Spring mathematics tournament – 1998

Problem 8'.1. Find all values of the real parameter a such that the inequalities $|x+1|+|2-x|<a$ and $\dfrac{5a-8}{6x-5a+5}<-\dfrac{1}{2}$ are equivalent.

Solution: We begin with the first inequality. $|x+1|+|2-x|<a$. If $x<-1$, it is equivalent to $-x-1-x+2<a$ or $x>\dfrac{1-a}{2}$, i.e., $\dfrac{1-a}{2}<x<-1$, which has a solution if $\dfrac{1-a}{2}<-1$, i.e., if $a>3$. If $x\in[-1;2]$, then $|x+1|+|2-x|=x+1+2-x=3$ and the above equation has a solution only when $a>3$. Finally, if $x>2$, then $x+1+x-2<a$ or $x<\dfrac{a+1}{2}$, i.e., $2<x<\dfrac{a+1}{2}$, which has a solution when $a>3$. Therefore when $a\leq 3$, the inequality has no solution, and when $a>3$, the solutions form the interval $\left(\dfrac{1-a}{2};\dfrac{1+a}{2}\right)$.

Let us rewrite the second inequality in the form $\dfrac{6x+5a-11}{6x-5a+5} < 0$. Its solutions form either the interval $\left(\dfrac{11-5a}{6}; \dfrac{5a-5}{6}\right)$ or the interval $\left(\dfrac{5a-5}{6}; \dfrac{11-5a}{6}\right)$, depending on which of the two numbers $\dfrac{5a-5}{6}$ and $\dfrac{11-5a}{6}$ is greater.

Therefore the two inequalities are equivalent if

$$\frac{1-a}{2} = \frac{11-5a}{6}, \qquad \frac{1+a}{2} = \frac{5a-5}{6}$$

or

$$\frac{1-a}{2} = \frac{5a-5}{6}, \qquad \frac{1+a}{2} = \frac{11-5a}{6}.$$

The first pair is satisfied by $a=4$, the second one by $a=1$. But if $a=1$, the first inequality has no solution, whereas the second one does. Thus the only solution is $a=4$.

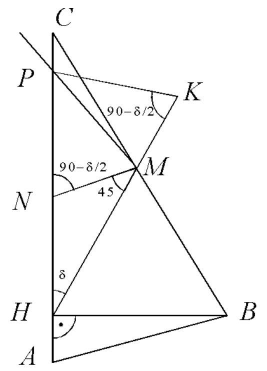
Problem 8'.2. Let M and N be the midpoints of the sides BC and AC and $BH, (H \in BC)$ the altitude in $\triangle ABC$. The straight line perpendicular to the bisector of $\angle HMN$ intersects the line AC in point P such that $HP = \frac{1}{2}(AB+BC)$ and $\angle HMN = 45\,\mathrm{deg}$.

a.) Prove that $\triangle ABC$ is isosceles.

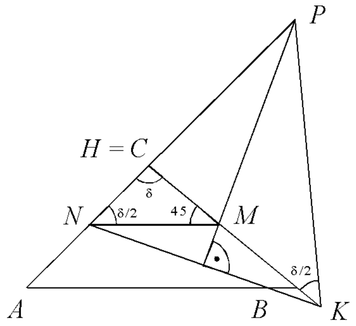
b.) Find the area of $\triangle ABC$ if $HM = 1$.

Solution: **a)** Since HM is a median to the hypothenuse BC of the right triangle $\triangle BHC$ and MN is a middle segment in $\triangle ABC$, it follows that $HM = \frac{1}{2}BC$ (if $H \equiv C$, again $HM = \frac{1}{2}BC$) and

$MN = \frac{1}{2}AB$. Therefore $HP = HM + MN$. Denote $\angle HMN = \delta$. There are two cases to consider for the points H, N and P: **1.** N lies between H and P; **2.** H lies between N and P.



