Spring mathematics tournament—Kazanlâk, 30 March–1 April 1999

Problem 8'1. Given an inequality $|x-1| < ax$, where a is a real parameter:

a) Solve the inequality.

b) Find all values of a such that the inequality has exactly two integer solutions.

Chavdar Lozanov, Kiril Bankov, Teodosi Vitanov

Solution: a) I. Let $x \geq 1$. Then the inequality is equivalent to $x - 1 < ax \iff (1-a)x < 1$.

1. $1 - a > 0, a < 1 \implies x < \dfrac{1}{1-a}, \dfrac{1}{1-a} > 1 \iff a > 0$. Therefore $0 < a < 1, 1 \leq x < \dfrac{1}{1-a}$.

2. $1-a=0 \Longrightarrow a=1 \Longrightarrow 0.x<1$. Therefore $a=1, x \geq 1$.

3. $1-a<0 \Longrightarrow 1<a \Longrightarrow x \geq 1$. Therefore $1<a, x \geq 1$.

II. Let $x<1$. Then the inequality is equivalent to $1-x<ax \iff 1<(a+1)x$.

1. $a+1>0 \iff a>-1, \dfrac{1}{a+1}<x<1, \dfrac{1}{a+1}<1 \iff a>0$. Therefore $a>0, \dfrac{1}{a+1}<x<1$, when $-1<a\leq 0$ no solution exists.

2. $a+1=0 \iff a=-1 \Longrightarrow 1<0\cdot x$, so no solution exists.

3. $a+1<0 \iff a<-1 \Longrightarrow x<\dfrac{1}{a+1}<0$. Therefore $a<-1, x<\dfrac{1}{a+1}$.

So when $a<-1$, then $x<\frac{1}{a+1}$; for $-1\leq a\leq 0$ no solution exists; when $0<a<1$, then $\frac{1}{a+1}<x<\frac{1}{1-a}$; when $1\leq a$, then $1\leq x$.

b) It follows from a) that the inequality could have two integer solutions only if $0<a<1$. Since in this case $0<\dfrac{1}{a+1}<1<\dfrac{1}{1-a}$, we find that there are exactly two integer solutions if and only if

$$2<\frac{1}{1-a}\leq 3.$$

Therefore the answer is $\dfrac{1}{2}<a\leq\dfrac{2}{3}$.

Problem 8'2. Let M be the midpoint of the side BC of $\triangle ABC$ and $\angle CAB = 45^\circ$; $\angle ABC = 30^\circ$.

a) Find $\angle AMC$.

b) Prove that $AM = \dfrac{AB \cdot BC}{2AC}$.

Chavdar Lozanov

Solution: a) Draw $CH \perp AB$. Now $\angle ACH = 45^\circ = \angle CAH$ and $\angle HCB = 60^\circ$. For $\triangle ACH$ it is true that $AH = HC$. Further it follows from $\triangle CHB$ that $CH = \dfrac{1}{2}CB = HM$. Therefore $AH = HM$, so $\angle MAH = \angle AMH = \dfrac{180^\circ - \angle AHM}{2}$. Note that $\angle CHM = 60^\circ$ and $\angle AHM = 90^\circ + 60^\circ = 150^\circ$. Therefore $\angle AMH = 15^\circ$. We obtain that $\angle AMC = \angle HMC - \angle AMH = 60^\circ - 15^\circ = 45^\circ$.

b) Let S be the area of $\triangle ABC$. We know that $S = \dfrac{AB \cdot CH}{2} = \dfrac{AB \cdot CB}{4}$. Since AM is a median, it follows that $S_{AMC} = \dfrac{S}{2}$. If $CP \perp AM$, then $\dfrac{S}{2} = \dfrac{AM \cdot CP}{2}$. But $\angle CAM = \angle CAB - \angle MAB = 45^\circ - 15^\circ = 30^\circ$, and thus $CP = \dfrac{AC}{2}$. Therefore $\dfrac{S}{2} = \dfrac{AM \cdot AC}{4}$. Now $\dfrac{AB \cdot CB}{4} = \dfrac{AM \cdot AC}{2}$, and we obtain $AM = \dfrac{AB \cdot BC}{2AC}$.

