WINTER MATHEMATICAL COMPETITION

1996

Grade 8

Problem 1. For which integer values of the parameter a the equation $|2x+1|+|x-2|=a$ has an integer solutions?

Solution. We shall consider the following cases:

I. $x>2$. Then $2x+1>0$, $x-2>0$ and the equation is equivalent to $2x+1+x-2=a$. Thus, $x=\dfrac{1+a}{3}$, which is a solution when $\dfrac{1+a}{3}>2$, i.e. when $a>5$.

II. $-\dfrac{1}{2}\leq x\leq 2$. Then $2x+1\geq 0$, $x-2\leq 0$ and the equation has the form $2x+1-(x-2)=a$. Thus, $x=a-3$, which is a solution when $-\dfrac{1}{2}\leq a-3\leq 2$, i.e. $\dfrac{5}{2}\leq a\leq 5$. In this interval the integers are $a=3,4,5$ and respectively we get $x=0,1,2$, which are integer solutions.

III. $x<-\dfrac{1}{2}$. Then $2x+1<0$, $x-2<0$ and the equation is equivalent to $-(2x+1)-(x-2)=a$ or $-3x=a-1$. Thus $x=\dfrac{1-a}{3}$, which is a solution, when $\dfrac{1-a}{3}<-\dfrac{1}{2}$, i.e. when $a>\dfrac{5}{2}$.

According to the considered cases the given equation has solution when $a>\dfrac{5}{2}$. When $a=3$ the only integer solution is $x=0$. When $a=4$ there are two integer solutions $x=1$ and $x=-1$. When $a=5$, $x=2$ is the only integer solution. When $a>5$ the equation has two solutions $x_1=\dfrac{1+a}{3}$ and $x_2=\dfrac{1-a}{3}$. For these solutions we have: if $a=3k-1$, then x_1 is integer only; if $a=3k+1$, then x_2 is integer only; if $a=3k$, there is no integer solution.

As a result, we get integer solutions when $a=3$ or $a=3k\pm 1$, where k is a positive integer, greater than 1.

Problem 2. The bisector AD ($D\in BC$) of the acute isosceles triangle ABC divides it into two isosceles triangles. Let O and I be the incenter and the circumcenter of $\triangle ABC$, respectively. AO meets BC in point E, while F is the intersection point of the lines BI and DO. Prove that:

a) the quadrilaterals $ABEF$ and $ADCF$ are rhombi with equal side lengths;

b) If H is the altitude center of $\triangle ABE$, then the points A, D, E, F, H are concyclic.

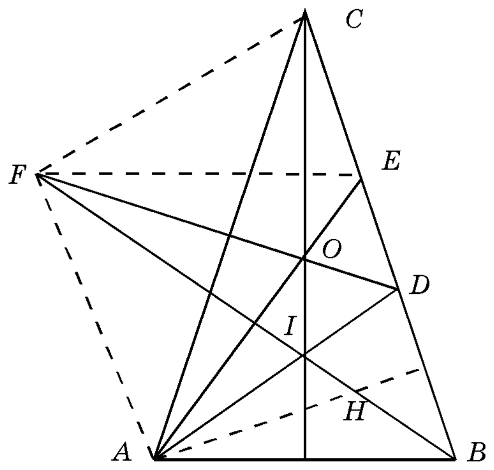
Solution. Firstly, let us justify the position of AD. If it is a bisector of the angle between the two equal sides, then it is perpendicular to BC, $\triangle ADB$ and $\triangle ADC$ are isosceles. Consequently $AD=BD=CD$ and $\angle BAD=\angle CAD=45^\circ$, i.e. $\angle BAC=90^\circ$, which contradicts to the condition that $\triangle ABC$ is acute. It follows that AD is bisector of the angle belonging to the base AB, while $\triangle ABD$ and $\triangle ACD$ are isosceles (Figure 1).

a) If $\angle BAD=\angle CAD=\alpha$, then $\angle ABC=2\alpha$. Since $\angle ADB>\angle CAD=\angle BAD$, the only possibility is $\angle ADB=\angle ABC=2\alpha$ and $AD=AB$. But $\angle ACD=\angle ADB-\angle CAD=2\alpha-\alpha=$

