Winter mathematics competition—Burgas, 1997

Problem 8.1. Let F be the set of points with coordinates (x, y) such that $||x| - |y|| + |x| + |y| = 2$.

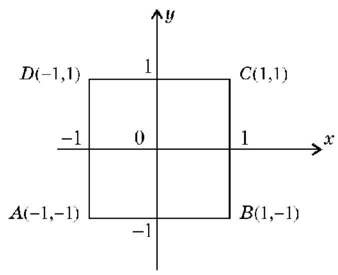
(a) Draw F.

(b) Find the number of points in F such that $2y = |2x - 1| - 3$.

Solution: (a) If $|x| \geq |y|$, then $||x| - |y|| + |x| + |y| = |x| - |y| + |x| + |y| = 2|x| = 2$, thus $|x| = 1$ and therefore $1 \geq |y|$, so $-1 \leq y \leq 1$. We conclude that the segments $-1 \leq y \leq 1$ on the lines $x = 1$ and $x = -1$ belong to F.

If $|x| \leq |y|$, then $||x| - |y|| + |x| + |y| = -|x| + |y| + |x| + |y| = 2|y| = 2$, thus $|y| = 1$ and therefore $1 \geq |x|$, so $-1 \leq x \leq 1$. We conclude that the segments $-1 \leq x \leq 1$ on the lines $y = 1$ and $y = -1$ also belong to F.

Thus we have determined that F consists of the sides of a square with vertices $A(-1, -1), B(1, -1), C(1, 1), D(-1, 1)$.



(b) We find the number of solutions of $2y = |2x-1|-3$ on each of the segments AB, BC, CD, DA.

The segment CD consists of all points (x, y) such that $-1 \leq x \leq 1$, $y = 1$. The equation $2 = |2x - 1| - 3$ has no solution x when $-1 \leq x \leq 1$. Therefore $2y = |2x - 1| - 3$ has no solution on CD.

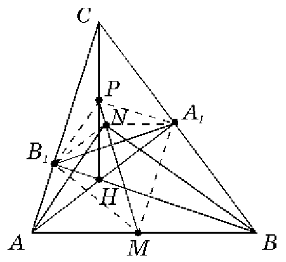
The segment AB consists of all points (x, y) such that $-1 \leq x \leq 1$, $y = -1$. The equation $-2 = |2x - 1| - 3$ has two solutions: $x = 0$ and $x = 1$. Therefore $2y = |2x - 1| - 3$ has two solutions on AB.

As above we get that $2y = |2x - 1| - 3$ has a unique solution on AD: $(x, y) = (-1, 0)$ and a unique solution on BC: $(x, y) = (1, -1)$. Note that the last one has already been obtained as a point on AB. Thus there are three solutions of $2y = |2x - 1| - 3$ in F: $(x, y) = (-1, 0), (0, -1), (1, -1)$.

Problem 8.2. Let H be the orthocentre of an acute triangle ABC. Prove that the midpoints of AB and CH and the intersecting point of the internal bisectors of $\angle CAH$ and $\angle CBH$ lie on a line.

Solution: Denote by M, P and N the midpoints of AB and CH and the intersecting point of the internal bisectors of $\angle CAH$ and $\angle CBH$ (fig. 1).

Let AA_1 ($A_1 \in BC$) and BB_1 ($B_1 \in AC$) be altitudes in $\triangle ABC$. We show first that M, N and P lie on the axis of symmetry l of A_1B_1. From $\angle CA_1H = \angle CB_1H = 90^\circ$ we get $PA_1 = PB_1 = \frac{CH}{2}$. Similarly, from $\angle AA_1B = \angle AB_1B = 90^\circ$ we get $MA_1 = MB_1 = \frac{AB}{2}$. Therefore $M \in l$ and $P \in l$. We prove now that $\triangle NMA_1 \cong \triangle NMB_1$. Now $\angle BAA_1 = 90^\circ - \beta$ and $\angle ABB_1 = 90^\circ - \alpha$, thus $\angle NBB_1 = \angle NBA_1 = \frac{1}{2}\angle A_1BB_1 = 45^\circ - \frac{\gamma}{2}$. By analogy $\angle NAA_1 = 45^\circ - \frac{\gamma}{2}$, so $\angle ANB = 180^\circ - \angle NAB - \angle NBA = 90^\circ$. Therefore $\triangle ABN$ is a right triangle and $MN = MA_1 = MB_1 = \frac{AB}{2}$.