Problem 8'3. Consider all points in the plane whose coordinates (x, y) in an orthogonal coordinate system are integer numbers and $1 \le x \le 19, 1 \le y \le 4$. Each point is painted green, red or blue.

Prove that there exists a rectangle with sides parallel to the coordinate axes whose vertices are all of the same colour.

Kiril Bankov

Solution: Since the number of coloured points is $4 \cdot 19 = 76$ and there are three different colours, it follows that there are at least 26 points of the same colour (say blue). Denote by $p_1, p_2, \ldots, p_{19}$ the lines parallel to the ordinate axis and passing through the points $(1,0), (2,0), \ldots, (19,0)$, respectively. Let $n_1, n_2, \ldots, n_{19}$ be the number of blue points on the lines $p_1, p_2, \ldots, p_{19}$. It is clear that $0 \leq n_i \leq 4$ for $i = 1, 2, \ldots, 19$. Without loss of generality assume $n_1 \geq n_2 \geq \cdots \geq n_{19}$. Since $n_1 + n_2 + \cdots + n_{19} = 26$, we obtain that $n_1 \geq 2$.

1. Let $n_1 = 4$. Then the remaining 22 blue points lie on 18 lines and so $n_2 \geq 2$. Therefore there are two blue points on each of the lines p_1 and p_2 all having the same ordinates. These points form the required rectangle.

2. Let $n_1 = 3$. Then the remaining 23 blue points lie on 18 lines and therefore $n_2 \geq 2$.

   a) If $n_2 = 3$, then there is a blue rectangle with vertices on p_1 and p_2.

   b) Let $n_2 = 2$. Then $n_3 = n_4 = n_5 = n_6 = 2$. The number of ways of choosing two blue points on p_1 is 3 and the corresponding number for each of $p_2, p_3, \ldots, p_6$ is 1. The total number of blue pairs is $3 + 5 \cdot 1 = 8$, which is greater that 6—the number of ways of choosing two horizontal lines out of 4 lines. Therefore there exists a blue rectangle.

3. Let $n_1 = 2$. Then $n_2 = n_3 = \ldots = n_7 = 2$. Since $7 > 6$, we apply the same reasoning as in 2b).

Problem 8.1. Find all rational numbers a such that $|4a-2| \le 1$ and $A = \dfrac{4a-1}{27a^4}$ is integer. *Ivan Tonov*

Solution: It follows from $|4a-2| \le 1$ that $\dfrac{1}{4} \le a \le \dfrac{3}{4}$. Also, it is clear that when $a \ge \dfrac{1}{4}$, then $A \ge 0$ and $A = 0$ only if $a = \dfrac{1}{4}$. Let k be a positive integer such that $A = k$. Then $27a^4 - 4al + l = 0$, where $l = \dfrac{1}{k}$. Multiply the above equality by 3 and write it in the following way:

$$81a^4 - 18a^2 + 1 + 18a^2 - 12al + 3l - 1 = 0 \iff$$

$$(9a^2-1)^2 + 2(3a-1)^2 - 12a(l-1) + 3(l-1) = 0,$$

so $(9a^2-1)^2 + 2(3a-1)^2 + 3(l-1)(1-4a) = 0$. Therefore $3(l-1)(1-4a) \le 0$, which is possible (recall $a > \dfrac{1}{4}$) only if $l \ge 1$ or $k \le 1$. But since k is a positive integer, it follows that $k = 1$ and $a = \dfrac{1}{3}$. Therefore the required values are $a = \dfrac{1}{4}$ and $a = \dfrac{1}{3}$.