α, i.e. $\angle ACD = \angle CAD$ and $CD = AD$. From the equality $2\alpha+2\alpha+\alpha = 180^\circ$ we define $\alpha = 36^\circ$. Thus, $\angle ACB = 36^\circ$, $\angle BAC = \angle ABC = 72^\circ$. The points O and I lie on the bisector of $\angle ACB$, which is a segment bisector of AB. We have $OA = OC$ and $\angle OAC = \angle OCA = 18^\circ = \angle OAD$, while $\angle OAB = 36^\circ + 18^\circ = 54^\circ$, $\angle AEB = 180^\circ - 72^\circ - 54^\circ = 54^\circ = \angle BAE$. It follows that $BE = AB$ and the bisector BI of $\angle ABC$ is the segment bisector of AE. In the triangle ACD, which is isosceles, we have that AO and CO are bisectors and it follows that DO is the bisector of $\angle ADC$ and the segment bisector of AC. Then F is the intersection point of the segment bisectors of AE and AC, i.e. F is the circumcenter of $\triangle ACE$, the radius of the circumcircle is $AF = EF = CF$. We have $\angle AFE = 2\angle ACE = 72^\circ$, while $\angle AFB = \angle BFE = 36^\circ = \angle ABF = \angle EBF$ and consequently $AF = AB = BE = EF$, i.e. $ABEF$ is a rhombus. In the same way $CF = AF = AB = AD = CD$ and $ADCF$ is a rhombus, which side is $AD = AB$.

b) It is clear that $\angle AEF = \angle AEB = 54^\circ$ and $\angle ADF = \dfrac{1}{2}\angle ADC = \ frac12(180^\circ - 72^\circ) = 54^\circ$, the segment AF is seen from the points D and E under 54°. We draw a line through A, perpendicular to BC, which intersects BI in the point H — the altitude center of $\triangle ABE$, because $BI \perp AE$. But in $\triangle ABD$ the altitude AH is a bisector of $\angle BAD = 36^\circ$. Thus, $\angle BAH = 18^\circ$, $\angle AHF = 18^\circ + 36^\circ = 54^\circ = \angle AEF = \angle AOF$. Note that the points D, E, H are in one and the same semiplane with respect to AF. Consequently, D, E, H together with A and F are concyclic.

Figure 1.



Problem 3. Every day a student preparing himself for the Winter competition in mathematics has been solving problems during a period of 5 weeks. He has been solving at least one problem daily but no more than 10 problems weekly.

a) Prove that during some consecutive days the student has solved 19 problems exactly.

b) If $1 \le n \le 34$ is a natural number, prove that during some consecutive days the student has solved n problems exactly.

Remark. Every week begins on Monday and ends on Sunday.

Solution. Since a) is a particular case of b), we shall solve b) only.

According to the condition the student has been solving problems during $5 \cdot 7 = 35$ days and has solved at most $5 \cdot 10 = 50$ problems. Let x_i be the number of the problems, solved during the i-th day ($i = 1, 2, \ldots, 35$).

Let $1 \le n \le 34$ be a fixed natural number. We want to prove that there exist such $k < l$, that $x_{k+1} + \cdots + x_l = n$. Denote $X_i = x_1 + \cdots + x_i$. Obviously,

$$1 \le X_1 < X_2 < \ldots < X_{35} \le 50$$

and the problem is to prove the existence of such $k < l$, that $X_l - X_k = n$.

Case 1. $1 \le n \le 19$. We consider the numbers

$$X_1 < X_2 < \ldots < X_{35}, \quad X_1 + n < X_2 + n < \ldots < X_{35} + n, \tag{1}$$

which are integers and their number is 70. Obviously they are in the interval $[1, 50 + n]$, in which there are $50 + n \le 50 + 19 < 70$ integers. Consequently, among the numbers (1) there are at least two equal. The first 35 of them as well as the next 35 are different from each other. Therefore, there exist such k and l, for which $X_l = X_k + n$, i.e. for which $X_l - X_k = n$.

