Spring mathematics tournament—1997

Problem 8'1. Given the equation $|x-a|+15=6|x+2|$, where a is a real parameter.

(a) Prove that for any value of a the equation has exactly two distinct roots x_1 and x_2.

(b) Prove that $|x_1-x_2|\geq 6$ and find all values of a for which $|x_1-x_2|=6$.

Solution: (a) When $x\leq -2$, the equation is equivalent to $|x-a|=-6x-27$, which has a solution only if $-6x-27\geq 0$, i. e., if $x\leq -\frac{9}{2}$. Considering the cases of both $x\geq a$ and $x\leq a$ shows that if $x\leq -2$, the given equation has a unique root x_1, and if $x>-2$, it has a unique root x_2:

$$x_1=\begin{cases}\frac{a-27}{7}, & a<-\frac{9}{2}\\ -\frac{a+27}{5}, & a\geq -\frac{9}{2}\end{cases}\qquad x_2=\begin{cases}\frac{3-a}{5}, & a\leq \frac{1}{2}\\ \frac{a+3}{7}, & a>\frac{1}{2}\end{cases}$$

(b) It follows from (a) that

$$|x_1 - x_2| = \begin{cases} \dfrac{156 - 12a}{35}, & a < -\dfrac{9}{2} \\ 6, & -\dfrac{9}{2} \le a \le \dfrac{1}{2} \\ \dfrac{12a + 204}{35}, & a > \dfrac{1}{2} \end{cases}$$

It remains to be seen that $\dfrac{156 - 12a}{35} > 6$ when $a < -\dfrac{9}{2}$ and $\dfrac{12a + 204}{35} > 6$ when $a > \dfrac{1}{2}$. Therefore $|x_1 - x_2| \ge 6$ and equality obtains only when $a \in [-\dfrac{9}{2}, \dfrac{1}{2}]$.

Problem 8'2. Let O be the intersecting point of the diagonals of the convex quadrilateral $ABCD$ and let $\angle DAC = \angle DBC$. The midpoints of AB and CD are respectively M and N and P and Q are points on AD and BC respectively such that $OP \perp AD$ and $OQ \perp BC$. Prove that $MN \perp PQ$.

Solution: Denote by E and F the midpoints of AO and BO. Then $\triangle PEM \cong \triangle MFQ$ since $MF = \dfrac{1}{2}AO = PE$, $QF = \dfrac{1}{2}OB = ME$ and $\angle MEP = \angle MEO + \angle OEP = \angle MEO + 2\angle DAC = \angle MFO + 2\angle DBC = \angle MFQ$ ($MFOE$ is a parallelogram). Therefore $MP = MQ$. The case when E and F are interior points for $\angle PMQ$ is treated similarly (prove that there are no other possibilities). By analogy we conclude that $NP = NQ$. Hence M and N lie on the axis of symmetry of PQ and so $MN \perp PQ$.

Problem 8'.3. Find all natural numbers n such that there exists an integer number x for which $499(1997^n + 1) = x^2+x$.

Solution: Let n be a solution of the problem. Then $(2x + 1)^2 = 1996 \cdot 1997^n + 1997$. If $n = 1$ we get $(2x + 1)^2 = 1997^2$ and so $2x + 1 = \pm 1997$. Therefore $x = 998$ and $x = -999$ satisfy the conditions of the problem. Let $n \geq 2$. Now $(2x + 1)^2$ is divisible by 1997, which is a prime number, and so $(2x + 1)^2$ is divisible by 1997^2. But this is impossible, since $1996 \cdot 1997^n + 1997$ is not divisible by 1997^2 when $n \geq 2$. The only solution is $n = 1$.

Problem 8.1. Find all values of the real parameter m such that the equation $(x^2 - 2mx - 4(m^2 + 1))(x^2 - 4x - 2m(m^2 + 1)) = 0$ has exactly three distinct roots.

Solution: Suppose m satisfies the conditions of the problem and the equations

$$\begin{aligned} x^2 - 2mx - 4(m^2 + 1) &= 0 \\ x^2 - 4x - 2m(m^2 + 1) &= 0 \end{aligned}$$

share the root x_0. After subtracting we get $(2m - 4)x_0 = (2m - 4)(m^2+1)$ and so $x_0 = m^2+1$ (note that if $m = 2$, the two equations coincide). Substituting x_0 in any of the equations gives the equation $(m^2 + 1)(m^2 - 2m - 3) = 0$ with roots $m = -1$ and $m = 3$. Direct verification shows that the condition is satisfied only for $m = 3$.

