WINTER MATHEMATICAL COMPETITION

1995

Grade 8 — First Group.

Problem 1. Prove that for every positive integer n the following proposition holds: "The number 7 is a divisor of $3^n + n^3$ if and only if 7 is a divisor of $3^n.n^3 + 1$."

Solution. If 7 is a divisor of n, then 7 is neither a divisor of $3^n + n^3$ nor a divisor of $3^n.n^3 + 1$.

Let 7 be not a divisor of n. In this case 7 divides $n^6 - 1 = (n^3 - 1)(n^3 + 1)$ and since 7 is a prime number, then 7 divides either $n^3 - 1$ or $n^3 + 1$. Now the above proposition follows from the equalities:

$$3^n.n^3 + 1 = (n^3 - 1)(3^n - 1) + (n^3 + 3^n)$$

and

$$3^n.n^3 + 1 = (n^3 + 1)(3^n + 1) - (n^3 + 3^n).$$

Problem 2. Let $ABCDE$ be a convex pentagon and let M, P, N, Q be the midpoints of the segments AB, BC, CD, DE respectively. If K and L are the midpoints of the segments MN and PQ respectively and the segment AE is of length a, find the length of the segment KL.

Figure 1.

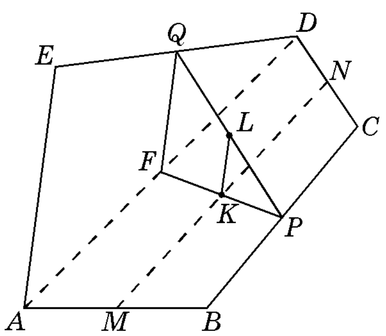
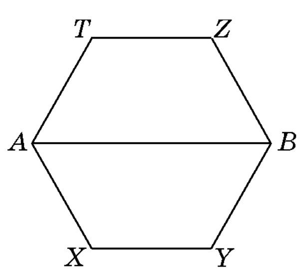


Figure 2.



Solution. Let F be the midpoint of the segment AD (Figure 1). Then the quadrilateral $MPNF$ is a parallelogram. Hence K is midpoint of the segment FP. It follows from here that $KL = \frac{1}{2}.FQ$. On the other hand $FQ = \frac{1}{2}.a$ (because F and Q are midpoints of ED and AD respectively).

Therefore $KL = \frac{1}{4}.a$.

Problem 3. Every point in the plane is colored either in black or in white. Prove that there exists a right angled triangle with hypotenuse of length 2 and an acute angle of $60°$, which vertices are colored in one and the same colour.

Solution. First we shall show that there exist two points which are colored in one and the same colour and the distance between them is 2. Indeed let ABC be a equilateral triangle of side 2. Obviously two of its vertices (say A and B) are colored in one and the same colour (e.g. white). Let $AXYBZT$ be a regular hexagon with a big diagonal AB (Figure 2).

If one of the vertices X, Y, Z, T (e.g. X) is white, then the vertices of the triangle ABX ($\angle AXB = 90°$, $\angle XAB = 60°$) are colored in one and the same colour. Otherwise the vertices of the triangle XYT ($\angle YXT = 90°$, $\angle XYT = 60°$) are colored in one and the same colour.

Grade 8.

Problem 1. Let $A = \dfrac{1}{\sqrt{4x^2+4x+1}}$ and $B = \dfrac{2x-2}{\sqrt{x^2-2x+1}}$. Find all integer values of x, for which the number $C = \dfrac{2A+B}{3}$ is an integer.

Solution. We have

$$A = \frac{1}{\sqrt{(2x+1)^2}} = \frac{1}{|2x+1|}, \qquad B = \frac{2(x-1)}{\sqrt{(x-1)^2}} = \frac{2(x-1)}{|x-1|},$$

and

$$C = \frac{2}{3}\cdot\left(\frac{1}{|2x+1|} + \frac{x-1}{|x-1|}\right),$$

1. Let $x > 1$. Then

$$C = \frac{2}{3}\cdot\left(\frac{1}{2x+1} + 1\right) = \frac{4(x+1)}{3(2x+1)} > 0$$

and

$$C - 1 = \frac{4(x+1)}{3(2x+1)} - 1 = \frac{1-2x}{3(2x+1)} < 0.$$

Hence $0 < C < 1$, i.e. C is not an integer for any $x > 1$.

2. Let $-\dfrac{1}{2} < x \leq 1$. Then $x = 0$ (because x is an integer) and $C = 0$. Thus $x = 0$ is a solution of the problem.

