Winter mathematics competition—Pleven, 6–8 February 1998

Dedicated to the One Hundredth Anniversary of the UBM

Problem 8.1. Let three numbers a, b and c be chosen so that $\dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{a}$.

a.) Prove that $a = b = c$.

b.) Find the sum $x + y$ if $\dfrac{x}{3y} = \dfrac{y}{2x - 5y} = \dfrac{6x - 15y}{x}$ and the expression $-4x^2 + 36y - 8$ has its maximum value.

Solution:

a.) It is obvious that $a \neq 0$, $b \neq 0$, $c \neq 0$. The first equality gives $b^2 = ac$, whence by multiplying both sides by b we get $b^3 = abc$. Similarly $a^3 = abc$ and $c^3 = abc$. Hence $a^3 = b^3 = c^3$ and therefore $a = b = c$.

b.) By multiplying both the numerator and the denominator of the second fraction by 3 and using the result of a.) we obtain $x = 3y$. Thus $-4x^2 + 36y - 8 = -9(4y^2 - 4y + 1) + 1 = -9(2y - 1)^2 + 1$, and its maximum value is 1 when $2y - 1 = 0$. Therefore $y = \frac{1}{2}$ and $x = \frac{3}{2}$, i. e., $x + y = 2$.

Problem 8.2. In the acute triangle $\triangle ABC$ with $\angle BAC = 45\,\text{deg}$, BE ($E \in AC$) and CF ($F \in AB$) are altitudes. Let H, M and K be the orthocentre of ABC and the midpoints of BC and AH, respectively.

a.) Prove that the quadrangle $MEKF$ is a square.

b.) Prove that the diagonals of the quadrangle $MEKF$ intersect at the midpoint of OH, where O is the circumcentre of $\triangle ABC$.

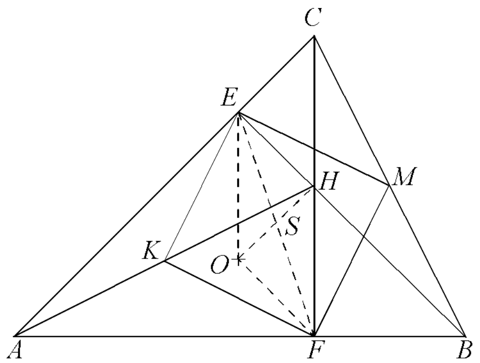
c.) Find the length of EF when the circumradius of $\triangle ABC$ is 1.

Solution:

a.) The segments EM and FM are medians to the hypotenuses of $\triangle BCE$ and $\triangle BCF$ and therefore $EM = FM = \frac{1}{2}BC$. Similarly, for $\triangle AHE$ and $\triangle AHF$ we get $EK = FK = \frac{1}{2}AH$. Since $\angle BAC = 45\,\text{deg}$, we find that $\triangle AEB$ and $\triangle CEH$ are isosceles. Hence $AE = BE$ and $EC = EH$, i.e., $\triangle AHE \cong \triangle BCE$. Therefore $EK = EM$. Thus $MEKF$ is a rhombus. Furthermore,

$$\begin{aligned} \angle MEK &= \angle MEB + \angle HEK = \angle CBE + \angle HEK \\ &= \angle EAH + \angle HEK = \angle EAH + \angle AHE = 90\,\text{deg}, \end{aligned}$$

i.e., the quadrangle is a square.



b.) It follows from a.) that the intersecting point S of the diagonals of the quadrangle $MEKF$ is the midpoint of both diagonals. Since $\triangle AEB$ is isosceles, E lies on the axis of symmetry of the segment AB and therefore $EO \perp AB$, i.e., $EO \| HF$. Similarly $FO \| EH$. Thus the quadrangle $EOFH$ is a parallelogram. From the above we conclude that S is the midpoint of OH.

c.) a.) implies that in the acute triangle $\triangle ABC$ with orthocentre H and $\angle BAC = 45\,\text{deg}$ it is true that $AH = BC$. $\triangle AFE$ is of the same type and therefore $EF = AO = 1$. (It follows from b.) that O is orthocentre of this triangle.)

Problem 8.3. Let 1998 points be chosen on the plane so that out of any 17 it is possible to choose 11 that lie inside a circle of diameter 1. Find the smallest number of circles of diameter 2 sufficient to cover all 1998 points.