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Further $\angle NMA_1 = \angle NMB - \angle A_1MB = 90^\circ - \gamma$. By analogy $\angle NMB_1 = 90^\circ - \gamma$ and therefore the considered triangles are identical. It follows now that $NA_1 = NB_1$, so $N \in l$.

Problem 8.3. The n points $A_0, A_1, \dots, A_{n-1}$ lie a circle in this order and divide it into equal arcs. Find an ordering $B_0, B_1, \dots, B_{n-1}$

of the same points such that the length of $B_0B_1\dots B_{n-1}$ is maximal.

Solution: Let first $n=2k+1$. Clearly a chord A_iA_j is of maximal length if $|i-j|=k$ or $k+1$. Cosider the following points:

$$A_0, A_k, A_{2k}, A_{k-1}, A_{2k-1}, A_{k-2}, A_{2k-2}, \dots, A_1, A_{k+1}.$$

Since each segment is of maximal length, it follows that the length of $A_0A_kA_{2k}A_{k-1}A_{2k-1}A_{k-2}A_{2k-2}\dots A_1A_{k+1}$ is maximal.

Let now $n=2k$. A chord A_iA_j is of maximal length if $|i-j|=k$. There are k such segments: $A_0A_k, A_1A_{k+1}, \dots, A_{k-1}A_{2k-1}$. The second longest chord A_iA_j is obtained when $|i-j|=k-1$ or $k+1$. Cosider the following points:

$$A_0, A_k, A_{2k-1}, A_{k-1}, A_{2k-2}, A_{k-2}, \dots, A_{k+1}, A_1.$$

It is easy to see that there are k segments of maximal length and $k-1$ segments of the second greatest length. Trivially, this is the required ordering.

Problem 9.1. Let $\alpha\neq\beta$ be the roots of the equation $x^2+px+q=0$. For any natural number n denote:

$$a_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}.$$

(a) Find p and q such that for any natural number n the following equality holds:

$$a_{n+1}a_{n+2}-a_na_{n+3}=(-1)^n.$$

(b) Prove that for these p and q it is true that

$$a_n + a_{n+1} = a_{n+2}$$

for any natural number n.

(c) Prove that for any natural number n, a_n is integer and if 3 divides n, then a_n is even.

Solution: (a) Since α and β are the roots of $x^2 + px + q = 0$, we know that $\alpha + \beta = -p$, $\alpha\beta = q$ and therefore:

$$\begin{aligned}
(-1)^n &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} \cdot \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} - \frac{\alpha^n - \beta^n}{\alpha - \beta} \cdot \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} \\
&= \frac{1}{(\alpha - \beta)^2}[-(\alpha + \beta)(\alpha\beta)^{n+1} + (\alpha^3 + \beta^3)(\alpha\beta)^n] \\
&= \frac{1}{p^2 - 4q}(pq^{n+1} - p(p^2 - 3q)q^n) \\
&= \frac{q^n}{p^2 - 4q}(-p^3 + 4pq) = -pq^n.
\end{aligned}$$

Thus

$$pq^n = (-1)^{n+1}. \qquad (1)$$

It follows from (1) for $n = 1$ and $n = 2$ that $pq = 1$ and $pq^2 = -1$ and so $p = -1$, $q = -1$. Direct verification shows that $p = -1$ and $q = -1$ satisfy (1) for any n. Also $\alpha \neq \beta$.

(b) Since α and β are the roots of $x^2 - x - 1 = 0$, we know that $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$. Therefore

$$a_n + a_{n+1} = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} = \frac{\alpha^n(1 + \alpha) - \beta^n(1 + \beta)}{\alpha - \beta}$$

$$= \quad \frac{\alpha^n.\alpha^2 - \beta^n.\beta^2}{\alpha - \beta} = \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} = a_{n+2},$$

which completes the proof of (b).