3. Let $x < -\dfrac{1}{2}$. Then $x \leq -1$ (because x is an integer). It is clear that

$$C = \frac{2}{3}\left(-\frac{1}{2x+1} - 1\right) = -\frac{4\,(x+1)}{3(2x+1)} \leq 0$$

and

$$C + 1 = \frac{2}{3}\left(-\frac{1}{2x+1} - 1\right) + 1 = 1 - \frac{4(x+1)}{3(2x+1)} = \frac{2x-1}{2\,(2x+1)} > 0.$$

Hence $-1 < C \leq 0$, i.e. $C = 0$ and $x = -1$.

Finally only $x = 0$ and $x = -1$ are solutions of the given problem.

Problem 2. Let M and N be the midpoints of the sides BC and AC of the triangle ABC, ($AB \neq AC$, $AB \neq BC$) and G be the intersection point of the lines AM and BN. The angle bisectors of $\angle BAC$ and $\angle ABC$ intersect BC and AC in the points D and E respectively. Prove that the quadrilateral $DEMN$ is inscribed in a circle if and only if there exists a circle, inscribed in the quadrilateral $CMGN$.

Figure 3.

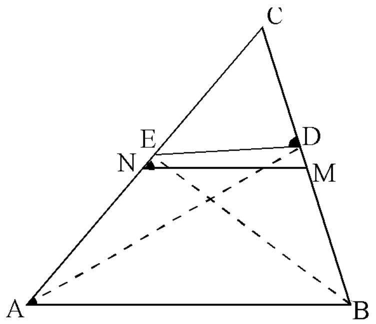
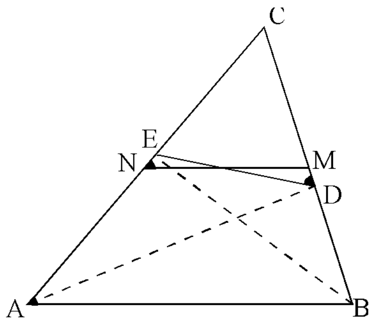


Figure 4.

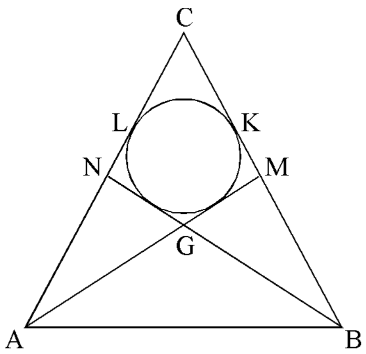


Solution. Without loss of generality we can suppose that N is between A and E. There are two possibilities for the points D and E which are shown in the Figure 3 and Figure 4.

The quadrilateral $DEMN$ is inscribed in a circle if and only if $\angle CNM = \angle CDE$, i.e. $\angle BAN + \angle BDE = 180^0$ (because $MN \| AB$ and $\angle CNM = \angle BAN$). Thus the quadrilateral $DEMN$ is inscribed in a circle iff the quadrilateral $ABDE$ is inscribed in a circle. This is equivalent to $\angle DAE = \angle DBE$, i.e. to $AC = BC$.

Therefore we should prove that there exists a circle, inscribed in the quadrilateral $CMGN$ if and only if $AC = BC$.

Figure 5.



Let $AC = BC$. Then $CM = CN$ and since G is the center of gravity of the triangle ABC, we have $GM = \frac{1}{3}AM = \frac{1}{3}BN = GN$. Hence $CM + GN = CN + GM$, i.e. there exists a circle, inscribed in the quadrilateral $CMGN$ (Figure 5).

Conversely if there exists a circle, inscribed in the quadrilateral $CMGN$, then $CM + GN = CN + GM$ and $\frac{1}{2}BC + \frac{1}{3}BN = \frac{1}{2}AC + \frac{1}{3}AM$. Hence $AM - BN = \frac{3}{2}(BC - AC)$. Let K and L be the points of contact of the circle and the sides CM and CN respectively. Obviously $CK = CL$.

On the other hand this circle is inscribed in the triangles ACM and BCN. Hence $CK = \frac{1}{2}(AC + CM - AM)$, $CL = \frac{1}{2}(BC + CN - BN)$. Thus $AM - BN = \frac{1}{2}(AC - BC)$. It follows from here that $\frac{3}{2}(BC - AC) = \frac{1}{2}(AC - BC)$, i.e. $AC = BC$.

Problem 3. Thirty points are given in the plane. Some of them are connected with segments as it is shown in the Figure 6. The points are labeled with different positive integers.

If a is a segment and p and q are the numbers, corresponding to its endpoints, we denote $\mu(a) = |p - q|$.

a) Construct an example of labeling of the points with the integers $1, 2, \ldots 30$, in which there exists exactly one segment a with $\mu(a) = 5$;

b) Prove that for every labeling there exists at least one segment a with $\mu(a) \geq 5$.

Figure 6.