1. Let us find a point K on the extension of HM such that $MK = MN$. Then $HP = HK$ and $\angle HKP = 90\deg - \frac{\delta}{2}$. The statement of the problem implies that MP is the bisector of $\angle NMK$ (external to $\triangle HMN$), i.e., $\angle NMP = \angle KMP$. Therefore $\triangle PNM \cong \triangle PKM$ and $\angle PNM = \angle PKM = 90\deg - \frac{\delta}{2}$. On the other hand, $\angle PNM = 45\deg + \delta$, i.e., $90\deg - \frac{\delta}{2} = 45\deg + \delta$, whence $\delta = 30\deg$ and $\angle HNM = 105\deg$. Also $\angle MHB = 60\deg$ and since $HM = MB$, it follows that $\angle HMB = 60\deg$. Thus $\angle BMN = 105\deg = \angle ANM$, i.e., $ABMN$ is a isosceles trapezoid $AN = BM$. Therefore $AC = BC$.



2. Let H lie between N and P and let K be a point such that $MK = MN$ and $HP = HK$. It follows from the isoscelesness of $\triangle KHP$ that $\angle HPK = \angle HKP = \dfrac{\delta}{2}$. Since MP is the bisector of $\angle NMK$ in the isosceles $\triangle NMK$, it follows that MP is the axis of symmetry of NK. Thus $NP = KP$ and $\triangle NMP \cong \triangle KMP$. Therefore $\angle PNM = \dfrac{\delta}{2}$ and since $\triangle NMP$, it follows that $\delta + \dfrac{\delta}{2} + 45\,\mathrm{deg} = 180\,\mathrm{deg}$. and hence $\delta = 90\,\mathrm{deg}$. Therefore $MH \perp AC$ and since $BH \perp AC$, we get that H coincides with C and $\angle BCA = 90\,\mathrm{deg}$, $\angle ABC = \angle BAC = 45\,\mathrm{deg}$ and so $AC = BC$.

b) 1. When $\delta = 30\,\mathrm{deg}$: $\angle ACB = 30\,\mathrm{deg}$, $HM = HB = 1$, $AC = BC = 2$. Then $S_{ABC} = \dfrac{1}{2}AC \cdot BH = 1$.

2. When $\delta = 90\,\mathrm{deg}$: $AC \perp BC$, $AC = BC = 2BH = 2$.

Then $S_{ABC} = \dfrac{1}{2}AC \cdot BC = 2$.

Problem 8'.3. (Problem for the UBM award)

Is it possible to find 100 straight lines in the plane such that there are exactly 1998 intersecting points?

Solution: Consider 99 lines such that 73 of them are parallel and the remaining 26 lines pass through a single point and intersect all 73 parallel lines. Then the total number of intersecting points is $73 \cdot 26 + 1 = 1899$. Choose the last line in such a way that it intersects all lines and does not pass through any of the points. Now there are $1899 + 99 = 1998$ intersecting points.

Problem 8.1. The graph of a linear function is parallel to the graph of $y = \frac{5}{4}x + \frac{95}{4}$, passing through $M(-1; -25)$, and intersects the coordinate axes Ox and Oy in A and B correspondingly.

(a) Find the coordinates of A and B.

(b) Consider the unity grid in the plane. Find the number of squares containing points of AB (in their interiors).

Solution: (a) The graph of a linear function is parallel to the graph of $y = \frac{5}{4}x + \frac{95}{4}$ when the linear function is of the form $y = \frac{5}{4}x + b$. From the condition that M belongs to its graph we determine $-25 = -\frac{5}{4} + b$, whence $b = -\frac{95}{4}$. The coordiantes of A and B are respectively the solutions of the systems

$$\left| \begin{array}{l} y = \frac{5}{4}x - \frac{95}{4} \\ y = 0 \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} y = \frac{5}{4}x - \frac{95}{4} \\ x = 0 \end{array} \right. ,$$

whence we get $A(19;0)$ and $B\Big(0;-\dfrac{95}{4}\Big)$.