Problem 8.2. Given a $\triangle ABC$. Let M be the midpoint of AB, $\angle CAB = 15^\circ$ and $\angle ABC = 30^\circ$.

a) Find $\angle ACM$.

b) Prove that $CM = \dfrac{AB \cdot BC}{2AC}$.

Chavdar Lozanov

Solution: a) Let $AH \perp BC$. Then $\angle HAB = 60^\circ$ and $AH = \dfrac{AB}{2} = HM$. It follows from $\angle HAC = \angle HAB - \angle CAB = 45^\circ$ that $AH = HC$. Thus $HM = HC$ and $\angle HCM = \dfrac{180^\circ - \angle MHC}{2}$. But $\angle MHC = \angle AHB - \angle AHM = 90^\circ - 60^\circ = 30^\circ$. Therefore $\angle HCM = 75^\circ$, so $\angle ACM = \angle HCM - \angle HCA = 75^\circ - 45^\circ = 30^\circ$.

b) Let S be the area of $\triangle ABC$. We know that $S = \dfrac{BC \cdot AH}{2} = \dfrac{BC \cdot AB}{4}$. Since CM is a median, we obtain $S_{ACM} = \dfrac{S}{2}$. If $MP \perp AC$, then $\dfrac{S}{2} = \dfrac{AC \cdot PM}{2}$. It follows from $\triangle PMC$ that $PM = \dfrac{1}{2}MC (\angle PCM = 30^\circ)$. Therefore $\dfrac{S}{2} = \dfrac{AC \cdot MC}{4}$, so $\dfrac{AC \cdot MC}{2} = \dfrac{BC \cdot AB}{4}$, which implies $MC = \dfrac{AB \cdot BC}{2AC}$.

Problem 8.3. Given n points on a circle denoted consecutively by $A_1, A_2, \ldots, A_n$ $(n \geq 3)$. Initially 1 is written at A_1 and 0 at all remaining points. The following operation is allowed: choose a point A_i where a 1 is written and replace the numbers a, b and c written at the points A_{i-1}, A_i and A_{i+1} by $1-a$, $1-b$ and $1-c$, respectively. (Here A_0 means A_n and A_{n+1} means A_1.)

a) If $n = 1999$, is it possible to have a 0 in all points after performing the described operation a finite number of times?

b) Find all values of n such that it is not possible to have a 0 in all points after finite number of operations.

Kiril Bankov

Solution: a) After performing the transformation from the conditions of the problem consecutively for the points $A_1, A_2, A_3, \ldots, A_{n-2}$, we have the following distribution of 0s and 1s:

$$\begin{array}{cccccccc} A_1 & A_2 & A_3 & \ldots & A_{n-3} & A_{n-2} & A_{n-1} & A_n \\ 1 & 1 & 1 & \ldots & 1 & 0 & 1 & 1 \end{array} \tag{1}$$

If $n = 1999$, then arrange the obtained 1998 ones in 666 groups of three 1s and then perform the operation on each of the groups. We obtain a zero in every point.

b) If $n = 3k + 1$ we can repeat the steps from a) and we get only zeroes in the points.

If $n = 3k+2$, then starting from (1) and performing the operation with A_{n-1} one obtains:

$$\begin{array}{ccccccc} A_1 & A_2 & A_3 & \ldots & A_{n-2} & A_{n-1} & A_n \\ 1 & 1 & 1 & \ldots & 1 & 0 & 0 \end{array}$$

There are $3k$ ones which can be arranged in k groups of 3 ones each and again to obtain anly zeroes.