(c) Since $a_1 = \dfrac{\alpha - \beta}{\alpha - \beta} = 1$ and $a_2 = \dfrac{\alpha^2 - \beta^2}{\alpha - \beta} = \alpha + \beta = 1$, it follows by induction from

$$a_n + a_{n+1} = a_{n+2} \tag{2}$$

that a_n is integer for any n. From (2) we obtain:

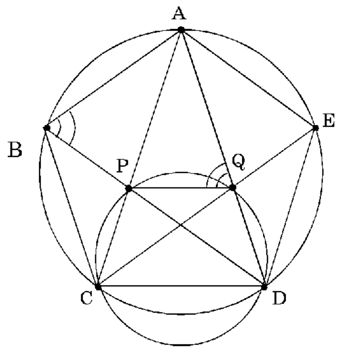
$$a_{n+3} = a_{n+2} + a_{n+1} = a_{n+1} + a_n + a_{n+1} = 2a_{n+1} + a_n.$$

Observe that $a_3 = a_1 + a_2 = 2$ is an even number. It is easy to see now (again by induction) that a_n is even for $n = 3k$.

Problem 9.2. A pentagon $ABCDE$ is inscribed in a circle. Let P be the intersecting point of AC and BD, and let Q be the intersecting point of AD and CE. Prove that if the triangles ABP, AEQ, CDP, CDQ and APQ have the same area, then $ABCDE$ is a regular pentagon.

Solution: It suffices to prove that the sides of $ABCDE$ are equal. Since $\triangle ABP$ and $\triangle CDP$ have equal areas, so do the triangles ACD and ADB. Therefore the quadrilateral $ABCD$ (fig. 1) is a trapezoid inscribed in a circle, so $AB = CD$. By analogy from $S_{AEQ} = S_{CDQ}$ we get $AC \| DE$ and $AE = CD$. Now, $\angle PCQ = \angle ACE = \dfrac{\overset{\frown}{AE}}{2} = \dfrac{\overset{\frown}{AB}}{2} = \angle ADB = \angle PDQ$. Therefore $CDQP$ is inscribed in a circle.

On the other hand it follows from $S_{CDP} = S_{CDQ}$ that $CDQP$ is a trapezoid. Thus $\angle CDP = \angle DCQ \Rightarrow \overset{\frown}{BC} = \overset{\frown}{DE}$ and therefore $BC = DE$. It remains to show that $AB = BC$. Consider the triangles ABP and APQ, whose areas are equal. They have AP as a common side and $\angle AQP = \angle APQ = \angle ACD = \angle ABD = \angle ABP$. It is easy to see now that $\triangle ABP \cong \triangle APQ$. Since $\angle APB = \angle ADP + \angle PAQ > \angle PAQ$,



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it follows that $\angle APB = \angle APQ = \angle AQP = \angle ABP$. Therefore $\overset{\frown}{AB} + \overset{\frown}{CD} = \overset{\frown}{AE} + \overset{\frown}{DE}$, so $\overset{\frown}{AB} = \overset{\frown}{DE} = \overset{\frown}{BC}$ or $AB = BC$, which completes the proof.

Problem 9.3. Given a rectangular table of 100 rows and 1997 columns. The table is filled with zeroes and ones in such a way that there are at least 75 ones in any column. Prove that it is possible to remove 95 rows in such a way that there is at most one column consisting of zeroes in the remaining table (5 rows and 1997 columns).

Solution: We show first that there is a row with at least 1498 ones. Assume the contrary. Denote by a_i the number of ones in the i-th row ($i = 1, 2, \ldots, 100$) and by b_i the number of ones in the i-th column ($i = 1, 2, \ldots, 1997$). Now $\sum_{i=1}^{1997} b_i = \sum_{i=1}^{100} a_i$. Note that the sum on the left-hand side is at least $1997 \cdot 75$, whereas the sum on the right-hand side is at most $1497 \cdot 100$, a contradiction.