(b) The coordinates of the points on AB satisfy the conditions

$$(1) \qquad \left| \begin{array}{l} y = \dfrac{5}{4}x - \dfrac{95}{4} \\ 0 \le x \le 19 \end{array} \right. .$$

Find the number of points with integer coordinates lying on the segment AB, i.e., the number of integer solutions of (1). It follows from $y = \dfrac{5}{4}x - \dfrac{95}{4}$ that $y = x - 23 + \dfrac{x-3}{4}$ and if x and y are integer, then $\dfrac{x-3}{4} = t$ is integer. Conversely, if t is integer, then $x = 4t + 3$ and $y = 5t - 20$ are integer. Since $0 \le x \le 19$, we find $0 \le 4t+3 \le 19$, i. e., $-\dfrac{3}{4} \le t \le 4$. This observation implies that the points with integer coordinates lying on AB satisfy the conditions $x = 4t+3$, $y = 5t - 20$ for $t = 0,\ 1,\ 2,\ 3,\ 4$. Therefore there are 5 such points.

Further: the segment AB exits a square and enters another one only when it intersects a line of the grid. The number of such intersections is $23 + 18 = 41$. But when AB passes through a knot (i.e., through a point with integer coordinates), the passage from a square to another one involves crossing two lines. This happens 4 times (not counting A). Therefore the required number is $41 - 4 + 1 = 38$.

Problem 8.2. Let l_1 and l_2 be the loci of the centroid G and the incentre I of the right triangle ABC whose hypotenuse AB is a given segment of length c.

(a) Find l_1 and l_2.

(b) Find the area of $\triangle ABC$ when the length of GI is minimal.

Solution: (a) The vertex C of $\triangle ABC$ could be placed in either semiplane with respect to AB. The centroid G lies on a circle k_1 of radius $\frac{c}{6}$ and centred at the midpoint O of segment AB. The incentre I lies on one of the two arcs which are the locus of the points X such that $\angle AXB = 135\,\text{deg}$. One of the arcs is part of a circle k_2' with centre Q' and radius R and the other one is part of a circle k_2'' with centre Q'' and the same radius and Q' and Q'' lie in different semiplanes with respect to AB.

Conversely: Let G be a point on k_1 distinct from the intersecting points M and N of k_1 and AB. Then the vertex C of $\triangle ABC$ is uniquely determined by $CO = \frac{c}{2}$. Since $AO = BO = CO$ it follows that $\triangle ABC$ is a right triangle. Choose that of the points Q' and Q'' which does not lie in the same semiplane as C does with respect to AB. Without loss of generality this is Q'. Denote the intersecting point of the bisector of $\angle BAC$ and k_2' by I. We shall prove that I is the incentre $\triangle ABC$. Since $\angle ABI = 45\,\text{deg} - \angle BAI$, we get $2 \cdot \angle ABI = 90\,\text{deg} - 2 \cdot \angle BAI = \angle ABC$.

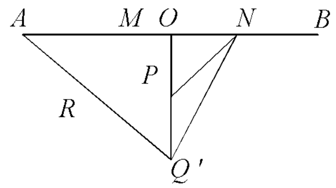
Therefore BI is the bisector of $\angle ABC$.

Let I be a point of k_2' such that $\angle AIB = 135\,\text{deg}$. Then the vertex C is uniquely determined by: AI is the bisector of $\angle BAC$, BI is the bisector of $\angle ABC$ and C lies in one and the same semiplane with I with respect to AB. Since $\angle BAI + \angle ABI = 45\,\text{deg}$, it follows that $\triangle ABC$ is a right triangle. Therefore $CO = \frac{c}{2}$ and hence the intersecting point G of CO and k_1 is the centroid of $\triangle ABC$.

These observations imply that l_1 is the circle k_1 without M and

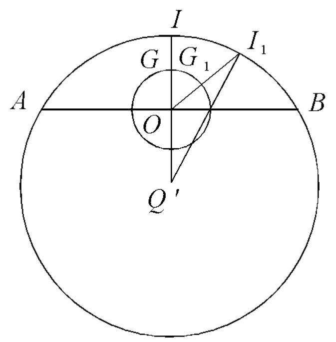
N, and l_2 consists of two arcs k_2' and k_2'' such that if $X \in k_2'$ or $X \in k_2''$, then $\angle AXB = 135\,\text{deg}$.