We shall prove now that if $n = 3k$, it is not possible to have only zeroes after a finite number of operations. Assume the opposite, i. e., that after a finite number of operations we have only zeroes. Denote the number of operations performed with the point A_i by a_i. Since each operation changes the number of ones by an odd number (1 or 3), it follows that the sum $S = a_1 + a_2 + \ldots + a_n$ of all operations is an odd number. On the other hand $S = (a_1 + a_2 + a_3) + (a_4 + a_5 + a_6) + \ldots + (a_{3k-2} + a_{3k-1} + a_{3k})$. Note that $a_1 + a_2 + a_3$ is equal to the number of changes (from 0 to 1 or *vice versa*) of the number written at A_2. Since at the beginning there is a 0 written at A_2 and a 0 again at the end, it follows that $a_1 + a_2 + a_3$ is an even number. The same applies for $a_4 + a_5 + a_6$ and so on. Therefore S is a sum of even numbers, a contradiction to the fact that S is odd. Answer to b): all numbers divisible by 3.

Problem 9.1. It is known that if the real parameter a equals any of the numbers $p < q < r$, then at least one of the remaining two is a root of the equation

$$x^2 - (2-a)x + a^2 - 2a - 7 = 0.$$

Prove that a) $p > -\frac{8}{3}$; b) $p < -1$. *Sava Grozdev*

Solution: a) It follows from the conditions of the problem that $D = (2-a)^2 - 4(a^2 - 2a - 7) \geq 0$, so $-\frac{8}{3} \leq a \leq 4$ and therefore $p \geq -\frac{8}{3}$. When $a = -\frac{8}{3}$, then the equation has an unique root $x = \frac{7}{3}$, and when $a = \frac{7}{3}$, the roots are $x = -\frac{8}{3}$ and $x = \frac{7}{3}$. Thus in the case of $p = -\frac{8}{3}$ three distinct numbers p, q and r satisfying the condition do not exist. Therefore $p > -\frac{8}{3}$.

b) Suppose that when $a = p$, then $x = q$ is a root. The case $x = r$ is treated in the same fashion. Since a and x are symmetric in the equality $x^2-(2-a)x+a^2-2a-7=0$, we get that when $a=q$, then $x=p$ is a root. When $a=r$, at least one of the numbers p and q is a root. Let that be p. Now when $a=p$, the equation has $x=r$ as a root (because of the symmetry). We obtain that when $a=p$, the roots are q and r. We conclude now that in all cases $p+q+r=2$. It is clear that the roots of $x^2-(2-p)x+p^2-2p-7=0$ are greater than p and therefore $p^2-(2-p)p+p^2-2p-7>0$, which gives $p<-1$ or $p>\frac{7}{3}$. But $3p<p+q+r=2$ and $p<\frac{2}{3}$. Therefore $p<-1$.

Problem 9.2. Through an interior point K of the non-equilateral $\triangle A_1A_2A_3$ lines $Q_2P_3 \parallel A_2A_3$, $Q_3P_1 \parallel A_3A_1$ and $Q_1P_2 \parallel A_1A_2$ are drawn. (Q_1, Q_2, Q_3 lie on A_3A_1, A_1A_2 and A_2A_3, respectively). Points $P_1, P_2, P_3, Q_1, Q_2, Q_3$ lie on a circle k. Prove that:

a) $\triangle P_1P_2P_3 \cong \triangle Q_1Q_2Q_3 \sim \triangle A_1A_2A_3$;

b) point K, the centre of k and the circumcentre $\triangle A_1A_2A_3$ lie on a line.

Rumen Kozarev

Solution: a) It follows from the conditions of the problem that the arcs $P_1\hat{Q}_1, P_2\hat{Q}_2$ and $P_3\hat{Q}_3$ are equal. Therefore $\angle P_3P_1P_2 = \frac{P_3Q_3P_2}{2} = \frac{P_3\hat{Q}_3}{2} + \frac{Q_3\hat{P}_2}{2} = \frac{P_1\hat{Q}_1}{2} + \frac{Q_3\hat{P}_2}{2}$. Since the quadrilateral $A_1P_1KQ_1$ is a parallelogram we get $\angle A_3A_1A_2 = \angle Q_1KP_1 = \frac{Q_1\hat{P}_1}{2} + \frac{Q_3\hat{P}_2}{2}$, so $\angle P_3P_1P_2 = \angle A_3A_1A_2$. Similarly, one can show that $\angle Q_3Q_1Q_2 = \angle A_3A_1A_2$, and the same equalities for the remaining pairs of angles. Thus $\triangle P_1P_2P_3 \sim \triangle Q_1Q_2Q_3 \sim \triangle A_1A_2A_3$ and since the first two triangles have the same circumcircle, they are identical.