Let now (1) and (2) share no roots. Since $D_1 = 4 + 5m^2 > 0$ (1) always has two distinct roots and therefore (2) should have equal roots. Thus $D_2 = 4 + 2m(m^2 + 1) = 0$ and so $m = -1$. But this case has already been considered. Thus we determine that $m = 3$.

Problem 8.2. The area of the equilateral triangle ABC is 7. Points M and N are chosen respectively on AB and AC so that $AN = BM$. Denote by O the intersecting point of the straight lines BN and CM. The area of BOC is 2.

(a) Prove that $MB : AB = 1 : 3$ or $MB : AB = 2 : 3$.

(b) Find $\angle AOB$.

Solution: (a) Denote $\dfrac{MB}{AB} = x$. Therefore $S_{ABN} = 7x = S_{BMC}$ and so $S_{BOM} = 7x - 2$ and $S_{AMON} = S_{BOC} = 2$. Further $S_{CON} = 7 - 2 - 2 - (7x - 2) = 5 - 7x$, $S_{ANO} = \dfrac{x}{1-x} \cdot S_{CNO} = \dfrac{x(5-7x)}{1-x}$, $S_{AMO} = \dfrac{1-x}{x} \cdot S_{BOM} = \dfrac{1-x}{x}(7x-2)$. It follows from $S_{AMON} = S_{ANO} + S_{AMO}$ that $2 = \dfrac{x(5-7x)}{1-x} + \dfrac{1-x}{x}(7x-2)$, and thus $9x^2 - 9x + 2 = 0$. The roots of the above equation are $x_1 = \dfrac{1}{3}$ and $x_2 = \dfrac{2}{3}$.

(b) Since $\triangle ABN \cong \triangle BMC$, we get $\angle BOM = \angle BCM + \angle CBO = \angle MBO + \angle CBO = 60°$. Further $\angle MAN + \angle MON = 180°$ and therefore the quadrilateral $AMON$ is inscribed in a circle. Let $\dfrac{MB}{AB} = \dfrac{1}{3}$, i. e., $AM = 2BM = 2AN$. Denote by Q the midpoint of AM. Triangle AQN is isosceles and has an angle equal to $60°$, so it is equilateral. Therefore Q is the circumcentre of $AMON$ and $\angle AOM = \angle ANM = 90°$. Thus $\angle AOB = 150°$. Similarly, if

$\dfrac{MB}{AB} = \dfrac{2}{3}$, i. e., $2AM = MB = AN$, we get $\angle AMN = \angle AON = 90°$, so $\angle AOB = 90°$.

Problem 8.3. Given n points, $n \geq 5$, in the plane such that no three lie on a line. John and Peter play the following game: On his turn each of them draws a segment between any two points which are not connected. The winner is the one after whose move every point is an end of at least one segment. If John is to begin the game, find the values of n for which he can always win no matter how Peter plays.

Solution: Call a point isolated if it is not an end of a segment. John wins exactly when there are 1 or 2 isolated points before his last move. Peter is forced to reach the above only if before his move there are exactly 3 isolated points and any of the remaining $n-3$ points are connected by a segment. Indeed, if there are at least 4 isolated points he could connect one of them with a non-isolated point. If the isolated points are 3 but not all of the remainig $n-3$ points are connected he could draw a missing segment. Since the number of segments with ends in $n-3$ points is $\dfrac{(n-3)(n-4)}{2}$, we determine that John wins only when $\dfrac{(n-3)(n-4)}{2}$ is an odd integer number. This is true when n is of the form $4k+1$ or $4k+2$.

Problem 9.1. Let $f(x) = x^2 - 2ax - a^2 - \dfrac{3}{4}$ where a is real parameter. Find all values of a such that the inequality $|f(x)| \leq 1$ holds for any x in the interval $[0,1]$.

Solution: Let M and m be the maximum and minimum values of $f(x)$ in the interval $[0,1]$. Then the condition of the problem is equivalent to $M \le 1$ and $m \ge -1$. There are three cases to consider.