Case 2. $20 \le n \le 34$. Firstly, we shall prove the following

Lemma. If the integers $z_1, z_2, \dots, z_m$ belong to the interval $[1, 2n]$ and if $m > n$, then among the numbers $z_1, z_2, \dots, z_m$ there are two, the difference of which is equal to n exactly.

Proof. With the numbers from the interval $[1, 2n]$ we construct the following pairs:

$$(1, n+1), (2, n+2), \dots, (n, 2n).$$

The number of these pairs is n and the difference of the numbers in each pair is equal to n. Since $m > n$, at least two of the numbers $z_1, z_2, \dots, z_m$ belong to one and the same pair. Therefore their difference is equal to n.

Let us finish now the solution of the problem.

If $n \ge 25$, then $2n \ge 50$, and thus all the numbers $X_1, X_2, \dots, X_{35}$ are in the interval $[1, 2n]$. On the other hand $n < 35$ and according to the lemma there are two numbers among $X_1, X_2, \dots, X_{35}$ which difference is n.

If $20 \le n \le 24$, we represent the interval $[1, 50]$ as an union of the intervals $[1, 2n]$ $[2n+1, 50]$. In the second one there are $50 - (2n + 1) + 1 = 50 - 2n$ integers. Then, the number of the integers among $X_1, X_2, \dots, X_{35}$, which belong to the interval $[1, 2n]$, is at least $35 - (50 - 2n) = 2n - 15 \ge 40 - 15 = 25 > n$. Consequently, we can apply the lemma again.

Remark. The case 1 can be solved by the lemma proved above.

Grade 9

Problem 1. Let $f(x) = x^3 - (p+5)x^2 - 2(p-3)(p-1)x + 4p^2 - 24p + 36$, where p is a real parameter.

a) Prove that $f(3 - p) = 0$.

b) Find all values of p, for which two of the roots of the equation $f(x) = 0$ are lengths of the cathetuses of a rectangle triangle which hypotenuse is equal to $4\sqrt{2}$.

Solution. a) We have $f(x) = (x + p - 3)(x^2 - 2(p+1)x + 4(p-3))$.

b) The roots of $f(x) = 0$ are $x_{1,2} = p + 1 \pm \sqrt{p^2 - 2p + 13}$ and $x_3 = 3 - p$. If $p > 3$, then $x_1 > 0$, $x_2 > 0$ and $x_3 < 0$. The equation $x_1^2 + x_2^2 = 32 = \left(4\sqrt{2}\right)^2$ gives $p = \pm 1$, which is impossible. If $p = 3$, then two of the roots are equal to 0 and this case gives no solution. If $p < 3$, then $x_1 > 0$, $x_3 > 0$ and $x_2 < 0$. Therefore, $32 = x_1^2 + x_3^2 = (p + 1 + \sqrt{p^2 - 2p + 13})^2 + (3 - p)^2$. This equation is equivalent to

$$(p+1)(3(p-3) + 2\sqrt{p^2 - 2p + 13}) = 0.$$

If $p \ne -1$, we get $2\sqrt{p^2 - 2p + 13} = 3(3 - p) > 0$, from where $5p^2 - 46p + 29 = 0$ and $p = \dfrac{23 \pm 8\sqrt{6}}{5}$. Since $\dfrac{23 + 8\sqrt{6}}{5} > 3$ and $\dfrac{23 - 8\sqrt{6}}{5} < 3$, we find $p_1 = -1$ and $p_2 = \dfrac{23 - 8\sqrt{6}}{5}$.

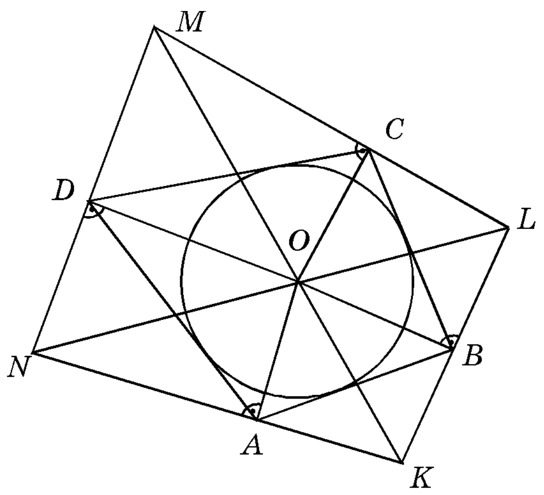
Problem 2. The incenter of the quadrilateral $ABCD$ is O. The lines $l_A \perp OA$, $l_B \perp OB$, $l_C \perp OC$ and $l_D \perp OD$ are drawn through the points A, B, C and D respectively. The lines l_A and l_B meet each other in the point K, l_B and l_C — in L, l_C and l_D — in M, l_D and l_A — in N.