Without loss of generality assume that the first row begins by 1498 ones. Consider the table formed by the last 499 columns. As above we prove that there is a row of this table (not necessarily distinct from the first one) with at least 375 ones. Let that be the second row (if it is not the first one) and let it begin with 375 ones in the new table. Now consider the table formed by the last 124 columns. Analogously, there exists a row having at least 93 ones. Let that be the third row (if it is not the first or the second one) and let it begin with 94 ones in the new table. Consider next the table formed by the last 31 columns and note that there exists a row that originally has 24 ones. Finally consider the table formed by the last 7 rows and note that there exists a row that originally has 6 ones.

We now have 5 rows (if there are fewer of them, we add arbitrary rows). Remove the remaining 95 rows of the original table. Since $1498+375+93+24+6 = 1996$, there is at most one column consisting of zeroes.

Problem 10.1. Find all real numbers x such that $\tan\left(\frac{\pi}{12}-x\right)$, $\tan\frac{\pi}{12}$ and $\tan\left(\frac{\pi}{12}+x\right)$ form (in some order) a geometric progression.

Solution: Denote $a=\tan\frac{\pi}{12}$ and $y=\tan x$. There are three cases to consider:

1. $\tan\left(\frac{\pi}{12}-x\right)\cdot\tan\left(\frac{\pi}{12}+x\right)=\tan^2\frac{\pi}{12}$. Now $\frac{a-y}{1+ay}\cdot\frac{a+y}{1-ay}=a^2$. Therefore $a^2-y^2=a^2(1-a^2y^2)$, and so $(a^4-1)y^2=0$. Since $a\neq\pm 1$ we get that $y=0$, so $\tan x=0$. Obviously all numbers x of the kind $x=k\pi$, $k\in\mathbb{Z}$ are solutions to the problem.

2. $\tan\frac{\pi}{12}\tan\left(\frac{\pi}{12}+x\right)=\tan^2\left(\frac{\pi}{12}-x\right)$. We obtain

   $$a\frac{a+y}{1-ay}=\left(\frac{a-y}{1+ay}\right)^2\Longrightarrow(a^2+1)y[ay^2+(a^2-1)y+3a]=0.$$

   The case of $y=0$ is settled in 1. Let y_1 and y_2 be the roots of the equation $ay^2+(a^2-1)y+3a=0$. Since $a=\tan 15\deg=\tan(45\deg-30\deg)=2-\sqrt{3}$ we get $y_1=y_2=\sqrt{3}$, so $\tan x=\sqrt{3}$. Obviously all x of the kind $x=\frac{\pi}{3}+k\pi$, $k\in\mathbb{Z}$ are solutions of the problem.

3. $\tan\frac{\pi}{12}\tan\left(\frac{\pi}{12}-x\right)=\tan^2\left(\frac{\pi}{12}+x\right)$. The substitution $z=-x$ transforms this case to the previous one. Therefore $x=-\frac{\pi}{3}+k\pi$, $k\in\mathbb{Z}$.

The required numbers are $x = k\pi$ and $x = \pm\dfrac{\pi}{3} + k\pi$, $k \in \mathbb{Z}$.

Problem 10.2. Two points C and M are given in the plane. Let H be the orthocentre of $\triangle ABC$ such that M is a midpoint of AB.

(a) Prove that $CH \cdot CD = |AM^2 - CM^2|$ where $D \in AB$ and $CD \perp AB$.

(b) Find the locus of points H when AB is of given length c.

Solution: (a) It is easy to see that the equality holds if $\triangle ABC$ is a right triangle. If $\triangle ABC$ is not a right triangle, then $\triangle BDH$ and $\triangle ADC$ exist and $\triangle BDH \sim \triangle ADC$. Therefore $\dfrac{DH}{AD} = \dfrac{BD}{CD}$ and so

$$CD \cdot DH = AD \cdot BD \tag{1}$$

There are three cases:

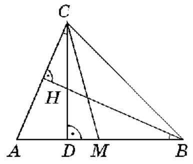
Case 1: $\triangle ABC$ is an acute triangle (fig. 1).

Case 2: $\triangle ABC$ is obtuse triangle and $\angle C > 90^\circ$ (fig. 2).