Case 1: $a \in [0,1]$. Then $m = f(a) = -2a^2 - \frac{3}{4}$ and $M = f(0) = -a^2 - \frac{3}{4}$ or $M = f(1) = -a^2 - 2a + \frac{1}{4}$. It follows from $m \ge -1$ and $M \le 1$ that $a \in [-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}]$ and $a \in (-\infty, -\frac{3}{2}] \cup [-\frac{1}{2}, \infty)$. In this case the solution is $a \in [0, \frac{\sqrt{2}}{4}]$.

Case 2: $a < 0$. Now $m = f(0)$ and $M = f(1)$. From $m \ge -1$ and $M \le 1$ we get that $a \in [-\frac{1}{2}, \frac{1}{2}]$. Therefore $a \in [-\frac{1}{2}, 0)$.

Case 3: $a > 1$. Now $m = f(1)$ and $M = f(0)$. It follows from $m \ge -1$ and $M \le 1$ that $a \in [-\frac{5}{2}, \frac{1}{2}]$ which is a contradiction with $a > 1$.

Thus the solution is $a \in [-\frac{1}{2}, \frac{\sqrt{2}}{4}]$.

Problem 9.2. Let I and G be the incentre and the centre of $\triangle ABC$ with sides $AB = c$, $BC = a$, $CA = b$.

(a) Prove that (if $a > b$) the area of CIG equals $\frac{(a-b)r}{6}$ where r is the inradius of ABC.

(b) If $a = c+1$ and $b = c-1$, prove that the segment IG is parallel to AB and find its length.

Solution: (a) We shall use the usual notation for a triangle. Let CL and CM be respectively the bisector and the median from C. It follows from $CG = \frac{2}{3}CM$ that $S_{CIG} = \frac{2}{3}S_{CIM}$. Thus $S_{CIM} = S_{CLM} - S_{ILM} = \frac{LM \cdot h_c}{2} - \frac{LM \cdot r}{2} = \frac{LM}{2}(h_c - r)$. We find from $AL + BL = c$ and $\frac{AL}{BL} = \frac{b}{a}$ that $AL = \frac{bc}{a+b}$ and so $LM = AM - AL = \frac{c}{2} - \frac{bc}{a+b} = \frac{c(a-b)}{2(a+b)}$. Also, $h_c - r = \frac{2S}{c} - r = \frac{2pr}{c} - r = \frac{r(2p-c)}{c} = \frac{r(a+b)}{c}$. Therefore $S_{CIG} = \frac{2}{3}S_{CIM} = \frac{2}{3} \cdot \frac{c(a-b)}{4(a+b)} \cdot \frac{r(a+b)}{c} = \frac{(a-b)r}{6}$.

(b) The distances from I and G to AB are respectively r and $\frac{h_c}{3}$. Hence $r = \frac{S}{p} = \frac{ch_c}{2p} = \frac{ch_c}{3c} = \frac{h_c}{3}$ and so $IG \| AB$. Therefore the altitude from C of triangle CIG equals $\frac{2}{3}h_c = 2r$. Thus $S_{CIG} = IG \cdot r$. On the other hand $S_{CIG} = \frac{(a-b)r}{6} = \frac{r}{3}$ and so $IG = \frac{1}{3}$.

Problem 9.3. Let $n \geq n$ be an even number and A be a subset of $\{1, 2, \ldots, n\}$. Consider the sums of the form $\varepsilon_1 x_1 + \varepsilon_2 x_2 + \varepsilon_3 x_3$, where x_1, x_2, x_3 are integer numbers in A (not necessarily distinct), $\varepsilon_1, \varepsilon_2, \varepsilon_3$ (at least one of which is not 0) belong to $\{-1, 0, 1\}$ and none of the elements of A appears with coefficients 1 and -1 in any of the sums. Call A a 'free' set if n divides none of the above sums.

(a) Construct a 'free' set having $[\frac{n}{4}]$ elements ($[x]$ is the least integer number less than or equal to x).

(b) Prove that no set of $[\frac{n}{4}]+1$ elements is 'free'.

Solution: (a) The set $A=\{1,3,\dots,2[\frac{n}{4}]-1\}$ is 'free' and has $[\frac{n}{4}]$ elements (prove it!).