(b) We shall show that $R > \dfrac{2}{3}c$ and it will follow (how?) that k_1 lies in the interior of both k_2' and k_2''. Let P be the intersecting point of OQ' and k_1 which lies between O and Q'.

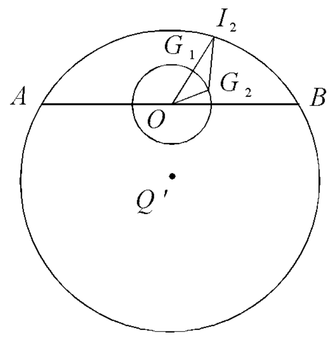


Since $PQ' = \dfrac{c}{3} = \dfrac{c}{6} + \dfrac{c}{6} = OP + ON$ and $\triangle PNO$, it follows that $OP + ON > PN$ and so $PQ' > PN$. Hence $\angle PNQ' > \angle PQ'N$. Therefore $\angle PNQ' + 45\,\text{deg} > \angle PQ'N + 45\,\text{deg}$, so $\angle ANQ' > \angle AQ'N$.

It follows for $\triangle AQ'N$ that $AQ' > AN$, i. e.,$R > \dfrac{2}{3}c$. Let $G \in l_1$ and $I \in l_2$ be such that Q', O, G and I lie in a line in this order. We shall show that GI is the least possible segment.



Case 1: $G_1 \in l_1$, $I_1 \in l_2$ and O lie on a line. Then $G_1I_1 > GI$. Indeed, it follows from $\triangle Q'I_1O$ that $OQ' + OI_1 > Q'I_1$, i. e., $OQ' + OG_1 + G_1I_1 > R = OQ' + OG + GI$ and so $G_1I_1 > GI$.



Case 2: $G_2 \in l_1$, $I_2 \in l_2$ and O are not on a line. Let OI_2 intersect k_1 in a point G_1. Then $G_2I_2 > G_1I_2$. Indeed, it follows from $\triangle OG_2I_2$ that $OG_2 + G_2I_2 > OI_2$. But $OI_2 = OG_1 + G_1I_2 = OG_2 + G_1I_2$ and therefore $G_2I_2 > G_1I_2$. But from case 1 we get that $G_1I_2 > GI$ and so $G_2I_2 > GI$.

Thus $\triangle ABC$ having the required property is such that Q', O, G and I lie on a straight line. Similarly, Q'' determines a triangle equal to the first one. Since $OQ' \perp AB$, $\triangle ABC$ is an isosceles triangle and so $S_{ABC} = \dfrac{c^2}{4}$.

Problem 8.3. (Problem for the UBM award)

Given n points on a circle such that no three chords with ends in the given points intersect in a point. Prove that there exists n such that there are $\dfrac{n^2 - 3n + 4}{2}$ chords with ends in the given points partitioning the interior of the circle into 1998 regions.

Solution: First we shall prove the following *Lemma:* The number of regions into which the interior of a circle is divided by drawing all $\binom{n}{2}$ chords with ends in n given points, provided no three chords intersect in a point, is $\binom{n}{4} + \binom{n}{2} + 1$.

Use induction by n. It is easy to see that the above formula holds for $n = 2$. Suppose it is true for some n. To obtain the result we count how many new regions are added when a new point appears on the circle. It is easily seen that if a chord intersects t other chords, then it 'adds' $t + 1$ new regions. Therefore the new regions are:

$$\sum_{k=0}^{n-1}(k(n-k-1)+1).$$

Given that $1+2+\cdots+n-1 = \dfrac{(n-1)n}{2}$ and $1^2+2^2+\cdots+(n-1)^2 = \dfrac{(n-1)n(2n-1)}{6}$, we easily obtain that the above sum equals

$$\binom{n+1}{4} + \binom{n+1}{2} + 1 - \left(\binom{n}{4} + \binom{n}{2} + 1\right),$$

which completes the proof.