Case 3: $\triangle ABC$ is obtuse triangle and $\angle A > 90^\circ$ or $\angle B > 90^\circ$ (fig. 3).

In cases 1 and 2 it follows from (1) that

$$CD \cdot DH = (AM \mp DM)(AM \pm DM) = AM^2 - DM^2 = AM^2 - (CM^2 - CD^2),$$



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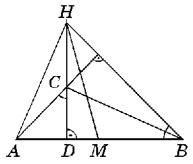

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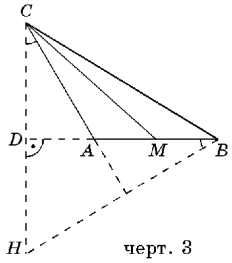


черт. 3

so

$$AM^2 - CM^2 = CD(DH - CD). \tag{2}$$

In case 1 we have $CD = CH + DH$ and from (2) we get

$$AM^2 - CM^2 = CD(DH - CH - DH) = -CH \cdot CD.$$

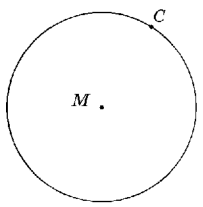
In case 2 we have $CD = DH - CH$ and from (2) we get

$$AM^2 - CM^2 = CD(DH - DH + CH) = CH \cdot CD.$$

In case 3 it follows from (1) that $CD \cdot DH = (DM \mp AM)(DM \pm AM) = DM^2 - AM^2 = CM^2 - CD^2 - AM^2$ and therefore $CM^2 - AM^2 = CD(DH + CD)$. Now $DH = CH - CD$ and so $CM^2 - AM^2 = CD(CH - CD + CD) = CH \cdot CD$. Summarising all three cases considered (and the case of a right triangle $\triangle ABC$) we get

$$CH \cdot CD = |AM^2 - CM^2|.$$

(b) It follows from the conditions of the problem that A and B are diametrically opposite in the circle (k) of centre M and radius $\frac{c}{2}$. When the diameter AB varies in (k) we obtain all triangles ABC with the fixed vertex C. There are three cases:



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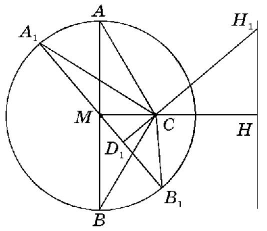
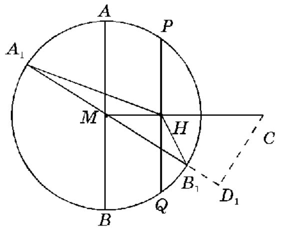


черт. 5

Case 1: $\frac{c}{2} = |CM|$ (fig. 4); Case 2: $\frac{c}{2} > |CM|$ (fig. 5); Case 3: $\frac{c}{2} < |CM|$ (fig. 6).

In case 1 C lies on (k) and triangles ABC are right triangles and their orthocentres coincide with C. In this case the locus consists of point C.



черт. 6

Consider now case 2. Let AB be a diameter in (k) perpendicular to CM. The orthocentre H of $\triangle ABC$ lies on line CM. Moreover H is an external point for (k) because $\triangle ABH$ is acute. The last follows from $\angle AHB = 180\deg - \angle ACB$ and $\angle ACB > 90\deg$.

Let A_1B_1 be a arbitrary diameter of (k) such that $C \notin A_1B_1$. Denote by H_1 the orthocentre of A_1B_1C and by D_1 the foot of the altitude from the vertex C. Since $\triangle A_1B_1C$ is obtuse, we get that $H_1 \neq C$. Consider $\triangle MD_1C$ and $\triangle CHH_1$ and use a). We are in case 2 of (a). For $\triangle ABC$ we obtain that $CH \cdot CM = \dfrac{c^2}{4} - CM^2$ and for $\triangle A_1B_1C$ that $CH_1 \cdot CD_1 = \dfrac{c^2}{4} - CM^2$. Therefore $\dfrac{CH}{CH_1} = \dfrac{CD_1}{CM}$, so $\triangle MD_1C \sim \triangle CHH_1$. Since $\triangle MD_1C$ is a right triangle, we know that $H_1H \perp CM$ and so H_1 lies on a line l through H which is perpendicular to CM. Let $H_1 \neq H$ be an arbitrary chosen point of l and let A_1B_1 be a diameter of (k) such that $AB \perp CH_1$.