b) Since $A_1P_1KQ_1$ is a parallelogram, we obtain $\angle A_1P_1Q_1 = \angle KQ_1P_1 = \dfrac{P_2\hat{Q}_2}{2} + \dfrac{P_1\hat{Q}_2}{2} = \angle P_1P_2P_3 = \angle A_1A_2A_3$. Similarly $\angle A_1Q_1P_1 = \angle A_1A_2A_3$.

Let O be the circumcentre of $\triangle A_1A_2A_3$. It is easy to see that $OA_1 \perp P_1Q_1, OA_2 \perp P_2Q_2$ and $OA_3 \perp P_3Q_3$. Denote the midpoint of OK by S. If R_1, R_2 and R_3 are the midpoints of P_1Q_1, P_2Q_2 and P_3Q_3 then $SR_1 \parallel OA_1; SR_2 \parallel OA_2; SR_3 \parallel OA_3$ (SR_1 is a middle segment in $\triangle OKA_1 \Longrightarrow SR_1 \parallel OA_1$). Therefore S lies on the axes of symmetry of $P_1Q_1; P_2Q_2$ and P_3Q_3, so S is the centre of the circle through the points $P_1, P_2, P_3, Q_1, Q_2, Q_3$.

Problem 9.3. Find all polynomials $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0, a_0 \neq 0$ with integer coefficients such that $f(a_i) = 0, i = 0, \ldots, n-1$. *Sava Grozdev*

Solution: It is clear that $n > 1$. Since $f(x) = (x - a_0)(x - a_1)\ldots(x - a_{n-1})$, it follows from $f(0) = a_0 = (-1)^n a_0 \ldots a_{n-1}$ that $|a_i| = 1$ for $i = 1, 2, \ldots, n-1$.

First case: $|a_0| = 1$. Now $f(x) = (x-1)^p(x+1)^q, p + q = n > 1$. We know that $(x-1)^p(x+1)^q = (x^p - px^{p-1} + \ldots)(x^q + qx^{q-1} + \ldots)$ and comparing the coefficients in front of x^{n-1} and x^{n-2} we obtain

$$\left| \begin{array}{l} q - p = a_{n-1} = \pm 1 \\ \frac{q(q-1)}{2} + \frac{p(p-1)}{2} - pq = a_{n-2} = \pm 1 \end{array} \right. .$$

It is easy to find now that $p+q=3$ and so $p=1, q=2$ or $p=2, q=1$. In the first case we get that $f(x)=x^3+x^2-x-1$, which is a solution and in the second one $f(x)=x^3-x^2-x+1$, which is not.

Second case: $|a_0|\geq 2$. Now $0=f(a_0)=|a_0^n+a_{n-1}a_0^{n-1}+\cdots+a_1a_0+a_0|\geq |a_0|^n-|a_0|^{n-1}-\cdots|a_0|^2-|a_0|-|a_0|=\frac{|a_0|(|a_0|-2)(|a_0|^{n-1}-1)}{|a_0|-1}\geq 0$. Therefore $|a_0|=2$. Moreover $a_{n-1}a_0^{n-1},\ldots,a_1a_0,a_0$ have the same negativity—the opposite to those of a_0^n.

