2\left(xz^2 + (y + 2x)z + x\right) = 0.$$

We have one of the following possibilities:

(i) The number $z = 1$ is the only to satisfy the equality $xz^2 + (y + 2x)z + x = 0$. This is true if $x + y + 2x + x = 0$ and $D = (y + 2x)^2 - 4x^2 = y(y + 4x) = 0$, from where $y = -4x$ and $y > 0$.

(ii) There is no real z, which satisfy the equality $xz^2 + (y + 2x)z + x = 0$. Then, $D = (y + 2x)^2 - 4x^2 = y(y + 4x) < 0$, from where $y < -4x$ and $y > 0$.

Therefore, every time when $y \leq -4x$ and $y > 0$, the point (x, y) belongs to the locus.

Case 2. Let $z = -1$. Analogously to the previous case we find that $y < 0$ and then the given equality takes the form

$$(z + 1)^2\left(xz^2 + (y - 2x)z + x\right) = 0.$$

We have the following possibilities:

(i) the number $z = -1$ is double root of the equation

$$xz^2 + (y - 2x)z + x = 0. \qquad (*)$$

(ii) the equation (*) has no real root.

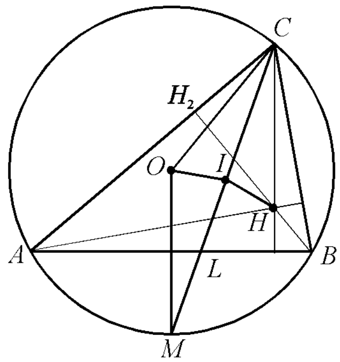
By computing we deduce that here $y \geq 4x$ and $y < 0$.

Finally, the locus (Figure 11) consists of the internal as well as of the boundary points of the angle which is defined by the graphs of the linear functions $y = 4x$ and $y = -4x$ when $x < 0$, without the points from the negative part of the $x-$ axis.

Problem 2. In the triangle ABC, h_a and h_b are the altitudes from A and B respectively, ℓ_c is the internal bisector of $\angle ACB$, while O, I and H are the circumcenter, the incenter and the altitude center, respectively. Prove that if $\dfrac{\ell_c}{h_a} + \dfrac{\ell_c}{h_b} = 2$, then $OI = IH$.

Solution. Let $BC = a$, $AC = b$, $\angle ACB = \gamma$, $CL = \ell_c$ (Figure 12). Since $S_{ABC} = S_{ALC} + S_{BLC}$, then $\dfrac{1}{2} \cdot \ell_c a \sin\dfrac{\gamma}{2} + \dfrac{1}{2} \cdot \ell_c b \sin\dfrac{\gamma}{2} = \dfrac{1}{2} \cdot ab\sin\gamma$, from where $\ell_c = \dfrac{2ab\cos\frac{\gamma}{2}}{a+b}$. But $h_a = b\sin\gamma$ and $h_b = a\sin\gamma$. Then $\dfrac{\ell_c}{h_a} + \dfrac{\ell_c}{h_b} = \dfrac{2a\cos\frac{\gamma}{2}}{(a+b)\sin\gamma} + \dfrac{2b\cos\frac{\gamma}{2}}{(a+b)\sin\gamma} = \dfrac{1}{\sin\frac{\gamma}{2}}\left(\dfrac{a}{a+b} + \dfrac{b}{a+b}\right) = \dfrac{1}{\sin\frac{\gamma}{2}}$, which shows that $\sin\dfrac{\gamma}{2} = \dfrac{1}{2}$, i.e. $\dfrac{\gamma}{2} = 30°$, because $\dfrac{\gamma}{2} < 90°$. Consequently, $\gamma = 60°$.

Figure 12.



Let CL meet the circumcircle of $\triangle ABC$ in M. Then $OM \perp AB$. But $CH \perp AB$, and therefore $\angle OMC = \angle MCH$. On the other hand $OM = OC = R$ (R is the circumradius). Consequently, $\angle OMC = \angle OCM = \angle MCH$. Let H_2 be the foot of the altitude from B. From $\triangle HCH_2$ we have $CH = \dfrac{CH_2}{\sin\alpha}$ (because $\angle H_2HC = 90° - \angle ACH = \angle CAB = \alpha$). But $CH_2 = a\cos\gamma$, i.e. $CH = \dfrac{a}{\sin\alpha} \cdot \cos\gamma = 2R\cos\gamma$. Since $\gamma = 60°$, then $CH = R$.

We consider $\triangle COI$ and $\triangle CHI$. Since the point I lies on CM, then $\angle OCI = \angle HCI$. Also, $CO = R = CH$ and consequently $\triangle COI \cong \triangle CHI$. Thus, $OI = IH$.