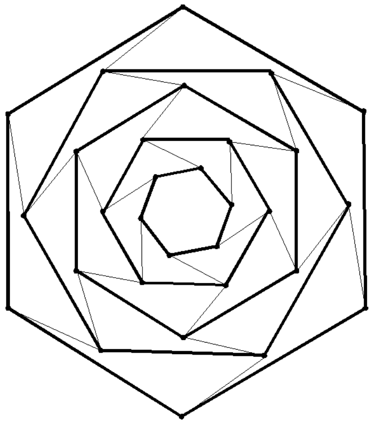
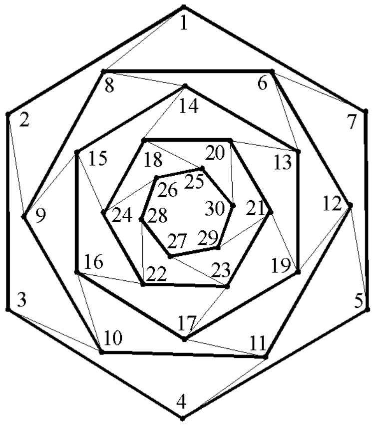


Figure 7.



Solution. a) A possible example is shown in the Figure 7.

b) Let the points be labeled with the positive integers $m = m_1 < m_2 < \ldots < m_{30} = M$. It is clear that $M \geq m + 29$.

Let A and B be the points labeled with m and M respectively and let $a_1 = AC_1$, $a_2 = C_1C_2$, …, $a_{k-1} = C_{k-2}C_{k-1}$, $a_k = C_{k-1}B$ be the shortest path of segments, connecting A and B. It is not difficult to see that $k \leq 7$. If we assume that $\mu(a_i) < 5$ for $i = 1, 2, \ldots k$, then $m + 29 \leq M \leq m + 4k \leq m + 28$, which is a contradiction.

Grade 9.

Problem 1. Let m be a real number, such that the roots x_1 and x_2 of the equation

$$f(x) = x^2 + (m-4)x + m^2 - 3m + 3 = 0$$

are real numbers.

a) Find all values of m for which $x_1^2 + x_2^2 = 6$.

b) Prove that

$$1 < \frac{mx_1^2}{1-x_1} + \frac{mx_2^2}{1-x_2} + 8 \leq \frac{121}{9}.$$

Solution. a) Since x_1 and x_2 are real numbers, then

$$D(f) = (m-4)^2 - 4\left(m^2 - 3m + 3\right) = -3m^2 + 4m + 4 \geq 0.$$

Hence $-\dfrac{2}{3} \leq m \leq 2$. On the other hand

$$6 = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1x_2 = -m^2 - 2m + 10.$$

We obtain from here that $m = -1 \pm \sqrt{5}$. But

$$-1 - \sqrt{5} < -\frac{2}{3} < -1 + \sqrt{5} < 2$$

and therefore only $m = \sqrt{5} - 1$ is a solution of the given problem.

b) We have

$$\begin{aligned}
\frac{mx_1^2}{1-x_1} + \frac{mx_2^2}{1-x_2} &= \frac{m[x_1^2(1-x_2) + x_2^2(1-x_1)]}{f(1)} \\
&= \frac{x_1^2 + x_2^2 - x_1x_2(x_1+x_2)}{m-2} \\
&= \frac{m^3 - 8m^2 + 13m - 2}{m-2} = m^2 - 6m + 1
\end{aligned}$$

Thus if $F = \dfrac{mx_1^2}{1-x_1} + \dfrac{mx_2^2}{1-x_2} + 8$, then $F = (m-3)^2$ and

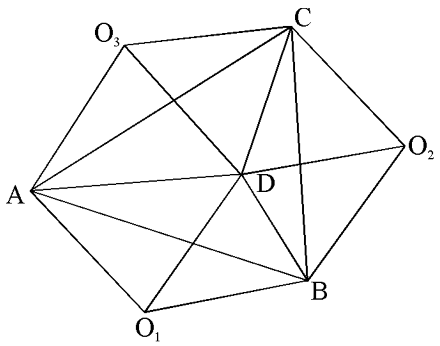
$$\frac{121}{9} = \left(-\frac{2}{3} - 3\right)^2 \geq F > (2-3)^2 = 1.$$

Problem 2. The point D lies inside the acute triangle ABC. Three of the circumscribed circles of the triangles ABC, ABD, BCD and CAD have equal radii. Prove that the fourth circle has the same radius.

Solution. There are two cases:

1. The radii of the circumscribed circles of the triangles ABD, BCD, CAD are equal.

Figure 8.



Let O_1, O_2 and O_3 be the centers of these circles (Figure 8). Obviously the quadrilaterals O_2CO_3D, O_3AO_1D and O_1BO_2D are rhombuses.