a) Prove that the lines KM and LN meet each other in the point O.

b) If the lengths of the segments OK, OL and OM are p, q and r respectively, find the length of the segment ON.

Solution. a) We shall prove that the points N, O and L are colinear. Denote $\angle ABC = \angle B$, $\angle BCD = \angle C$, $\angle CDA = \angle D$ and $\angle DAC = \angle A$. Since O is incenter, the segments OA,OB,OC and OD are the bisectors of the corresponding angles of the quadrilateral $ABCD$. Note that each of the quadrilaterals $AKBO$, $BLCO$, $CMDO$ and $DNAO$ is inscribed.

Figure 2.



Consequently: $\angle NOK + \angle KOL = \pi - \angle ONA - \angle OKA + \pi - \angle OKB - \angle OLB = \pi - \angle ADO - \angle ABO + \pi - \angle BAO - \angle BCO = 2\pi - (\frac{\angle D}{2} + \frac{\angle B}{2} + \frac{\angle A}{2} + \frac{\angle C}{2}) = 2\pi - \pi = \pi$. It follows from here that the points N,O and L are colinear. Analogously, the points K,O and M are colinear. Therefore, O is the intersection point of the diagonals of the quadrilateral $KLMN$.

b) Firstly, we shall prove that the quadrilateral $KLMN$ is inscribed. Indeed,

$$\begin{aligned} & \angle NKL + \angle NML \\ = & \angle AKO + \angle OKB + \angle DMO + \angle OMC \\ = & \frac{\angle B}{2} + \frac{\angle A}{2} + \frac{\angle C}{2} + \frac{\angle D}{2} = 2\pi. \end{aligned}$$

Thus, $OK.OM = OL.ON$, from where

$$ON = \frac{OK.OM}{OL} = \frac{p.r}{q}.$$

Problem 3. A square with side length 5 is divided into unit squares by parallel to its sides lines. Let A be the set of the vertexes of the unit squares which are not on the sides of the given square. How many points from A can be chosen at most in a way that no three of them are vertexes of isosceles rectangle triangle?

Solution. We shall prove that the maximal number is 6. Let us enumerate the points in the way, shown on the table 1.

It is easy to be seen that no 3 of the points 1,2,3,8,12 and 16 are vertexes of a isosceles rectangle triangle. Assume that there exists a set of 7 points with the desired property. Note that if 4 points form a square, then at most 2 of them can be among the already chosen ones. The points 1,4,16 and 13; 2,8,15 and 9; 3,12,14 and 5 form squares.

13	14	15	16
9	10	11	12
5	6	7	8
1	2	3	4

Table 1:

Consequently, at most 6 of the chosen points lie on the contour. It follows from here that at least one of the points 6,7,10 and 11 is from the chosen ones. Due to the symmetry we may assume that this is the point 7. Since the points 7,16 and 9; 1,7 and 14 form isosceles rectangle triangles, then at most two of the points 1,9,14 and 16 are from the chosen ones. The points 5,7,13 and 15 form a square and therefore at most one of the points 5,13 and 15 is from the chosen ones. It follows from here that at least 3 points are chosen from 2,3,4,6,8,10,11 and 12. By the pigeonhole principle we deduce that at least two points are chosen in one of the sets 3,6,11,8 and 2,4,10,12. It is easy to see that if the two points are in the first set, then we have two possibilities — 3 and 11 or 6 and 8 (in both cases it is not possible to chose more points on the square which encounters 7). Analogously, if the two points are in the second set, then the possibilities are two again — 2 and 12 or 4 and 10 (in both cases it is not possible to chose more points on the square which encounters 7). The contradiction shows that the maximal number of points which can be chosen is equal to 6.