We conclude that $a_0=-2, n$ is an even number and $a_i=(-1)^{i+1}$ for $i=1,2,\ldots,n-1$. If $n>2$, then $a_2=-1$ and $0=f(-1)=(-1)^n+(-1)^n(-1)^{n-1}+\cdots+(-1)^2(-1)-2=-n-2\neq 0$, which is impossible. Therefore $n=2$ and $f(x)=x^2+x-2$, which is a solution.

Answer:

$$f(x)=x^2+x-2 \qquad \text{and} \qquad f(x)=x^3+x^2-x-1.$$

Problem 10.1. Prove that the inequality

$$x(2\cdot 3^x-\frac{4x^2+x+2}{x^2+x+1})\geq 0$$

holds for any real number x. *Rumen Kozarev*

Solution: 1) Let $x\leq 0$. Then $2\cdot 3^x\leq 2$. We shall show that $\frac{4x^2+x+2}{x^2+x+1}\geq 2$. The last inequality is equivalent to $4x^2+x+2\geq 2x^2+2x+2 \iff x(2x-1)\geq 0 \iff x\in(-\infty;0]\cup\left[\frac{1}{2};+\infty\right)$ and therefore it holds for $x\leq 0$.

2) Let $x > 0$. We prove that $\frac{4x^2+x+2}{x^2+x+1}) < 2 \cdot 3^x$. Assume the opposite, i. e., $\frac{4x^2+x+2}{x^2+x+1} \geq 2 \cdot 3^x \Longrightarrow \frac{4x^2+x+2}{x^2+x+1} > 2 \cdot 3^0 = 2 \iff x(2x-1) > 0 \iff x \in (-\infty; 0) \cup (\frac{1}{2}; +\infty)$. Since $x > 0$, we obtain that $x \in \left(\frac{1}{2}; +\infty\right)$. Therefore $\frac{4x^2+x+2}{x^2+x+1} \geq 2 \cdot 3^x > 2 \cdot 3^{\frac{1}{2}} > 3 \iff 4x^2+x+2 > 3x^2+3x+2 \iff x^2-2x-1 > 0 \iff x \in (-\infty; 1-\sqrt{2}) \cup (1+\sqrt{2}; +\infty) \Longrightarrow x \in (1+\sqrt{2}; +\infty)$, because $x \geq \frac{1}{2}$. Thus $\frac{4x^2+x+2}{x^2+x+1} \geq 2 \cdot 3^x > 2 \cdot 3^{1+\sqrt{2}} > 2 \cdot 3^2 = 18$. Since obviously $\frac{4x^2+x+2}{x^2+x+1} < \frac{4x^2+4x+4}{x^2+x+1} = 4$ for any $x > 0$, we get a contradiction.

Problem 10.2. Let M be an interior point in the square $ABCD$. Denote the second points of intersection of the lines AM, BM, CM, DM with the circumcircle of $ABCD$ by A_1, B_1, C_1, D_1, respectively. Prove that

$$A_1B_1 \cdot C_1D_1 = A_1D_1 \cdot B_1C_1.$$

Emil Kolev

Solution: Since $\triangle ABM \sim \triangle A_1B_1M$; $\triangle BCM \sim \triangle B_1C_1M$; $\triangle CDM \sim \triangle C_1D_1M$; $\triangle DAM \sim \triangle D_1A_1M$, we obtain

$$\frac{AB}{A_1B_1} = \frac{BM}{A_1M}; \frac{BC}{B_1C_1} = \frac{BM}{C_1M}; \frac{CD}{C_1D_1} = \frac{DM}{C_1M}; \frac{DA}{D_1A_1} = \frac{DM}{A_1M}.$$

It is easy to see now that

$$\frac{AB}{A_1B_1}\cdot\frac{CD}{C_1D_1}=\frac{BM\cdot DM}{A_1M\cdot C_1M}=\frac{BC}{B_1C_1}\cdot\frac{DA}{D_1A_1}.$$

Using that $AB = BC = CD = DA$ it follows from the above that $A_1B_1 \cdot C_1D_1 = A_1D_1 \cdot B_1C_1$, *Q. E. D.*