Hence $O_2C \| O_3D \| AO_1$ and $O_2C = AO_1$. Thus the quadrilateral CAO_1O_2 is a parallelogram. It follows from here that $AC \| O_1O_2$ and since $O_1O_2 \perp DB$, then $BD \perp AC$. Analogously $CD \perp AB$ and $AD \perp BC$. Therefore D is the altitude center of $\triangle ABC$.

Now it is easy to see that $\angle BDC = 180^0 - \angle BAC$, which implies that the circumscribed circles of $\triangle BCD$ and $\triangle ABC$ are symmetric according to the line BC.

Therefore their radii are equal.

2. The radii of the circumscribed circles of the triangles ABC, ACD and BCD are equal.

In this case the circumscribed circles of $\triangle ABC$ and $\triangle BCD$ are symmetric according the line BC. Hence $\angle BDC = 180^0 - \angle BAC$. Analogously $\angle ADC = 180^0 - \angle ABC$. Therefore $\angle ADB = 360^0 - \angle BDC - \angle ADC = 180^0 - \angle ACB$ and the circumscribed circles of $\triangle ABD$ and $\triangle ABC$ are symmetric according to the line AB. Thus they have equal radii.

Problem 3. Let A be a set with 8 elements. Find the maximal number of 3-element subsets of A, such that the intersection of any two of them is not a 2-element set.

Solution. Let $B_1, B_2, \dots, B_n$ be subsets of A such that $|B_i| = 3$, $|B_i \cap B_j| \neq 2$ $(i, j = 1, \dots, n)$. Assume that there exists an element $a \in A$, which belongs to four of the subsets $B_1, B_2, \dots, B_n$ (e.g. $a \in B_1, B_2, B_3, B_4$). Then $|B_i \cap B_j| \geq 1$ $(i, j = 1, \dots, 4)$. But $B_i \neq B_j$ if $i \neq j$, i.e. $|B_i \cap B_j| \neq 3$. Thus $|B_i \cap B_j| = 1$ $(i, j = 1, 2, 3, 4)$. It follows from here that $|A| \geq 1 + 4.2 = 9$, which is a contradiction. Therefore every element of A belongs to at most three of the subsets $B_1, B_2, \ldots, B_n$. Then $3n \le 8.3$, i.e. $n \le 8$.

If $A = \{a_1, a_2, \ldots, a_8\}$, then the subsets

$$\begin{array}{llll} B_1 = \{a_1, a_2, a_3\}, & B_2 = \{a_1, a_4, a_5\}, & B_3 = \{a_1, a_6, a_7\}, & B_4 = \{a_8, a_3, a_4\}, \\ B_5 = \{a_8, a_2, a_6\}, & B_6 = \{a_8, a_5, a_7\}, & B_7 = \{a_3, a_5, a_6\}, & B_8 = \{a_2, a_4, a_7\} \end{array}$$

provide an example of exactly eight 3-element subsets of A, such that $|B_i \cap B_j| \ne 2$.

Therefore the searched number is $n = 8$.

Grade 10.

Problem 1. Find all positive roots of the equation

$$\log_{(x+a-1)} \frac{4}{x+1} = \log_a 2,$$

where $a > 1$ is a real number.

Solution. It is clear that if $a > 1$ and $x > 0$, then $x + a - 1 > 0$ and $\dfrac{4}{x+1} > 0$. Hence in this case $\log_{(x+a-1)} \dfrac{4}{x+1}$ is well defined. Since

$$\log_{(x+a-1)} \frac{4}{x+1} = \frac{\log_a \dfrac{4}{x+1}}{\log_a (x+a-1)},$$

then the given equation is equivalent to

$$\frac{4}{x+1} = 2^{\log_a(x+a-1)}. \tag{1}$$

The function $\dfrac{4}{x+1}$ is strictly decreasing in the interval $(0, +\infty)$. The function $\log_a(x+a-1)$ is strictly increasing in the interval $(0, +\infty)$ and obviously the same is true for the function $2^{\log_a(x+a-1)}$. Therefore the equation (1) has no more than one root in the interval $(0, +\infty)$.

On the other hand it is easy to check that $x = 1$ is a root of this equation.

Problem 2. A circle k with center O and diameter AB is given. The points C and D are moving along the arc $\overset{\frown}{AB}$ so that C is between B and D and if $\angle BOC = 2\beta$ and $\angle AOD = 2\alpha$, then $\tan\alpha = \tan\beta + \dfrac{3}{2}$. Prove that the lines, which are perpendicular to CD and divide CD in ratio $1:4$ measured from C, pass through a fixed point of the given circle.

Solution. Let E be such a point on the arc $\overset{\frown}{AB}$, not containing C and D, that if $\angle BOE = 2\delta$, then $\tan\delta = 2$ (Figure 9).