(b) Let $n=4m$ and suppose that $A\subset\{1,2,\dots,n\}$ is a 'free' set having $[\frac{n}{4}]+1=m+1$ elements. Without loss of generality assume that $A\subset\{1,2,\dots,2m\}$ (since we can replace $x\in A$ by $n-x$). We shall show that there exist two elements of A whose sum is equal to another element of A. Indeed, let $a_1<a_2<\dots<a_{m+1}$ be the elements of A and consider the set $B=\{a_1+a_i : i=1,2,\dots,m+1\}$. There are $2m+2$ integer numbers $a_1,a_2,\dots,a_{m+1},2a_1,a_1+a_2,\dots,a_1+a_{m+1}$ from $A\cup B$ and they lie in the interval $[a_1,a_1+a_{m+1}]$, which contains exactly $a_{m+1}+1\le 2m+1$ integer numbers. This gives $a_i=a_1+a_j$ for some i,j. But then $a_1+a_j-a_i=0$, which is impossible. The case $n=4m+2$ is settled in a similar fashion. Notice that $2m+1$ cannot be an element of a 'free' set.

Problem 10.1. Find the least natural number a such that the equation $\cos^2\pi(a-x)-2\cos\pi(a-x)+\cos\frac{3\pi x}{2a}\cdot\cos(\frac{\pi x}{2a}+\frac{\pi}{3})+2=0$ has a root.

Solution: The roots of the our equations are the common roots of $\cos\pi(x-a)=1$ and $\cos\frac{3\pi x}{2a}\cdot\cos(\frac{\pi x}{2a}+\frac{\pi}{3})+1=0$. The roots of the first one are $x=a+2n,\ n=0,\pm1,\pm2,\dots$ and the roots of the second one are $x=2a(k-\frac{1}{3}),\ k=0,\pm1,\dots$. Therefore $a=\frac{6n}{6k-5}$

for some integer numbers n and k. It is easy to see now that the least natural number with the required property is $a = 6$.

Problem 10.2. Point F lies on the base AB of a trapezoid $ABCD$ and is such that $DF = CF$. Let E be the intersecting point of AC and BD and O_1 and O_2 are circumcentres of ADF and FBC respectively. Prove that the straight lines FE and O_1O_2 are orthogonal.

Solution: Let k_1 and k_2 be circles with centres O_1 and O_2 and let the intersecting points of the two circles be points P and Q. It is well known that $PQ \perp O_1O_2$. On the other hand, if L is an arbitrary point and two lines through L intersect k_1 and k_2 in points A, B and C, D respectively. Then $L \in PQ$ if and only if $LA \cdot LB = LC \cdot LD$. Let k_1 and k_2 be the circumscribed circles of $\triangle AFD$ and $\triangle FBC$ and let G be the intersecting point of FE with CD. Denote by C_1 and D_1 those points on DC for which $AD_1||CF$ and $BC_1||DF$, i. e., such that the quadrilaterals $AFCD_1$ and $BFDC_1$ are parallelograms. Using that $FD = FC$ we get $\angle CFB = \angle FCD = \angle FDC = 180° - \angle BC_1C$. This means that F, B, C and C_1 lie on a circle and so the line DC intersects k_2 in C and C_1. By analogy $\angle AFD = \angle FDC = \angle FCD = \angle AD_1D$ and so line DC meets k_1 in points D and D_1. In accordance with the initial notes FE is perpendicular to OO_1 if and only if $GC \cdot GC_1 = GD \cdot GD_1$. It follows from $\triangle GCE \sim \triangle FAE$, $\triangle GDE \sim \triangle FBE$ and $\triangle DCE \sim \triangle BAE$ that $\frac{GC}{AF} = \frac{CE}{EA} = \frac{DC}{AB}$ and $\frac{GD}{BF} = \frac{DE}{EB} = \frac{DC}{AB}$. Thus $GC = \frac{DC \cdot AF}{AB}$ and $GD = \frac{BF \cdot DC}{AB}$. On the other hand $GC_1 = |DC_1 - DG| = |BF - DG| = BF|1 - \frac{DG}{BF}| = BF|1 - \frac{DC}{AB}| = \frac{BF}{AB} \cdot |AB - DC|$,

$GD_1 = |CD_1 - CG| = |AF - CG| = AF|1 - \dfrac{CG}{AF}| = AF|1 - \dfrac{DC}{AB}| = \dfrac{AF}{AB}\cdot|AB - DC|$. Therefore $GC\cdot GC_1 = \dfrac{DC \cdot AF \cdot BF}{AB^2}\cdot|AB - DC| = GD \cdot GD_1$.

Problem 10.3. Find all natural numbers n for which a convex n-gon can be partitioned into triangles through its diagonals in such a way that there is an even number of diagonals from each vertex. (If there is a vertex with no digonals through it, assume that there is an even number (zero) of diagonals from this vertex).