Problem 3. Let A_1, A_2, ..., A_n $(n \geq 4)$ be n points in the plane, no 3 of which are colinear.

a) Prove that there is at most one point A_s, such that all triangles $A_sA_iA_j$ $(i, j = 1, 2, \dots, n)$ are acute.

b) Let among A_1, A_2, ..., A_n be a point, which is a vertex of an acute triangles only. We consider the angles defined by the given points. Denote by N_k the number of the acute angles $\angle A_iA_kA_j$ $(i, j = 1, 2, \dots, n)$ for which the point A_k is their vertex. Find the minimal value of N_k.

Solution. a) Assume that there is more than one point with the given property and let A and B be such two points, while X and Y be any two of the remaining points. There are two possibilities for the points A, B, X and Y:

(i) A, B, X and Y define a convex quadrilateral. Because the sum of its internal angles is 360°, then at least one of these angles is not acute. Consequently, at least one of the triangles with vertex A or B is not acute.

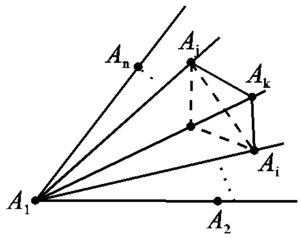
(ii) One of the points A, B, X and Y is inside the triangle defined by the other three. Then, the sum of the three angles with a vertex inside is 360°. Consequently, in this case a triangle with vertex in A or B is not acute again.

The contradictions show that it is not possible to exist more than one point which is a vertex of acute triangles only.

b) Let A_1 be the point which is a vertex of acute triangles only. We consider the angles $\angle A_k A_1 A_s$ (Figure 13). Let $A_2 A_1 A_n$ be the biggest one. Because all angles with vertex A_1 are acute, the points A_3, A_4, ..., A_{n-1} are inside the acute angle $\angle A_2 A_1 A_n$. We can assume that the points are enumerate in such a way that $\angle A_2 A_1 A_k < \angle A_2 A_1 A_{k+1}$ for $k = 3, 4, \ldots, n$.

Consider $\angle A_i A_k A_j$, where $2 \leq i < k < j \leq n$.

Figure 13.



We assume that A_k is internal for $\triangle A_1 A_i A_j$. Then

$$\angle A_1 A_k A_i + \angle A_1 A_k A_j > 180°$$

and at least one of the angles $\angle A_1 A_k A_i$ and $\angle A_1 A_k A_j$ will not be acute, which is a contradiction.

Therefore, the points A_1, A_i, A_k, A_j define a convex quadrilateral. If we assume that $\angle A_i A_k A_j \leq 90°$, then there exists an angle of this quadrilateral with vertex A_1, A_i or A_j which is $\geq 90°$. This is impossible.

Therefore, $\angle A_i A_k A_j > 90°$, and it is clear that $\angle A_i A_k A_j < 90°$ when $i, j < k$ or $i, j > k$.

Particularly, it follows from here that no of the angles $A_i A_k A_j$ is right. The number of all angles with vertex A_k is equal to $\dfrac{(n-1)(n-2)}{2}$. If T_k is the number of the obtuse angles with vertex A_k, then $N_k = \dfrac{(n-1)(n-2)}{2} - T_k$. Thus, N_k is minimal when T_k is maximal.

It is easy to see that $T_1 = T_2 = T_n = 0$. Let $3 \leq k \leq n-1$. It is clear that the number of the points A_i for which $2 \leq i < k$ is $k-2$, and the number of the points A_j for which $k < j \leq n$ is $n-k$. Then $T_k = (k-2)(n-k)$ for $k = 3, 4, \ldots, n-1$.

But $(k-2)(n-k) \leq \dfrac{(n-2)^2}{4}$, and the equality is reached when $k-2 = n-k$, i.e. when $k = \dfrac{n+2}{2}$. We have two possibilities:

(i) The number n is even. Then $\dfrac{n+2}{2}$ is integer and consequently the maximal value of T_k is $T_{\frac{n+2}{2}} = \dfrac{(n-2)^2}{2}$. Therefore, the minimal value of N_k in this case is $N_{\frac{n+2}{2}} = \dfrac{(n-1)(n-2}{2} - \dfrac{(n-2)^2}{2} = \dfrac{n(n-2)}{4}$.