We shall show that the point E satisfies the problem's conditions.

It is enough to prove that if F is the foot of the perpendicular from E to CD, then the point F is between C and D and $\dfrac{CF}{FD} = \dfrac{1}{4}$.

We have

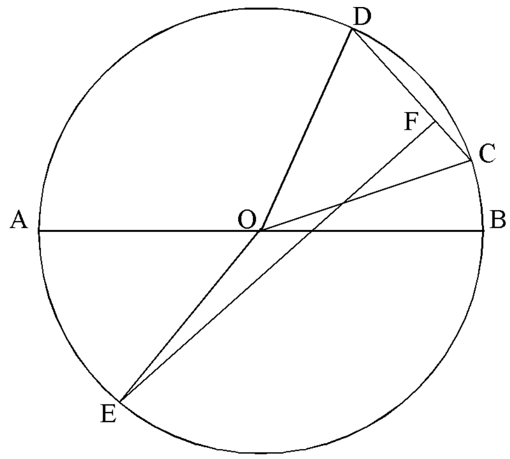
$$\tan(\angle ECD) = \tan(\angle ECA + \angle ACD) = \tan(\frac{\pi}{2} - \delta + \alpha)$$

$$= \frac{\tan\left(\frac{\pi}{2}-\delta\right)+\tan\alpha}{1-\tan\left(\frac{\pi}{2}-\delta\right).\tan\alpha} = \frac{\frac{1}{2}+\tan\alpha}{1-\frac{\tan\alpha}{2}} = \frac{1+2\tan\alpha}{2-\tan\alpha}$$

$$= \frac{1+2\left(\tan\beta+\frac{3}{2}\right)}{2-\left(\tan\beta+\frac{3}{2}\right)} = \frac{4+2\tan\beta}{\frac{1}{2}-\tan\beta} = \frac{8+4\tan\beta}{1-2\tan\beta},$$

$$\tan(\angle EDC) = \tan(\angle EDB + \angle BDC) = \tan(\delta+\beta) = \frac{\tan\delta+\tan\beta}{1-\tan\delta.\tan\beta} = \frac{2+\tan\beta}{1-2\tan\beta}.$$

Figure 9.



Thus

$$\tan(\angle ECD) = 4\tan(\angle EDC).$$

It follows from the last equality that either

$$\angle ECD \geq \frac{\pi}{2} \quad \text{and} \quad \angle EDC \geq \frac{\pi}{2}$$

or

$$\angle ECD < \frac{\pi}{2} \quad \text{and} \quad \angle EDC < \frac{\pi}{2}.$$

But these angles belong to the triangle EDC. Hence they are acute angles. Therefore the point F is between the points C and D. Since

$$\tan(\angle ECD) = \frac{EF}{FC} \quad \text{and} \quad \tan(\angle EDC) = \frac{EF}{FD},$$

then $\frac{EF}{FC} = 4\frac{EF}{FD}$, i.e. $FD = 4.FC$.

Problem 3. Find all prime numbers p, for which the number $p(2^{p-1}-1)$ is a k-th power $(k>1)$ of a positive integer.

Solution. Let $p(2^{p-1}-1) = x^k$ ($x>0$ is an integer). It is clear that $p \neq 2$, i.e. $p = 2q+1$ is an odd number. Since p/x, then $x = p.y$ (y is a positive integer) and $(2^q-1)(2^q+1) = p^{k-1}y^k$. At least one of the numbers 2^q-1 and 2^q+1 is a k-th power of an integer, because they are relatively prime numbers.

1. Let $2^q - 1 = z^k$, i.e. $2^q = z^k+1$. If k is even, then z^k+1 is not divisible by 4 . Hence $q=1$, $p=3$ and $p(2^{p-1}-1) = 3^2$.

If $k = 2l+1$ then $2^q = (z+1)(z^{2l} - z^{2l-1} + \cdots - z + 1)$, i.e. $z+1 = 2^\alpha$, where $0 \leq \alpha < q$. On the other hand

$$2^q = (2^\alpha-1)^{2l+1}+1 = 2^{2\alpha}.A + 2^\alpha(2l+1),$$

(A is an integer). The last equality contradicts with $\alpha < q$.

2. Let $2^q+1 = z^k$, i.e. $2^q = z^k - 1$. If k is odd, we obtain a contradiction as in the previous case.

If $k = 2l$, then $(z^l-1)(z^l+1) = 2^q$ and since $\text{GCD}(z^l-1, z^l+1) = 2$, we have $z^l - 1 = 2$, i.e. $q=3$, $p=7$, $p(2^{p-1}-1) = 7.63 = 21^2$.

The sought numbers are $p=3$ and $p=7$.

Grade 11.