Solution: It is easy to see by induction that if an n-gon is partitioned into triangles through d non-intersecting diagonals then $n = d + 3$. Let n be a natural number and $A_1A_2\cdots A_n$ is a convex n-gon which can be partitioned into triangles through d diagonals in a way that there is an even number of diagonals through each vertex. Since $n = 3$ is a solution we may assume that $n \geq 4$.

It is clear that at least one side of each triangle is a diagonal of the n-gon. We say that a triangle is of type t_k, $(k = 1, 2, 3)$ if exactly k of its sides are diagonals. Denote by x_k the number of triangles of type t_k. It is easy to see that $2x_1 + x_2 = n = d + 3$ abd $x_1 + 2x_2 + 3x_3 = 2d$. It follows now that $x_1 = x_3 + 2$, so $x_1 > 0$. Therefore there exists a triangle two of whose sides are sides of the n-gon. Let that be $A_{j-1}A_jA_{j+1}$. Diagonal $A_{j-1}A_{j+1}$ is a side of another triangle—e. g., $A_{j-1}A_{j+1}A_s$. Assume that $A_{j-1}A_s$ or $A_{j+1}A_s$ is a side of the n-gon. If it is $A_{j-1}A_s$ then $s = j - 2$. It follows now that there are no diagonals from A_{j-1} distinct from $A_{j-1}A_{j+1}$ because such a diagonal intersects $A_{j+1}A_s$. This contradicts the premise that there is an even number of diagonals from each vertex. Therefore both $A_{j-1}A_s$ and $A_{j+1}A_s$ are diagonals so $A_{j-1}A_{j+1}A_s$ is of type t_3. Hence there is a triangle of type t_3 adjacent to each triangle of type t_1. If distinct triangles of type t_1 are adjacent to distinct triangles of type t_3 then $x_1 \leq x_3 = x_1 - 2 < x_1$, a contradiction. Therefore there are at least two triangles of type t_1 adjacent to one and the same triangle of type t_3. Without loss of generality assume these are the triangles $A_1A_nA_{n-1}$ and $A_{n-1}A_{n-2}A_{n-3}$. Consider the polygon $A_1A_2\cdots A_{n-3}$. Obviously the diagonals partition this polygon into triangles and there is an even number of diagonals through each vertex.

Conversely, if the polygon $A_1A_2\cdots A_{n-3}$ can be partitioned in the required way, then adding the vertices A_{n-2}, A_{n-1}, A_n and diagonals $A_{n-3}A_{n-1}$ and A_1A_{n-1} shows that the same is true for the polygon $A_1A_2\cdots A_n$.

Therefore a natural number $n \geq 6$ is a solution if and only if $n-3$ is a solution. It is easy to see that $n = 3$ is a solution, whereas $n = 4$ and $n = 5$ are not. Thus all natural numbers satisfying the conditions of the problem are $n = 3k$, $k = 1, 2, \ldots$.

Problem 11.1. For any real number b denote by $f(b)$ the maximal value of $|\sin x + \dfrac{2}{3+\sin x} + b|$. Find the minimal value of $f(b)$.

Solution: Substitute $t = \sin x$ and $g(t) = t + \dfrac{2}{3+t} + b$. Since $g(t)$ is an increasing function in the interval $[-1, 1]$, it follows that $f(b) = \max(|g(-1)|, |g(1)|) = \max(|b|, |b + \frac{3}{2}|)$. Now from the graph of the function $f(b)$ we conclude that $\min f(b) = f(-\frac{3}{4}) = \frac{3}{4}$.

Problem 11.2. A convex quadrilateral $ABCD$ is such that $\angle DAB = \angle ABC = \angle BCD$. Let H and O be respectively the orthocentre and the circumcentre of $\triangle ABC$. Prove that H, O and D lie on a line.