Problem 10.3. Consider n points in the plane such that no three lie on a line. What is the least number of segments having their ends in the given points such that for any two points A and B there exists a point C connected to both A and B? *Emil Kolev*

Solution: Denote the points by $A_1, A_2, \ldots, A_n$. Draw segments connecting A_1 with all remaining points. Also, draw segments A_2A_3, A_4A_5, ..., $A_{n-1}A_n$ when n is odd and A_2A_3, A_4A_5, ..., $A_{n-2}A_{n-1}$, A_2A_n when n is even. It is easy to see that the condition of the problem is met and that there are $\left\lceil\frac{3n-3}{2}\right\rceil$ segments ($\lceil M\rceil$ denotes the least natural number which is greater or equal to M).

Suppose it is possible to draw less than $\left\lceil\frac{3n-3}{2}\right\rceil$ segments and to meet the condition of the problem. Obviously each point is connected by a segment with another one. If each point is connected to at least three others, we will have that the number of segments is at least $\frac{3n}{2}$, which is greater than $\left\lceil\frac{3n-3}{2}\right\rceil$. Therefore there exists a point (let that be A_1) connected with at most two others. If A_1 is connected to exactly one point (let that be A_2), a point connected to both A_1 and A_2 does not exist. Therefore A_1 is connected to exactly two points (let them be A_2 and A_3). It is easy to see that A_2 and A_3

are connected by a segment. Consider the pairs A_1 and A_i for any $i > 3$. It is clear that the point connected to both A_1 and A_i could be either A_2 or A_3. In both cases A_i is connected to A_2 or A_3. Since there are at least two segments from each point $A_i, i > 3$ then the number of segments from $A_i, i > 3$ is at least $2(n-3)$. Further, since at least $n-3$ from these points connect some point of $A_i, i > 3$ with A_2 or A_3 (and therefore they are counted once) the total number of drawn segments is at least $3 + n - 3 + \left\lceil \frac{n-3}{2} \right\rceil = \left\lceil \frac{3n-3}{2} \right\rceil$.

This is a contradiction with the number of drawn segments. Therefore the answer is $\left\lceil \frac{3n-3}{2} \right\rceil$.

Problem 11.1. Given a function $f(x)$ defined for any real x and $f(\mathrm{tg} x) = \sin 2x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Find the minimum and the maximum of the function $f(\sin^3 x) \cdot f(\cos^3 x)$.

Oleg Mushkarov, Nikolai Nikolov

Solution: Let $t = \mathrm{tg} x$. Then $\sin 2x = \frac{2t}{1+t^2}$ and it follows from the conditions of the problem that $f(t) = \frac{2t}{1+t^2}$ for any t. Therefore

$$f(\sin^3 x) \cdot f(\cos^3 x) = \frac{4 \sin^3 x \cdot \cos^3 x}{(1 + \sin^6 x)(1 + \cos^6 x)} =$$

$$\frac{4(\sin x \cdot \cos x)^3}{2 - 3(\sin x \cdot \cos x)^2 + (\sin x \cdot \cos x)^6}$$

Let $u = \sin x \cdot \cos x = \frac{1}{2}\sin 2x$. Then $u \in [-\frac{1}{2}, \frac{1}{2}]$ and we have to find the minimum and the maximum of the function $g(u) = \frac{4u^3}{2 - 3u^3 + u^6}$ in the interval $[-\frac{1}{2}, \frac{1}{2}]$. We obtain

$$g'(u) = \frac{12u^2(1-u^2)(u^4+u^2+1)}{(2-3u^2+u^6)^2} > 0$$

when $u \in [-\frac{1}{2}, \frac{1}{2}]$ and so $g(u)$ is an increasing function in the interval $[-\frac{1}{2}, \frac{1}{2}]$. It follows now that $\max\limits_{u\in[-\frac{1}{2},\frac{1}{2}]} g(u) = g(\frac{1}{2}) = \frac{32}{81}$ and $\min\limits_{u\in[-\frac{1}{2},\frac{1}{2}]} g(u) = g(-\frac{1}{2}) = -\frac{32}{81}$.