(ii) The number n is odd. Then, the nearest integers to $\dfrac{n+2}{2}$ are $\dfrac{n+1}{2}$ and $\dfrac{n+3}{2}$. It is easy to see that now the maximal value of T_k is $T_{\frac{n+1}{2}} = T_{\frac{n+3}{2}} = \left(\dfrac{n+1}{2} - 2\right)\left(n - \dfrac{n+1}{2}\right) = \dfrac{(n-1)(n-3)}{4}$. Consequently, the minimal value of N_k in this case is

$$N_{\frac{n+1}{2}} = N_{\frac{n+3}{2}} = \frac{(n-1)(n-2)}{2} - \frac{(n-1)(n-3)}{4} = \frac{(n-1)^2}{4}.$$

Grade 11

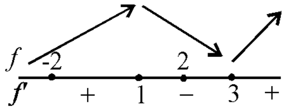
Problem 1. Find the values of the real parameter a, for which the inequality $x^6 - 6x^5 + 12x^4 + ax^3 + 12x^2 - 6x + 1 \geq 0$ is satisfied for all real x.

Solution. When $x = 0$ the given inequality is satisfied for all a. Thus, it is enough to find such a, that $\left(x^3+\frac{1}{x^3}\right) - 6\left(x^2+\frac{1}{x^2}\right) + 12\left(x+\frac{1}{x}\right) + a \geq 0$ for all $x > 0$ and $\left(x^3+\frac{1}{x^3}\right) - 6\left(x^2+\frac{1}{x^2}\right) + 12\left(x+\frac{1}{x}\right) + a \leq 0$ for all $x < 0$.

Denote $t = x + \frac{1}{x}$. It is clear that $x > 0 \iff t \geq 2$ and $x < 0 \iff t \leq -2$. But $x^2 + \frac{1}{x^2} = t^2 - 2$ and $x^3 + \frac{1}{x^3} = t^3 - 3t$.

We consider the function $f(t) = \left(t^3 - 3t\right) - 6\left(t^2 - 2\right) + 12t + = t^3 - 6t^2 + 9t + 12 + a$. The problem is reduced to find such a, that $f(t) \geq 0$ for all $t \geq 2$ and $f(t) \leq 0$ for all $t \leq -2$ simultaneously.

Figure 14.



But $f'(t) = 3t^2 - 12t + 9 = 3(t-1)(t-3)$, from where it is easy to obtain that (Figure 14) $f(t) \geq 0$ for all $t \geq 2$ iff $f(3) \geq 0$ and $f(t) \leq 0$ for all $t \leq -2$ iff $f(-2) \leq 0$. Since $f(3) = a + 12$ and $f(-2) = a - 38$, we find for a: $a \geq -12$ and $a \leq 38$. Finally, $-12 \leq a \leq 38$.

Problem 2. The point D lies on the arc $\overset{\frown}{BC}$ of the circumcircle of $\triangle ABC$ which does not contain the point A and $D \neq B$, $D \neq C$. On the rays $BD^{\rightarrow}$ and $CD^{\rightarrow}$ there are taken points E and F, such that $BE = AC$ and $CF = AB$. Let M be the midpoint of the segment EF.

a) Prove that $\angle BMC$ is right.

b) Find the locus of the points M, when D describes the arc $\overset{\frown}{BC}$.

Figure 15.

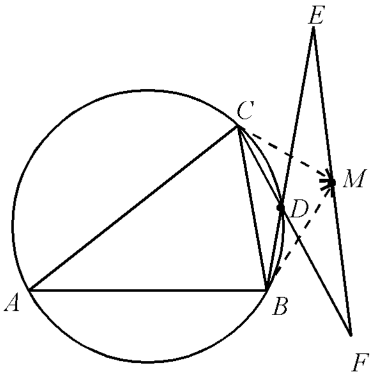
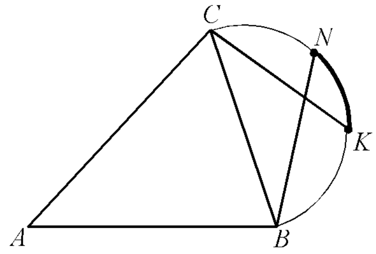


Figure 16.



Solution. Denote by α, β, γ the angles corresponding to the vertexes A, B, C of $\triangle ABC$, and by a, b, c the lengths of the sides BC, CA, AB, respectively. Let $\angle BCD = \varphi$. It follows from $D \in \overset{\frown}{BC}$ that $0 < \varphi < \alpha$, i.e. $\varphi \in (0, \alpha)$.