We show now that if $n = 17$ it is possible to draw $\dfrac{n^2-3n+4}{2} = 121$ chords such that there are 1998 regions. Let us draw all chords with ends in 16 of the given points (there are 120 such chords). It follows then that the interior of the circle is divided into $\binom{16}{4} + \binom{16}{2} + 1 = 1941$ regions. Draw a chord connecting the 17th point with one of the first 16 in a way that there are 8 and 7 points on the two sides of the drawn chord. This chord intersects $8 \cdot 7 = 56$ chords and therefore there are 57 new regions. Therefore the total number of regions is $1941 + 57 = 1998$.

Problem 9.1. Find all parameters a such that the inequality $|ax^2-3x-4| \le 5-3x$ holds for any $x \in [-1;1]$.

Solution: Observe that the inequality is equivalent to the system

$$\left|\begin{array}{l} ax^2-9 \le 0 \\ ax^2-6x+1 \ge 0. \end{array}\right.$$

It follows from the second inequality that $a > 0$, because if $a \le 0$ then $ax^2-6x+1 \ge 0$ is not true for $x=1$. Further, the second inequality gives $x^2 \le \frac{9}{a}$. Since this inequality is true for $x \in [-1;1]$, we get $1 \le \frac{9}{a} \Rightarrow a \le 9$. Let $D = 9-a$ be the discriminant of ax^2-6x+1. There are two cases to consider:

1) $D > 0$. The solution of the second inequality is $x \in (-\infty; x_1] \cup [x_2; +\infty)$, where $x_1 < x_2$ are the roots of $ax^2-6x+1=0$. Therefore $[-1;1] \subset (-\infty; x_1]$ or $[-1;1] \subset [x_2; +\infty)$.

1.1) $[-1;1] \subset (-\infty; x_1]$. It follows from the above that

$$\left|\begin{array}{l} a(-1)^2-6(-1)+1 \ge 0 \\ a \cdot 1^2 - 6 \cdot 1 + 1 \ge 0 \\ \dfrac{x_1+x_2}{2} > 1 \end{array}\right. \iff \left|\begin{array}{l} a \ge -7 \\ a \ge 5 \\ \dfrac{3}{a} > 1 \end{array}\right. \iff \left|\begin{array}{l} a \ge -7 \\ a \ge 5 \\ a < 3 \end{array}\right. ,$$

which is impossible.

1.2) $[-1;1] \subset [x_2; +\infty)$. Therefore

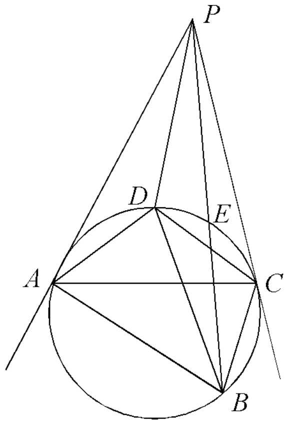
$$\left|\begin{array}{l} a(-1)^2-6(-1)+1 \ge 0 \\ a.1^2-6.1+1 \ge 0 \\ \dfrac{x_1+x_2}{2} < -1 \end{array}\right. \iff \left|\begin{array}{l} a \ge -7 \\ a \ge 5 \\ \dfrac{3}{a} < -1 \end{array}\right. ,$$

which is also impossible.

2) $D \leq 0 \Rightarrow$ the inequality $ax^2 - 6x + 1 \geq 0$ holds for any real value of x. It then follows that $9 - a \leq 0 \iff a \geq 9$. Therefore the solution is $a = 9$.

Problem 9.2. The quadrangle $ABCD$ is inscribed in a circle. The tangents to the circle passing through A and C intersect at the point P. If $PA^2 = PD \cdot PB$ and P does not lie on DB, prove that the intersecting point of AC and BD is the midpoint of AC.