Problem 1. Find all the values of the real parameter p, for which the range of the function

$$f(x) = \frac{2(1-p)+\cos x}{p-\sin^2 x}.$$

contains the interval $[1,2]$.

Solution. Let $y = \cos x$. The problem is to find all values of p, such that for every $k \in [1,2]$ the equation

$$\frac{2(1-p)+y}{p-1+y^2} = k$$

has at least one root $y_0 \in [-1,1]$, i.e. we should find all values of $q = 1-p$ such that for every $k \in [1,2]$ the equation

$$ky^2 - y - q(k+2) = 0$$

has at least one root $y_0^2 \neq q$, $y_0 \in [-1,1]$.

If $y_0^2 = q$, then $-y_0 - 2q = 0$, $y_0 = -2q$ and $4q^2 = q$, i.e. $q_1 = 0$ or $q_2 = \frac{1}{4}$.

If $q = 0$ the roots of the equation $ky^2 - y = 0$ are $y_1 = 0$, $y_2 = \frac{1}{k}$. But $y_2 = \frac{1}{k} \in [-1,1]$ and $y_2^2 \neq q = 0$ for every $k \in [1,2]$. Thus $q = 0$ satisfies the problem's conditions.

If $q = \frac{1}{4}$, then the roots of the equation $ky^2 - y - \frac{1}{4}(k+2) = 0$ are $y_1 = -\frac{1}{2}$, $y_2 = \frac{1}{2} + \frac{1}{k}$. But $y_2 \in [-1,1]$ only if $k = 2$. Thus $q = \frac{1}{4}$ doesn't satisfy the problems conditions.

Let $q \neq 0$ and $q \neq \frac{1}{4}$. The equation $ky^2 - y - q(k+2) = 0$ has real roots iff $D = 1 + 4k(k+2)q \geq 0$, i.e. iff $q \geq -\frac{1}{4k(k+2)}$ for every $k \in [1,2]$. Hence $q \geq -\frac{1}{32}$. The vertex of the parabola $g(y) = ky^2 - y - q(k+2)$ has as its first coordinate $y' = \frac{1}{2k} \in (0,1] \subset [-1,1]$. Therefore the equation $g(y) = 0$ has at least one root $y_0 \in [-1,1]$ iff at least one of the inequalities $g(-1) \geq 0$ and $g(1) \geq 0$ holds. It is easy to obtain from here that $q \leq \frac{k+1}{k+2}$ for every $k \in [1,2]$. Hence $q \leq \frac{2}{3}$, i.e. $q \in \left[-\frac{1}{32}, \frac{2}{3}\right]$ and $q \neq \frac{1}{4}$.

Finally $p \in \left[\frac{1}{3}, \frac{33}{32}\right]$, $p \neq \frac{3}{4}$.

Problem 2. The point O is on the edge AB of the tetrahedron $ABCD$. The circumscribed sphere of the tetrahedron $AOCD$ intersects the edges BC and BD in the points M and N ($M \neq C$, $N \neq D$) respectively. The circumscribed sphere of the tetrahedron $BOCD$ intersects the edges AC and AD in the points P and Q ($P \neq C$, $Q \neq D$) respectively. Prove that the triangles OMN and OQP are similar.

Solution. Since the points A, C, M and O are in one and the same plane and are lying on a sphere, then the quadrilateral $ACMO$ is inscribed in a circle (Figure 10). Then $\triangle BOM \sim \triangle BCA$, and $\frac{OM}{CA} = \frac{BM}{BA}$ (Figure 11). Analogously from $\triangle ABD$: $\frac{OQ}{BD} = \frac{AQ}{AB}$. Hence

$$\frac{OM}{OQ} = \frac{BM}{AQ} \cdot \frac{AC}{BD}. \tag{1}$$

From $\triangle BCD$ we have $\dfrac{MN}{CD} = \dfrac{BM}{BD}$, and from $\triangle ACD$: $\dfrac{PQ}{CD} = \dfrac{AQ}{AC}$. Thus

Figure 10.

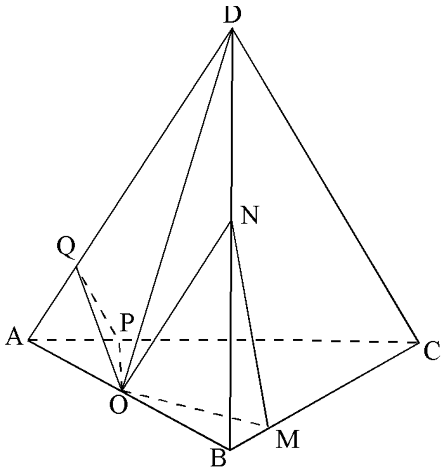
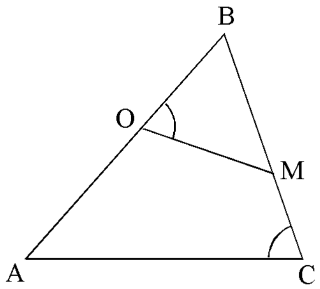


Figure 11.



$$\frac{MN}{PQ} = \frac{BM}{AQ} \cdot \frac{AC}{BD}. \tag{2}$$

It follows from (1) and (2) that $\dfrac{OM}{OQ} = \dfrac{MN}{PQ}$. Similarly $\dfrac{ON}{OP} = \dfrac{MN}{PQ}$.
Therefore $\triangle OMN \sim \triangle OQP$.