Since H is an external point for (k) and $l \perp CM$ through H, it follows that all points of l are external for (k). In particular $H_1 \neq C$ and therefore A_1B_1 exists. The orthocentre of $\triangle A_1B_1C$ lies on CH_1

and it follows from the above that it lies on l. Since H_1 is the intersecting point of these two lines it is obvious that H_1 is the orthocentre of $\triangle A_1B_1C$.

Therefore the locus is a line l perpendicular to CM through a point H on the ray opposite to $\overrightarrow{CM}$ and of distance $\dfrac{c^2}{4|CM|} - |CM|$ from point C.

Consider case 3. Let AB be a diameter of (k) perpendicular to CM. The orthocentre H of $\triangle ABC$ lies on CM but now H is an internal point for (k). Consider an arbitrary diameter A_1B_1 of (k) such that $H \in A_1B_1$. Let H_1 be the orthocentre of $\triangle A_1B_1C$ and CD_1– the altitude. Consider $\triangle MD_1C$ and $\triangle CHH_1$. It is essential to show that $\triangle CHH_1$ is uniquely determined (for $\triangle MD_1C$ it is obvious). It suffices to prove that $H_1 \neq C$. If the contrary is true, then $\triangle A_1B_1C$ is a right triangle with its right angle at C and since A_1B_1 is a diameter, C must lie on (k), a contradiction. Since $\triangle ABC$ is acute, we apply case 1 of (a), so $CH \cdot CM = CM^2 - \dfrac{c^2}{4}$. There are three cases for $\triangle A_1B_1C$:

1. if it is an acute triangle, we apply case 1 of (a) and therefore $CH_1 \cdot CD_1 = CM^2 - \dfrac{c^2}{4}$;
2. if it is an obtuse triangle, we apply case 3 of (a) and we get the same equality (note that $\triangle A_1B_1C$ is not obtuse at C because C is an external point for (k));
3. if it is a right triangle (say at B_1) then $D_1 \equiv B_1 \equiv H_1$ and therefore $CH_1 \cdot CD_1 = CB_1^2 = CM^2 - \dfrac{c^2}{4}$.

The further considerations follow those from case 2. We conclude that the locus is a line perpendicular to CM and passing through $H \in CM^{\rightarrow}$ of distance $|CM| - \dfrac{c^2}{4|CM|}$ from C.

In particular it follows that $PQ \perp CM$, a well-known property.

Problem 10.3. How many natural numbers $\overline{a_1 a_2 \dots a_{2n}}$ exist such that:

(a) none of the digits a_i is zero;

(b) the sum $a_1 a_2 + a_3 a_4 + \dots + a_{2n-1} a_{2n}$ is an even number?

Solution: Denote the required number by A_n. The product $a_{2i-1}a_{2i}$ is even if at least one of the digits a_{2i-1} and a_{2i} is even. Therefore there are $5 \cdot 4 + 4 \cdot 5 + 4 \cdot 4 = 56$ choices for a_{2i-1} and a_{2i} such that $a_{2i-1}a_{2i}$ is even. Similarly, $a_{2i-1}a_{2i}$ is odd when both a_{2i-1} and a_{2i} are odd. There are $5 \cdot 5 = 25$ choices for a_{2i-1} and a_{2i} such that $a_{2i-1}a_{2i}$ is odd. The number of $\overline{a_1 a_2 \dots a_{2n}}$ such that i of the items $a_{2i-1}a_{2i}$ are odd is $\binom{n}{i} 25^i 56^{n-i}$. Therefore $A_n = \sum_i \binom{n}{2i} 25^{2i} 56^{n-2i}$. Let $B_n = \sum_i \binom{n}{2i+1} 25^{2i+1} 56^{n-2i-1}$. Obviously

$$A_n + B_n = \sum_{i=0}^{n} \binom{n}{i} 25^i 56^{n-i} = (56 + 25)^n = 81^n,$$

$$A_n - B_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} 25^i 56^{n-i} = (56 - 25)^n = 31^n.$$

Thus $2A_n = 81^n + 31^n$ and $A_n = (81^n + 31^n)/2$.