a) We have (Figure 15) $\overrightarrow{BM} = \frac{1}{2}\cdot\left(\overrightarrow{BF} + \overrightarrow{BE}\right) = \frac{1}{2}\cdot\left(\overrightarrow{CF} + \overrightarrow{BE} + \overrightarrow{BC}\right)$ and $\overrightarrow{CM} = \frac{1}{2}\cdot\left(\overrightarrow{CF} + \overrightarrow{CE}\right) = \frac{1}{2}\cdot\left(\overrightarrow{CF} + \overrightarrow{BE} - \overrightarrow{BC}\right)$. Then we find for the scalar product $\overrightarrow{BM}\cdot\overrightarrow{CM}$:

$$\overrightarrow{BM}\cdot\overrightarrow{CM} = \frac{1}{4}\cdot\left(\left(\overrightarrow{CF} + \overrightarrow{BE}\right)^2 - \overrightarrow{BC}^2\right)$$

$$\begin{aligned} &= \frac{1}{4}\cdot\left(\overrightarrow{CF}^2+\overrightarrow{BE}^2+2\cdot\overrightarrow{CF}\cdot\overrightarrow{BE}-a^2\right)\\ &= \frac{1}{4}\cdot\left(c^2+b^2+2bc\cos(\pi-\alpha)-a^2\right)\\ &= \frac{1}{4}\cdot\left(c^2+b^2-2bc\cos\alpha-a^2\right)=0. \end{aligned}$$

(We have used that $|BE|=|AC|=b$, $|CF|=|AB|=c$ and $\angle(\overrightarrow{BE},\overrightarrow{CF})=\angle BDC=\pi-\alpha$.)

Consequently, $\overrightarrow{BM}\perp\overrightarrow{CM}$, i.e. the angle $\angle BMC$ is right.

b) It is clear that the points M and A are in different semiplanes with respect to the line BC. According to a) $\angle BMC$ is a right angle and consequently if the point M is from the locus, then M lies on a semicircle k with diameter BC, k and A are in different semiplanes with respect to the line BC. Let $\angle BCM=\psi$. Then $CM=a\cos\psi$ (because $\angle BMC=\frac{\pi}{2}$) and

$$\overrightarrow{CM}\cdot\overrightarrow{CB}=a(a\cos\psi)\cdot\cos\psi=(a\cos\psi)^2.$$

On the other hand by the sine theorem we find:

$$\begin{aligned} \overrightarrow{CM}\cdot\overrightarrow{CB} &= \frac{1}{2}\cdot\left(\overrightarrow{CB}+\overrightarrow{BE}+\overrightarrow{CF}\right)\cdot\overrightarrow{CB}\\ &= \frac{1}{2}\cdot\left(a^2+ab\cos\left(\pi-(\alpha-\varphi)\right)+ac\cos\varphi\right)\\ &= \frac{1}{2}a^2\left(1-\frac{b}{a}\cos(\alpha-\varphi)+\frac{c}{a}\cos\varphi\right)\\ &= \frac{1}{2}a^2\left(1+\frac{\sin\gamma\cos\varphi-\sin\beta\cos(\alpha-\varphi)}{\sin\alpha}\right)\\ &= \frac{1}{2}a^2\left(1+\frac{\sin(\gamma-\varphi)-\sin(\beta-\alpha+\varphi)}{2\sin\alpha}\right)\\ &= \frac{1}{2}a^2\left(1+\cos(\beta+\varphi)\right)=\left(a\cos\frac{\beta+\varphi}{2}\right)^2. \end{aligned}$$

Consequently, $\cos^2\psi=\cos^2\frac{\beta+\varphi}{2}$. But $\frac{\beta+\varphi}{2}<\frac{\beta+\alpha}{2}<\frac{\pi}{2}$ and therefore $\psi=\frac{\beta+\varphi}{2}$, i.e. $\angle BCM=\frac{\beta+\varphi}{2}$ and $\angle BCM\in\left(\frac{\beta}{2},\frac{\beta+\alpha}{2}\right)$. In addition we have $\angle CBM=\frac{\pi}{2}-\angle BCM=\frac{\alpha+\gamma-\varphi}{2}$ and $\angle CBM\in\left(\frac{\gamma}{2},\frac{\alpha+\gamma}{2}\right)$.

Let K and N be points from the semicircle k, for which $\angle BCK=\frac{\beta}{2}$ and $\angle CBN=\frac{\gamma}{2}$. Then $angleBCN=\frac{\pi}{2}-\frac{\gamma}{2}=\frac{\alpha+\beta}{2}$ and $\angle CBK=\frac{\pi}{2}-\frac{\beta}{2}=\frac{\alpha+\gamma}{2}$.

It follows from the above considerations that when the point D describes the arc $\overset{\frown}{BC}$, then $CM^{\rightarrow}$ describes the interior of $\angle(CK^{\rightarrow},CN^{\rightarrow})$, while $BM^{\rightarrow}$ describes the interior of $\angle(BK^{\rightarrow},BN^{\rightarrow})$. Consequently, the locus, we are looking for is the arc $\overset{\frown}{NK}$ from the semicircle k (Figure 16).

Problem 3. is the same as problem 3, grade 10.