Problem 3. Solve in positive integers the equation:

$$1 + 5^x = 2^y + 2^z.5^t.$$

Solution. If $y \geq 2$ and $z \geq 2$, then the right side of the given equation is divisible by 4. But $1 + 5^x \equiv 2 \pmod 4$ and hence $\min(y, z) = 1$. On the other hand $2^y \equiv 1 \pmod 5$. Thus y is divisible by 4 (4 is the index of 2 modulo 5). It follows from here that $y \geq 4$, $z = \min(y, z) = 1$ and the equation is $1 + 5^x = 2^{4y_0} + 2.5^t$, where $y = 4y_0$ (y_0 is a positive integer).

If $t = 1$ then $5^x - 2^{4y_0} = 9$. Using congruence modulo 3 we obtain that x is even, i.e. $x = 2x_0$. Hence $(5^{x_0} - 2^{2y_0})(5^{x_0} + 2^{2y_0}) = 9$, from where $x_0 = y_0 = 1$, i.e. $x = 2$ and $y = 4$.

Let $t > 1$. Then $16^{y_0} \equiv 1 \pmod{25}$ and y_0 is divisible by 5 (5 is the index of 3 modulo 25). Hence $y_0 = 5y_1$ and $1 + 5^x = 2^{20y_1} + 2.5^t$. On the other hand $2^{10} \equiv 1 \pmod{11}$ and $5^x \equiv 2.5^t \pmod{11}$. Obviously $x > t$. Thus $5^{x-t} \equiv 2 \pmod{11}$, which is not true.

Therefore the given equation has a unique solution $(x, y, z, t) = (2, 4, 1, 1)$.

Grade 12.

Problem 1. For every real number x we denote by $f(x)$ the maximal value of the function $\sqrt{t^2 + 2t + 2}$ in the interval $[x - 2, x]$.

a) Prove that $f(x)$ is an even function and find its minimal value;

b) Prove that the function $f(x)$ is not differentiable for $x = 0$.

c) Prove that the sequence $a_n = \{f(n)\}$, $n = 1, 2, \ldots$ is convergent and find its limit.

(For every real number a we denote with $\{a\}$ the unique real number in the interval $[0, 1)$ for which the number $a - \{a\}$ is an integer.)

Solution. It is clear that the maximal value of the function $\sqrt{t^2+2t+2}$ in the interval $[x-2,x]$ is reached at the endpoints of this interval. Since the inequality $(x-2)^2+2(x-2)+2 \le x^2+2x+2$ is equivalent to $x \ge 0$, then

$$f(x) = \begin{cases} \sqrt{x^2-2x+2} & \text{if} \quad x \le 0 \\ \sqrt{x^2+2x+2} & \text{if} \quad x \ge 0. \end{cases}$$

a) Obviously $f(x) = f(-x)$, i.e. $f(x)$ is an even function and its minimal value is $f(0) = 2$.

b) Since

$$\lim_{\substack{x\to 0\\ x<0}} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x<0}} \frac{\sqrt{x^2-2x+2}-\sqrt{2}}{x} = -\frac{\sqrt{2}}{2}$$

and

$$\lim_{\substack{x\to 0\\ x>0}} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{\sqrt{x^2+2x+2}-\sqrt{2}}{x} = \frac{\sqrt{2}}{2},$$

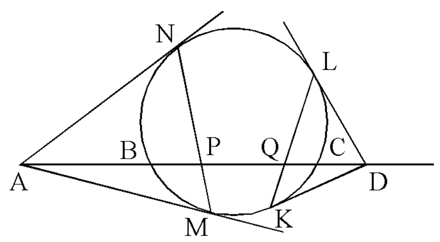
then $f'(0)$ doesn't exist.

c) It is clear that $a_n = \left\{\sqrt{n^2+2n+2}\right\}$. On the other hand $n+1 < \sqrt{n^2+2n+2} < n+2$, i.e. $a_n = \sqrt{n^2+2n+2} - (n+1)$.

Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+2n+2}+n+1} = 0.$$

Figure 12.