Solution: Let $\angle CAB = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$. Note that $\alpha < \beta$ and $\gamma < \beta$. There are three cases to consider: $\beta < 90°$, $\beta = 90°$ and $\beta > 90°$. Suppose first that $\beta < 90°$. Then O and H are interior points for $\triangle ABC$ and $\angle ACO = \angle CAO = \angle HCB = \angle HAB = 90° - \beta$. Therefore O is an interior point for $\triangle HAC$ and $\angle HAO = \beta - \gamma = \angle ACD$, $\angle HCO = \beta - \alpha = \angle CAD$, $\angle HAD = \angle HCD = 2\beta - 90°$. It follows from the Sine Theorem for $\triangle AHD$, $\triangle CHD$ and $\triangle ACD$ that $\dfrac{\sin \angle AHD}{\sin \angle HAD} = \dfrac{AD}{HD}$, $\dfrac{\sin \angle HCD}{\sin \angle CHD} = \dfrac{HD}{CD}$, $\dfrac{\sin \angle CAD}{\sin \angle ACD} = \dfrac{CD}{AD}$. By multiplying the above equalities we get

$$\sin \angle AHD \cdot \sin \angle HCO \cdot \sin \angle CAO = \sin \angle CHD \cdot \sin \angle HAO \cdot \sin \angle ACO.$$

It follows now from Ceva's Theorem that AO, CO and HD intersect in a point and so H, O and D lie on a line. In the case of $\beta = 90°$ we obtain that $H \equiv B$, O is the midpoint of AC and $AHCD$ is a rectangle. Therefore H, O and D lie on a line. Finally, let $\beta > 90°$. In this case B and O are interior points for $\triangle AHC$ and $\triangle ADC$ respectively. Similarly to the case $\beta < 90°$ we get that the points H, O and D lie on a line.

Problem 11.3. For any natural number $n \geq 3$ denote by $m(n)$ the maximum number of points which can be placed inside or on the outline of a regular n-gon with side 1 in a way that the distance between any two of them is greater than 1. Find all n for which $m(n) = n - 1$.

Solution: We prove first that $m(n)=n-1$ for $n=4,5,6$. Let n be one of the above numbers and let $B_1, B_2, \cdots, B_n$ be points satisfying the conditions of the problem for the regular n-gon $A_1A_2\cdots A_n$ of side 1 and centre O. It is obvious that $OB_i \le 1$ and therefore no three points O, B_i, B_j, $1 \le i \ne j \le n$ lie on a line. Furthermore at least one of the angles OB_iB_j is less than $\dfrac{2\pi}{n} \le 90°$ and it follows from the Cosine Theorem that $B_iB_j^2 \le OB_i^2 + OB_j^2 - 2OB_iOB_j \cos\dfrac{2\pi}{n} = B_i'B_j'^2$ where B_i' and B_j' are points on the segments OA_1 and OA_2 such that $OB_i' = OB_i$, $B_j' = OB_j$. When $n=4,5,6$ the greatest side in $\triangle OA_1A_2$ is $A_1A_2 = 1$ and therefore $B_iB_j \le B_i'B_j' \le 1$, which is a contradiction. Thus $m(n) \le n-1$ if $n=4,5,6$. It is easily seen that for these n there exist $n-1$ points on the outline of a regular n-gon $A_1A_2\cdots A_n$ with the required property. Therefore $m(n)=n-1$ if $n=4,5,6$. We shall prove now that if $n \ge 7$ then $m(n) \ge n$. Let $B_i \in A_iA_{i+1}$ be points such that $A_iB_i = 2^{i-1}\varepsilon$, $1 \le i \le n-1$ where $0 < \varepsilon < \dfrac{1}{2^{n-1}}$ is arbitrary chosen. From $n \ge 7$ we obtain $\cos\angle A_1A_2A_3 = -\cos\dfrac{2\pi}{n} < -\dfrac{1}{2}$. It follows now that $B_1B_2{}^2 > (1-\varepsilon)^2 + 4\varepsilon^2 + 2\varepsilon(1-\varepsilon)^2 = 1 + 3\varepsilon^2 > 1$. Similarly $B_iB_{i+1} > 1$ when $1 \le i \le n-2$. Further, it is clear that $B_1B_{n-1} > A_1A_n = 1$. Since $OA_1 = OA_2 = \cdots = OA_n > 1$, $A_iA_j > 1$ when $|i-j| \ge 2$ and $B_i \to A_i$ when $\varepsilon \to 0$, it follows that we can make ε so small that $OB_i > 1$ when $1 \le i \le n-1$ and $B_iB_j > 1$ when $|i-j| \ge 2$. Then the points $B_1, B_2, \cdots, B_{n-1}, O$ satisfy the conditions of the problem and so $m(n) \ge n$ when $n \ge 7$. Since it is obvious that $m(3)=1$, we come to the conclusion that $m(n)=n-1$ only when $n=4,5,6$.