Grade 10

Problem 1. Let p and q be such integers that the roots x_1 and x_2 of the quadratic equation $x^2 + px + q = 0$ are real numbers. Prove that if the numbers 1, x_1, x_2 (in some order) form a geometric progression, then the number q is a perfect cube.

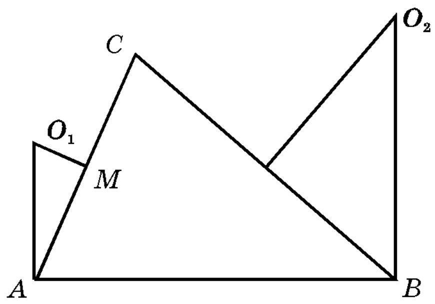
Solution. There are two possibilities for the order of the numbers in the geometric progression: $x_1, 1, x_2$ and $1, x_1, x_2$. In the first case we get $q = x_1x_2 = 1^2$, i.e. $q = 1$ is a perfect cube. Let now $x_2 = x_1^2$. We have $-p = x_1 + x_2 = x_1 + x_1^2$, i.e. x_1 satisfies the equation $x_1^2 + x_1 + p = 0$. On the other hand x_1 satisfies also $x_1^2 + px_1 + q = 0$. From these two equations we get $(p-1)x_1 + (q-p) = 0$. If $p \neq 1$, then x_1 is rational and $q = x_1x_2 = x_1x_1^2 = x_1^3$ is the cube of a rational number. Since q is integer, then it is a perfect cube. If $p = 1$, then $q = p$ and the quadratic equation becomes $x^2 + x + 1 = 0$. The last equation has no real root.

Problem 2. A triangle ABC with a radius R of the circumcircle is given. Let R_1 and R_2 be the radii of the circles k_1 and k_2, respectively, which pass through C and are tangent to the line AB in A and B, respectively.

a) Prove that the numbers R_1, R and R_2 form a geometric progression.

b) Find the angles of $\triangle ABC$, if the radius of the circle which is tangent to k_1, k_2 and the line AB, is equal to $\dfrac{R}{4}$.

Figure 3.



Solution. a) Let $AB = c$, $BC = a$ and $CA = b$. Denote by O_1 and O_2 the centers of the circles k_1 and k_2 (Figure 3). Then O_1 is the intersection point of the perpendicular from A to AB and the segment bisector of AC. Since $\angle MAO_1 = |90^\circ - \angle A|$ (M is the midpoint of AC), then $R_1 = AO_1 = \dfrac{AM}{\cos|90^\circ - \angle A|} = \dfrac{AC}{2\sin\angle A}$ and by the sine theorem it follows that

$$R_1 = R \cdot \frac{b}{a}. \tag{1}$$

Analogously,

$$R_2 = R \cdot \frac{a}{b}. \tag{2}$$

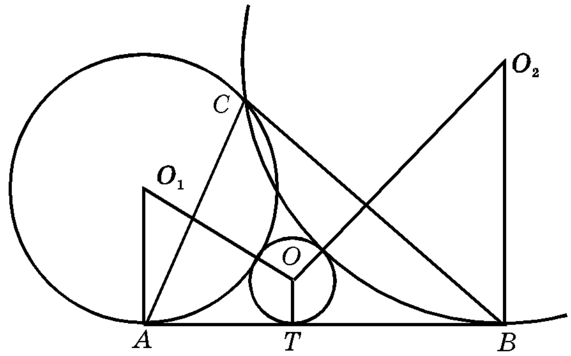
From here $R_1R_2 = R^2$ and the proposition is proved.