Problem 2. The points A, B, C and D lie on a straight line in the given order. A circle k passes through the points B and C and AM, AN, DK and DL are tangents to k.

a) Prove that the points $P = MN \cap BC$ and $Q = KL \cap BC$ don't depend on the circle k.

b). If $AD = a$, $BC = b$ $(a > b)$ and the segment BC is moving along AD, find the minimal length of the segment PQ.

Solution. a) From the Steward's formula for the triangle AMN and the segment AP (Figure 12) we have:

$$\begin{aligned} AP^2.MN &= AM^2.NP + AN^2.MP - MN.MP.NP \\ &= (AM^2 - MP.NP).MN \end{aligned}$$

(here $AM = AN$ because they are tangents to a circle).

Hence

$$\begin{aligned} AP^2 = AM^2 - MP.NP &= AB.AC - BP.CP \\ &= AB.AC - (AC-AP)(AP-AB) \\ &= 2.AB.AC - AP(AB+AC) + AP^2, \end{aligned}$$

i.e. $AP = \dfrac{2AB.AC}{AB+AC}$. Analogously $DQ = \dfrac{2DB.DC}{DC+DB}$. These equalities show that the position of the points P and Q doesn't depend on the circle k.

b) Let us denote $AB = x$, $BC = y$ and $CD = z$. It follows from a) that

$$\begin{aligned} PQ = AD - AP - DQ &= x + y + z - \frac{2x(x+y)}{y+2x} - \frac{2z(y+z)}{y+2z} \\ &= (x+y)\left(1 - \frac{2x}{y+2x}\right) + z\left(1 - \frac{2(y+z)}{y+2z}\right) \\ &= \frac{y^2(x+y+z)}{(y+2x)(y+2z)}. \end{aligned}$$

Having in mind that $y = b$, $x + y + z = a$, we obtain

$$PQ = \frac{b^2 a}{(a+x-z)(a+z-x)} = \frac{b^2 a}{a^2 - (x-z)^2} \geq \frac{b^2}{a}.$$

Therefore the minimal length of the segment PQ is $\dfrac{b^2}{a}$ and this length is reached iff $AB = CD = x = z = \dfrac{a-b}{2}$.

Problem 3. Find all prime numbers p and q, such that the number $2^p + 2^q$ is divisible by $p.q$.

Solution.

Lemma. If $k > 1$, then k doesn't divide $2^{k-1} + 1$.

Proof. Assume that k divides $2^{k-1} + 1$. Obviously k is odd. Let $k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, where $p_1, p_2, \dots, p_r$ are odd prime numbers ($p_i \neq p_j$ if $i \neq j$), $\alpha_1, \alpha_2, \dots, \alpha_r$ are positive integers and $r \geq 1$. Let $p_i - 1 = 2^{m_i}.t_i$, where t_i are odd integers ($i = 1, 2, \dots, r$). Let m_1 be the smallest number in the sequence $m_1, m_2, \dots, m_r$. It follows from $p_i \equiv 1 (\mathrm{mod}(p_i - 1))$ that $p_i \equiv 1 \pmod{2^{m_i}}$ and $p_i^{\alpha_i} \equiv 1 \pmod{2^{m_i}}$, $i = 1, 2, \dots, r$. Hence $k - 1 = 2^{m_1}.u$ (u is an integer). If $2^{k-1} \equiv -1 \pmod{k}$, then $2^{2^{m_1}.u} \equiv -1 \pmod{k}$ and $2^{(p_1-1).u} \equiv -1 \pmod{p_1}$, because t_1 is odd. But $2^{p_1-1} \equiv 1 \pmod{p_1}$ — a contradiction.

Let $2^p + 2^q$ is divisible by $p.q$. We have three cases:

1. p and q are odd prime numbers. Then $2^p + 2^q \equiv 0 \pmod{p}$ and since $2^p \equiv 2 \pmod{p}$, then $2^q \equiv -2 (\mathrm{mod} p)$ and $2^{pq} \equiv (-2)^p \equiv -2 \pmod{p}$. Similarly $2^{pq} \equiv -2 \pmod{q}$. Thus $2^{pq-1} \equiv -1 \pmod{pq}$, which is a contradiction with the lemma.

2. $p = 2$, $q > 2$. Then $4 + 2^q \equiv 0 (\mathrm{mod} q)$ and it follows from $2^q \equiv 2 \pmod{q}$ that $6 \equiv 0 (\mathrm{mod} q)$ and $q = 3$. It is clear that $2^2 + 2^3 = 12 \equiv 0 \pmod{2 \cdot 3}$.

3. $p = q = 2$. Then $2^2 + 2^2 = 8 \equiv 0 \pmod{2 \cdot 2}$.

Therefore the sought numbers are: $p = q = 2$; $p = 2$, $q = 3$; $p = 3$, $q = 2$.