Solution:



Let E be the second intersecting point of PB and the circle. Then $PA^2 = PE \cdot PB \Rightarrow PD = PE$. Hence $\angle APD = \angle EPC$. (If O is the centre of the circle, the above follows from the similarity of $\triangle ODP$ and $\triangle OEP$). Therefore $\triangle ADP \sim \triangle PCB$ because $\angle APD = \angle BPC$ and $\dfrac{AP}{BP} = \dfrac{DP}{CP}$, so $\dfrac{AD}{BC} = \dfrac{AP}{BP}$. (1)

Also, $\triangle APB \sim \triangle DCP$ because $\angle APB = \angle DPC = \angle APC - \angle APD$ and $\dfrac{AP}{DP} = \dfrac{BP}{CP}$ and so $\dfrac{AB}{DC} = \dfrac{BP}{CP}$. (2)

From (1) and (2) we get $\dfrac{AB \cdot AD}{BC \cdot DC} = \dfrac{AP \cdot BP}{BP \cdot CP} = 1$. On the other hand,

$$\frac{S_{ABD}}{S_{CBD}} = \frac{\frac{1}{2}AB \cdot AD \sin \angle DAB}{\frac{1}{2}BC \cdot DC \sin \angle DCB} = \frac{AB \cdot AD}{BC \cdot DC} = 1$$

$$(\angle DAB = 180\deg - \angle DCB).$$

Therefore $S_{ABD} = S_{CBD}$, i.e., the diagonal BD halves AC.

Problem 9.3. (Problem for the UBM award)

See problem 8.3.

Problem 10.1. Find all values of the real parameter a such that the inequality $x^4 + ax^3 + (a+3)x^2 + ax + 1 > 0$ holds for all real values of x.

Solution: If $x = 0$, the inequality holds for any a. Suppose $x \neq 0$. Dividing both sides of the inequality by x^2 gives $x^2 + ax + a + 3 + \dfrac{a}{x} + \dfrac{1}{x^2} > 0$ or $\Big(x + \dfrac{1}{x}\Big)^2 + a\Big(x + \dfrac{1}{x}\Big) + a + 1 > 0$. Substitute $t = x + \dfrac{1}{x}$. If $x > 0$, it is true that $t \geq 2$, and if $x < 0$, it is true that $t \leq -2$. Therefore we want to find all a such that the inequality $t^2 + at + a + 1 > 0$ holds for any $t \in (-\infty; -2] \cup [2; +\infty)$. Denote $f(t) = t^2 + at + a + 1$. The discriminant of $f(t)$ is $D = a^2 - 4a - 4$.

Let $D < 0$, i.e., $a \in (2 - 2\sqrt{2}; 2 + 2\sqrt{2})$. Then the inequality $f(t) > 0$ holds for any real t, in particular for $t \in (-\infty; -2] \cup [2; +\infty)$.

Let $D \geq 0$, i.e., $a \in (-\infty; 2 - \sqrt{2}] \cup [2 + \sqrt{2}; +\infty)$. Then the inequality $f(t) > 0$ holds for any $t \in (-\infty; -2] \cup [2; +\infty)$ if and only if

$$\left|\begin{array}{l} -2 < -\dfrac{a}{2} < 2 \\ f(2) > 0 \\ f(-2) > 0 \end{array}\right.$$

The solutions of this system when $a \in (-\infty; 2-\sqrt{2}] \cup [2+\sqrt{2}; +\infty)$ are

$a \in \left(-\dfrac{5}{3}; 2-2\sqrt{2}\right]$. So $a \in \left(-\dfrac{5}{3}; 2+2\sqrt{2}\right)$.

Problem 10.2. A quadrangle with perpendicular diagonals AC and BD is inscribed in a circle with centre O and radius 1.

a.) Calculate the sum of the squares of the sides of the quadrangle.

b.) Find the area of $ABCD$ if a circle with centre I is inscribed in it and $OI = \dfrac{1}{\sqrt{3}}$.