b) Let O be the center of the circle which touches k_1, k_2 and the line AB (Figure 4). Denote by T the tangent point of this circle with AB, and by r its radius. From the rectangle trapezoid $ATOO_1$ it follows that $AT = \sqrt{(R_1+r)^2 - (R_1-r)^2} = 2\sqrt{rR_1}$. Analogously, $BT = 2\sqrt{rR_2}$. Then $c = AT + TB = 2\sqrt{rR_1} + 2\sqrt{rR_2}$ and from here we find $r = \dfrac{c^2}{4(\sqrt{R_1}+\sqrt{R_2})^2}$. Using (1) and (2), we get

$$r = \frac{abc^2}{4R(a+b)^2}. \tag{3}$$

Since $\dfrac{ab}{(a+b)^2} \le \dfrac{1}{4}$ and $c^2 \le 4R^2$, it follows from (3) that $r \le \dfrac{R}{4}$. The equation is reached when $a = b$ and $c = 2R$. Now, it follows that $\angle A = \angle B = 45^\circ$ and $\angle C = 90^\circ$.

Figure 4.



Problem 3. A positive integer n and a real number φ are given in a way that $\cos\varphi = \dfrac{1}{n}$. Find all positive integers k, for which the number $\cos k\varphi$ is an integer.

Solution. **Case 1.** $n = 1$. Then $\cos\varphi = 1$ and $\varphi = 2m\pi$ $(m = 0, \pm 1, \ldots)$. For all $k \in \mathbb{N}$ the number $\cos k\varphi$ is an integer.

Case 2. $n = 2$. Then $\cos\varphi = \dfrac{1}{2}$ and $\varphi = \pm\dfrac{\pi}{3} + 2m\pi$ $(m = 0, \pm 1, \ldots)$. It is clear that for all $k \in \mathbb{N}$, which are divisible by 3, the number $\cos k\varphi$ is an integer.

Case 3. $n \ge 3$. We shall prove that for all $k \in \mathbb{N}$ the number $\cos k\varphi$ is not an integer. Let n be odd. We have $\cos\varphi = \dfrac{1}{n}$, $\cos 2\varphi = 2\cos^2\varphi - 1 = \dfrac{2-n^2}{n^2}$ and $(2-n^2, n) = 1$. We shall prove by induction that $\cos k\varphi = \dfrac{a}{n^k}$, where $(a, n) = 1$. Assume that the assertion is true for all integers from 1 to k. We shall check it for $k+1$. From

$$\cos(k+1)\varphi + \cos(k-1)\varphi = 2\cos k\varphi \cos\varphi$$

it follows that $\cos(k+1)\varphi = \dfrac{2}{n}\cdot\dfrac{a}{n^k} - \dfrac{b}{n^{k-1}} = \dfrac{2a-bn^2}{n^{k+1}}$, where $\cos k\varphi = \dfrac{a}{n^k}$ and $\cos(k-1)\varphi = \dfrac{b}{n^{k-1}}$. We have $(a,n) = 1$ and $(b,n) = 1$ according to the inductive assumption. It is clear that $(2a - bn^2, n) = 1$ and this ends the proof. The case when n is even is analogous. Now $\cos k\varphi$ is expressed by a fraction which denominator is equal to $2p^k$ with $n = 2p$, while the nominator has no common divisor with p.

Answer: if $n = 1$ $\forall k \in \mathbb{N}$;
if $n = 2$ $k = 3q$, where $q \in \mathbb{N}$;
if $n \ge 3$ there is no solution.

Grade 11

Problem 1. Find the values of the real parameter a, for which the function

$$f(x) = x^2 - 2x - |x - 1 - a| - |x - 2| + 4$$

has nonnegative values for all real x.

Solution. Firstly let $1 + a \le 2$, i.e. $a \le 1$. Then

$$f(x) = \begin{cases} x^2 + 1 - a, & x \le 1 + a \\ x^2 - 2x + 3 + a, & 1 + a \le x \le 2 \\ x^2 - 4x + 7 + a, & x \ge 2. \end{cases}$$

If $1+a \geq 2$, i.e. $a \geq 1$, we find that

$$f(x) = \begin{cases} x^2+1-a, & x \leq 2 \\ x^2-2x+5-a, & 2 \leq x \leq 1+a \\ x^2-4x+7+a, & x \geq 1+a. \end{cases}$$

Hence the smallest value of $f(x)$ is reached in one of the points 0, 1, 2, $1+a$.