Problem 11.2. A circle is tangent to the circumcircle of $\triangle ABC$ and to the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ at points M and N, respectively. Prove that the excentre to side BC of $\triangle ABC$ lies on the segment MN.

Oleg Mushkarov, Nikolai Nikolov

Solution: Let O be the circumcentre of $\triangle ABC$ and L be the centre of the circle tangent to the circumcircle of $\triangle ABC$. First we shall find the radius ρ of this circle. For $\triangle OAL$ we get $AL = \frac{\rho}{\sin\frac{A}{2}}$, $AO = R$, $OL = R + \rho$, $\angle OAL = \frac{|B - C|}{2}$. From the Cosine Law we obtain $(R+\rho)^2 = R^2 + \frac{\rho^2}{\sin^2\frac{A}{2}} - \frac{2R\rho\cos\frac{B-C}{2}}{\sin\frac{A}{2}}$. After simplification the above becomes

$$\rho\cos^2\frac{A}{2} = 2R\sin\frac{A}{2}(\sin\frac{A}{2} + \cos\frac{B-C}{2}) = 2R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Since $\sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}}$, $\cos\frac{B}{2} = \sqrt{\frac{p(p-b)}{ac}}$, $\cos\frac{C}{2} = \sqrt{\frac{p(p-c)}{ab}}$ and $abc = 4RS$, it follows from the previous equality that $\rho = \frac{r_a}{\cos^2\frac{A}{2}}$, where r_a is the exradius to side BC of $\triangle ABC$. Let $I = AL \cap MN$ and let T be the projection of I on the line AB. Since $AL \perp MN$, it follows that $IT = \frac{AI \cdot IM}{AM} = \frac{AI \cdot AM \cdot LM}{AM \cdot AL} = \frac{AI \cdot LM}{AL} = LM\cos^2\frac{A}{2} = r_a$. Since I lies on the bisector of $\angle A$, we conclude that I is the excentre to side BC of $\triangle ABC$.

Problem 11.3. Given an orthogonal coordinate system with origin O in the plane. Distinct real numbers are written at the points with integer coordinates. Let A be a nonempty finite set of integer points which is central-symmetric regarding O and $O \notin A$. Prove that there exists an integer point X such that if A_X is the image of A under translation defined by $\vec{OX}$, then at least half of the numbers written at the points of A_X are greater than the number written at X. *Avgustin Marinov*

Solution: Let us denote the number of points in A, which is obviously even, by $2s$. Connect all integer points X with the points from A_X by arrows so that the arrow points to the smaller number. Suppose no point X with the required property exists. Then there are at least $s+1$ arrows pointing out of any point X. For every natural number n denote the square with vertices $(n,n),(-n,n),(-n,-n)$ and $(n,-n)$ by K_n.

Since A is a finite set, there exists a natural number d such that $A \subset K_d$. For every n denote the number of arrows within the square K_n by S_n. Since there are at least $s+1$ arrows pointing out of every integer point of K_n (and these arrows are within the square K_{n+d}), it follows that $(2n+1)^2(s+1) \leq S_{n+d}$. On the other hand, since A_X is a central-symmetric set, there are at most $s-1$ arrows pointing to every integer point of K_{n+d}. Therefore $S_{n+d} \leq (2n+2d+1)^2(s-1)$. Thus $(2n+1)^2(s+1) \leq (2n+2d+1)^2(s-1)$, so $s+1 \leq (1+\frac{2d}{2n+1})^2(s-1)$ for any n. When $n \to \infty$ one obtains $s+1 \leq s-1$, a contradiction. Therefore a point X with the required property does exist.