Solution: a.) Denote $\angle BAC = \alpha$, $\angle CAD = \beta$. It follows from the Sine Theorem for $\triangle ABC$ and $\triangle ABD$ that $BC = 2\sin\alpha$, $AD = 2\sin\angle ABD = 2\sin(90\deg - \alpha) = 2\cos\alpha$. Analogously $CD = 2\sin\beta$, $AB = 2\cos\beta$. Therefore $AB^2 + BC^2 + CD^2 + DA^2 = 4\sin^2\alpha + 4\cos^2\alpha + 4\sin^2\beta + 4\cos^2\beta = 8$.

b.) The statement of the problem implies that $AB + CD = AD + BC$ or $2\cos\beta + 2\sin\beta = 2\cos\alpha + 2\sin\alpha$, i. e. $\sin(\beta + 45\deg) = \sin(\alpha + 45\deg)$. Therefore $\beta = \alpha$ or $\beta = 90\deg - \alpha$. We shall consider only the case $\beta = \alpha$ (the case $\beta = 90\deg - \alpha$ is settled in a similar fashion). Now $AD = AB$, $CD = BC$ and $\angle ABC = \angle ADC = 90\deg$. The points O and I lie on AC and when $\alpha \geq 45\deg$, I lies on the segment AO. In $\triangle ACD$ (DI is a bisector) determine $AI = \dfrac{2\cos\alpha}{\sin\alpha+\cos\alpha}$. Consequently $OI = AO - AI = \dfrac{\sin\alpha-\cos\alpha}{\sin\alpha+\cos\alpha} = \dfrac{1}{\sqrt{3}}$. Raising both sides to the 2nd power gives $\dfrac{1-\sin 2\alpha}{1+\sin 2\alpha} = \dfrac{1}{3}$. Hence $\sin 2\alpha = \dfrac{1}{2}$. It then follows that the area of $ABCD$ is equal to $S = 2S_{ABC} = AB \cdot BC = 4\cos\alpha\sin\alpha = 2\sin 2\alpha = 1$.

Problem 10.3. (Problem for the Atanas Radev award)

Find all natural numbers n such that: If a and b are natural numbers and n divides a^2b+1, then n divides a^2+b.

Solution: Obviously the condition holds for $n = 1$ and suppose $n \geq 2$. Let a be a natural number coprime to n. It follows from Bezout's Theorem that there exists a natural number b such that a^2b+1 is divisible by n. Further, n divides a^2+b and since $a^4-1 = a^2(a^2+b)-(a^2b+1)$, it follows that n divides a^4-1.

Let $n = 2^\alpha k$, where k is an odd number and $\alpha \geq 0$. Suppose $k \geq 3$. Then n and $k-2$ are coprime and therefore $2^\alpha k$ divides $(k-2)^4-1$, so k divides 15. Hence n is of the form $n = 2^\alpha \cdot 3^\beta \cdot 5^\gamma$, where $\alpha \geq 0$, $0 \leq \beta$, $\gamma \leq 1$. Consequently n divides 11^4-1 and we conclude that $\alpha \leq 4$.

It is easy to see now that n divides $2^4 \cdot 3 \cdot 5 = 240$.

Conversely, let n be a divisor of 240. Then n satisfies the condition of the problem. Indeed, if 3 divides a^2b+1, then 3 divides both a^2-1 and a^2+1 and therefore 3 divides a^2+b. Similarly, if 5 divides a^2b+1, then 5 divides a^2+b. If we can show that the same property holds for 2, 4, 8 and 16, we will be done. Assume 2^k divides a^2b+1, where $1 \leq k \leq 3$. Then a is an odd integer number and therefore a^2-1 is divisible by 8. Consequently $b+1$ is divisible by 2^k and thus a^2+b is divisible by 2^k as well. Assume 16 divides a^2b+1. Then a is an odd integer number and it is easy to verify that a^2 is congruent to 1 or 9 modulo 16. Further, b should be congruent to 15 or 7 modulo 16 correspondingly and again a^2+b is divisible by 16.

The required numbers are all divisors of 240.

Problem 11.1.

a.) Let p be a positive real parameter. Find the least values of the functions $f(x) = x + \dfrac{p}{x}$ and $g(x) = x + \dfrac{p}{x^2}$ in the interval $(0; +\infty)$.

b.) Let $a_1,\ a_2,\ a_3$ be positive real numbers. Prove that $3(a_1 + \sqrt{a_1a_2} + \sqrt[3]{a_1a_2a_3}) \leq 4(a_1+a_2+a_3)$.