Problem 11.1. The sequence $\{x_n\}_{n=1}^{\infty}$ is defined as:

$$x_1 = 3, \quad x_{n+1} = x_n^2 - 3x_n + 4, \quad n = 1, 2, 3, \dots$$

(a) Prove that $\{x_n\}_{n=1}^{\infty}$ is monotone increasing and unbounded.

(b) Prove that the sequence $\{y_n\}_{n=1}^{\infty}$ defined as $y_n = \dfrac{1}{x_1 - 1} + \dfrac{1}{x_2 - 1} + \cdots + \dfrac{1}{x_n - 1}$, $n = 1, 2, 3, \dots$, is convergent and find its limit.

Solution: (a) $x_{n+1} - x_n = (x_n - 2)^2 \geq 0$ and so $\{x_n\}_{n=1}^{\infty}$ is a monotone increasing function. We now prove by induction that $x_n \geq n + 2$. Obviously this equality holds for $n = 1$. Suppose it is true for $n = k \geq 1$. Then

$$x_{k+1} = x_k(x_k - 3) + 4 \geq (k + 2)(k - 1) + 4 \geq k + 3.$$

Therefore $x_n \geq n + 2$ when $n = 1, 2, 3, \dots$ and so the sequence is unbounded.

(b) It follows from the recursive definition of our sequence that $x_{k+1} - 2 = (x_k - 1)(x_k - 2)$. Hence

$$\frac{1}{x_{k+1} - 2} = \frac{1}{(x_k - 1)(x_k - 2)} = \frac{1}{x_k - 2} - \frac{1}{x_k - 1},$$

so
$$\frac{1}{x_k - 1} = \frac{1}{x_k - 2} - \frac{1}{x_{k+1} - 2}.$$

By adding the above equalities for $k = 1, 2, \dots, n$ we get

$$y_n = \frac{1}{x_1 - 2} - \frac{1}{x_{n+1} - 2} = 1 - \frac{1}{x_{n+1} - 2}.$$

Since $0 \leq \dfrac{1}{x_{n+1}-2} \leq \dfrac{1}{n}$ it follows that $\lim\limits_{n\to\infty} \dfrac{1}{x_{n+1}-2} = 0$, so $\lim\limits_{n\to\infty} y_n = 1$.

Problem 11.2. Given $\triangle ABC$ such that $\angle ABC \geq 60\,\text{deg}$ and $\angle BAC \geq 60\,\text{deg}$. Let BL ($L \in AC$) be the internal bisector of $\angle ABC$ and $AH, H \in BC$ be the altitude from A. Find $\angle AHL$ if $\angle BLC = 3\angle AHL$.

Solution: Denote $\theta = \angle AHL$, $\alpha = \angle BAC$, $\beta = \angle ABC$ and $\gamma = \angle ACB$. From the Sine Law for $\triangle AHL$ and $\triangle CHL$ we get $\dfrac{\sin\theta}{AL} = \dfrac{\sin\angle ALH}{AH}$ and $\dfrac{\sin(90\,\text{deg} - \theta)}{CL} = \dfrac{\sin\angle CLH}{CH}$. Since $\dfrac{CL}{AL} = \dfrac{BC}{BA} = \dfrac{\sin\alpha}{\sin\gamma}$ and $\dfrac{AH}{CH} = \tan\gamma$ (from $\angle ACB < 90\,\text{deg}$ it follows that H lies on a ray $CB^{\to}$), we get