We have $f(0) = 2-|a+1|$, $f(1) = 2-|a|$, $f(2) = 4-|1-a|$, $f(1+a) = a^2-|a-1|+3$. These four numbers must be nonnegative. We find that $a \in [-2,1]$. Then $|a-1| = 1-a$ and $f(1+a) = a^2+a+2 > 0$ for all a.

Let now $a \notin [-2,1]$. If $a < -2$, then $x^2-2x+3+a < 0$ when $x = 1$. Analogously, if $a > 1$, then $x^2+1-a < 0$ when $x = 0$.

Finally, $a \in [-2,1]$.

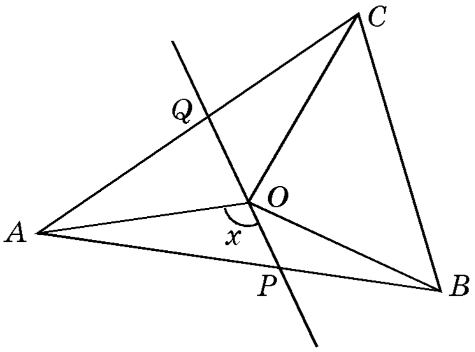
Problem 2. The point O is circumcenter of the acute triangle ABC. The points P and Q lie on the sides AB and AC respectively. Prove that O lies on the line PQ if and only if

$$\sin 2\alpha = \frac{PB}{PA}\sin 2\beta + \frac{QC}{QA}\sin 2\gamma,$$

where $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$.

Solution. Firstly, let O be on PQ. Denote $x = \angle AOP$ (Figure 5). By the sine theorem for the triangles AOP, BOP, AOQ and COQ we find that

Figure 5.



$$\frac{PA}{\sin x} = \frac{PO}{\sin(90^\circ-\gamma)},$$

$$\frac{PB}{\sin(2\gamma-x)} = \frac{PO}{\sin(90^\circ-\gamma)},$$

$$\frac{QA}{\sin(180^\circ-x)} = \frac{QO}{\sin(90^\circ-\beta)},$$

$$\frac{QC}{\sin(2\beta-180^\circ+x)} = \frac{QO}{\sin(90^\circ-\beta)}.$$

Then

$$\frac{PB}{PA}\sin 2\beta + \frac{QC}{QA}\sin 2\gamma = \frac{\sin(2\gamma-x)}{\sin x}\sin 2\beta - \frac{\sin(2\beta+x)}{\sin x}\sin 2\gamma =$$
$$\frac{1}{\sin x}((\sin 2\gamma\cos x - \sin x\cos 2\gamma)\sin 2\beta - (\sin 2\beta\cos x + \sin x\cos 2\beta)\sin 2\gamma)$$
$$= -\sin 2\beta\cos 2\gamma - \sin 2\gamma\cos 2\beta = -\sin 2(\beta+\gamma) = \sin 2\alpha.$$

Conversely, let the given equality be satisfied. Denote by Q' the intersection point of OP and AC. It follows from the above that $\sin 2\alpha = \dfrac{PB}{PA}\sin 2\beta + \dfrac{Q'C}{Q'A}\sin 2\gamma$. Then $\dfrac{Q'C}{Q'A} = \dfrac{QC}{QA}$. Since the points Q' and Q lie on the segment AC, then $Q' = Q$.

Problem 3. Find all functions $f(x)$ with integer values and defined in the set of the integers, such that

$$3f(x) - 2f(f(x)) = x$$

for all integers x.

Solution. The function $f(x) = x$ satisfies the condition of the problem.

Let $f(x)$ be a function which satisfies the condition. Let $g(x) = f(x) - x$. The condition can be written in the form

$$2f(f(x)) - 2f(x) = f(x) - x,$$

which is equivalent to

$$g(x) = 2g(f(x)).$$

From here we obtain

$$g(x) = 2g(f(x)) = 2^2 g(f(f(x))) = 2^3 g(f(f(f(x)))) = 2^4 g(f(f(f(f(x))))) = \ldots.$$

Since the numbers $g(f(f \ldots f(x)) \ldots)$ are integer, then $g(x)$ is divisible by 2^n for all integers x and all natural numbers n. This is possible only if $g(x) = 0$. Thus, $f(x) = x$ is the only solution of the problem.