Solution: a.) Since $f'(x) = 1 - \dfrac{p}{x^2} = \dfrac{(x-\sqrt{p})(x+\sqrt{p})}{x^2}$, we get that the function $f(x)$ decreases in the interval $(0; \sqrt{p})$ and increases in the interval $(\sqrt{p}; +\infty)$. Therefore the minimal value of $f(x)$ in $(0; +\infty)$ equals $f(\sqrt{p}) = 2\sqrt{p}$. Analogously it follows from $g'(x) = 1 - \dfrac{2p}{x^3}$ that the least value of $g(x)$ in $(0; +\infty)$ equals $g(\sqrt[3]{2p}) = \dfrac{3\sqrt[3]{2p}}{2}$.

b.) After substituting $x = a_1,\ y = \sqrt{a_1a_2},\ z = \sqrt[3]{a_1a_2a_3}$ our inequality becomes $0 \leq \dfrac{1}{3}x + \dfrac{4}{3}\dfrac{y^2}{x} - y + \dfrac{4}{3}\dfrac{z^3}{y^2} - z$. It follows from a.)

that $\frac{4}{3}y \le \frac{1}{3}x + \frac{4}{3}\frac{y^2}{x}$ and $\frac{1}{3}y + \frac{4}{3}\frac{z^3}{y^2} \ge z$. Therefore $\frac{1}{3}x + \frac{4}{3}\frac{y^2}{x} - y + \frac{4}{3}\frac{z^3}{y^2} - z \ge \frac{1}{3}y + \frac{4}{3}\frac{z^3}{y^2} - z \ge 0$, which completes the proof. Note that equality occurs when $x = 2y = 4z$, i. e., $a_1 = 4a_2 = 16a_3$. Note further that b.) could also be solved by the following inequalities:

$$a_1 + \sqrt{a_1a_2} + \sqrt[3]{a_1a_2a_3} = a_1 + \frac{1}{2}\sqrt{a_14a_2} + \frac{1}{4}\sqrt[3]{a_14a_216a_3} \le$$

$$\le a_1 + \frac{1}{4}(a_1 + 4a_2) + \frac{1}{12}(a_1 + 4a_2 + 16a_3) = \frac{4}{3}(a_1 + a_2 + a_3).$$

Problem 11.2. Let I and r are the incentre and inradius of $\triangle ABC$, and N is the midpoint of the median through C. Prove that if $r = CN - IN$, then $AC = BC$ or $\angle ACB = 90\deg$.

Solution: Use the standart notation for the elements of $\triangle ABC$ and apply the formula for a median in a triangle. Since IN and IM are medians in $\triangle CIM$ and $\triangle AIB$, respectively, we get $IN^2 = \frac{1}{4}(2CI^2 + 2MI^2 - CM^2) = = \frac{1}{4}(2CI^2 + AI^2 + BI^2 - \frac{1}{2}AB^2 - CM^2)$. Therefore

$$CN^2 - IN^2 = \frac{1}{4}(2CM^2 + \frac{1}{2}AB^2 - 2CI^2 - AI^2 - BI^2) =$$

$$\frac{1}{4}(a^2 + b^2 - 2(p-c)^2 - 2r^2 - (p-a)^2 - r^2 - (p-b)^2 - r^2) = \frac{(p-c)c}{2} - r^2.$$

It follows from the statement of the problem that $IN^2 = (CN - r)^2$, so $CN^2 - IN^2 + r^2 = 2CN \cdot r$. It follows from what we have proved above that $2CM \cdot r = (p-c)c$. Taking the square of both sides of this equality and using the formulæ $4CM^2 = 2a^2 + 2b^2 - c^2$ and $r^2 = \dfrac{(p-a)(p-b)(p-c)}{p}$, we get $(2a^2+2b^2-c^2)(p-a)(p-b) = p(p-c)c^2$. After some simple calculations the above equality becomes $(a^2 + b^2 - c^2)(a - b)^2 = 0$. Therefore $a = b$, i.e., $AC = BC$ or $a^2 + b^2 = c^2$, i.e., $\angle ACB = 90\,\mathrm{deg}$.

Problem 11.3. (Problem for the Atanas Radev award)

See problem 10.3.