$$(1) \qquad \frac{\cos\theta}{\sin\theta} = \frac{\sin\alpha}{\cos\gamma}.$$

It follows from the conditions of the problem that $3\theta = \alpha + \dfrac{\beta}{2}$, so $6\theta - \alpha = \alpha + \beta$. Thus $\cos\gamma = -\cos(6\theta - \alpha)$ and (1) is equivalent to $\cos\theta\cos(6\theta - \alpha) + \sin\theta\sin\alpha = 0 \iff \cos(7\theta - \alpha) + \cos(5\theta - \alpha) + \cos(\theta - \alpha) - \cos(\theta + \alpha) = 0 \iff -\sin 4\theta \sin(3\theta - \alpha) + \cos(3\theta - \alpha)\cos 2\theta = 0 \iff \cos 2\theta(2\sin 2\theta \sin(3\theta - \alpha) - \cos(3\theta - \alpha)) = 0$. There are two cases to consider:

1. $\cos 2\theta = 0$. From $0 < \theta < 90\,\text{deg}$ it follows that $\theta = 45\,\text{deg}$. (Working backwards we get that $135\,\text{deg} = \angle BLC = 3\angle AHL$

for any $\triangle ABC$ such that $\alpha + \dfrac{\beta}{2} = 135\,\mathrm{deg}$).

2. $\cos 2\theta \neq 0$. It follows from $3\theta - \alpha = \dfrac{\beta}{2}$ that

$$2\sin 2\theta = \mathrm{cotg}\frac{\beta}{2}. \tag{2}$$

But $60\,\mathrm{deg} \leq \beta < 180\,\mathrm{deg}$ and so

$$\mathrm{cotg}\frac{\beta}{2} < \mathrm{cotg} 30\,\mathrm{deg} = \sqrt{3}. \tag{3}$$

On the other hand, also from the conditions of the problem we get $180\,\mathrm{deg} > 3\theta = \alpha + \dfrac{\beta}{2} \geq 90\,\mathrm{deg}$, so $30\,\mathrm{deg} \leq \theta < 60\,\mathrm{deg}$. Therefore

$$2\sin 2\theta \geq 2\sin 60\,\mathrm{deg} = \sqrt{3}. \tag{4}$$

The inequalities (3) and (4) show that (2) holds only if $\theta = 30\,\mathrm{deg}$. In this case $\alpha = \beta = \gamma = 60\,\mathrm{deg}$. (It is obviously true that $90\,\mathrm{deg} = \angle BLC = 3\angle AHL$ for any equilateral $\triangle ABC$).

Problem 11.3. Find all integer numbers $m, n \geq 2$ such that

$$\frac{1 + m^{3^n} + m^{2.3^n}}{n}$$

is integer.

Solution: Let m and n satisfy the conditions of the problem. Since n is an odd integer number, we get $(m, n) = 1$ and $n > 2$. When $n = 3$, all $m \geq 4$ such that $m \equiv 1 (\mathrm{mod}\ 3)$ are solutions, because if $m \equiv -1 (\mathrm{mod}\ 3)$, then $1 + m^{3^n} + m^{2 \cdot 3^n} \equiv 1 - 1 + 1 \equiv 1 (\mathrm{mod}\ 3)$. Let now $n > 3$. It follows that $m^{3^n} \not\equiv 1 (\mathrm{mod}\ n)$, because otherwise $1 + m^{3^n} + m^{2 \cdot 3^n} \equiv 3 (\mathrm{mod}\ n)$, i. e., $n/3$. On the other hand $1 + m^{3^n} + m^{2 \cdot 3^n} = \dfrac{m^{3^{n+1}} - 1}{m^{3^n} - 1}$ and therefore $m^{3^{n+1}} \equiv 1 (\mathrm{mod}\ n)$. Let k be the least natural number such that $m^k \equiv 1 (\mathrm{mod}\ n)$. Further, $k/3^{n+1}$ and $k \not/ 3^n$, so $k = 3^{n+1}$. Let $\varphi(n)$ be Euler's function. From $(m, n) = 1$ it follows that $m^{\varphi(n)} \equiv 1 (\mathrm{mod}\ n)$, so $k \leq \varphi(n)$. Therefore $3^{n+1} \leq \varphi(n) \leq n - 1$, which is impossible.

The required numbers are: $n = 3$ and all $m \geq 4$ such that $m \equiv 1 (\mathrm{mod}\ 3)$.