(We say that a circle covers a certain number of points if all points lie inside the circle or on its outline.)

Solution: Consider a regular hexagon with a side of length 3. Choose 1998 points as follows: the 6 vertices of the hexagon and 1992 points inside a circle of diameter 1 centred at the centre of the hexagon. It is clear that the above 1998 points satisfy the condition of the problem. Moreover any circle of radius 1 covers at most one of the vertices of the hexagon. Therefore the required number is *no less than 7* (in our case: 6 circles for each vertex and a single circle for the remaining points).

Now we shall prove that the required number is *no greater than 7*. Arbitrarily choose 8 points and add other 9, for a total of 17. It is clear that there is a circle of diameter 1 covering at least 11 of these 17 points. At most 6 points lie outside the circle and therefore at least 2 of the initially chosen 8 points lie inside the circle. The distance between these two points is no greater than 1. We have proved that among any 8 points there always exist 2 such that the distance between them is no greater than 1.

Now choose a circle of radius 1 centred in one of the points. If the remaining points lie inside the circle, the required number is 1 and thus no greater than 7. If this is not the case, take another point outside the first circle. If all points lie in the two circles, then the required number is 2 and thus no greater than 7. Continuing in this way we either obtain no more than 7 circles covering all points or have 7 circles and a point that lies outside all circles. Consider this point and the centres of the chosen circles. There exist 2 points among these 8 such that the distance between them is no greater than 1. But this is impossible because of the way we chose our points.

Together the two parts of the proof demonstrate that the required number is 7.

Problem 9.1. Find all quadratic functions $f(x) = x^2 - ax + b$ with integer coefficients such that there exist distinct integer numbers m, n, p in the interval $[1, 9]$ for which $|f(m)| = |f(n)| = |f(p)| = 7$.

Solution: Let $f(x)$ be a function satisfying the conditions of the problem. Such a function cannot take one and the same value for three different arguments (otherwise we would have a quadratic equation having three distinct roots). Therefore two of the numbers $f(m)$, $f(n)$ and $f(p)$ equal 7 (or -7) and the third one equals -7 (or 7).

Case 1. $f(m) = f(n) = 7$, $f(p) = -7$. Without loss of generality we may assume that $m > n$. Since m, n are roots of $x^2 - ax + b - 7 = 0$, we obtain that $a = m + n$, $b = mn + 7$.

Subtracting the two equalities

$$\begin{aligned} m^2 - am + b &= 7 \\ p^2 - ap + b &= -7, \end{aligned}$$

we find

$$14 = m^2 - p^2 - a(m-p) = (m-p)(m+p-a) = (m-p)(p-n).$$

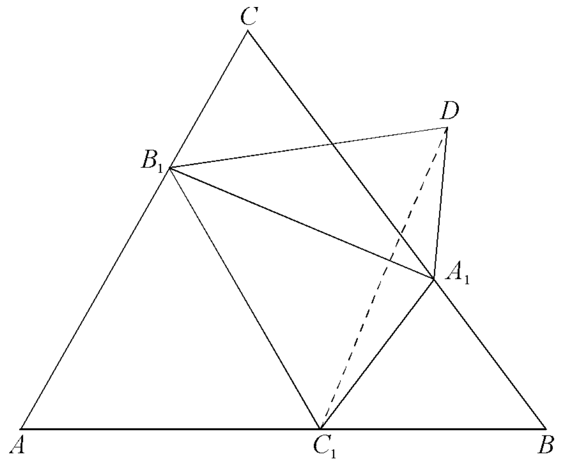
Thus the numbers $m-p$ and $p-n$ are either both positive or both negative and since $m > n$, they are positive. Moreover they are integer and therefore are equal to 1 and 14 or to 2 and 7. But since $m, n, p \in [1, 9]$ it follows that neither $m-p$ nor $p-n$ is 14. There are two cases to consider: $m-p=2$, $p-n=7$ and $m-p=7$, $p-n=2$, i. e., either $m=p+2$, $n=p-7$ or $m=p+7$, $n=p-2$. It is obvious that in both cases at least one of m, n, p lies outside the interval $[1, 9]$.

Case 2. $f(m) = f(n) = -7$, $f(p) = 7$. As in Case 1 we get $a = m+n$, $b = mn-7$ and $(m-p)(p-n) = -14$. Using similar arguments we obtain that either $m-p=2$, $p-n=-7$ or $m-p=-2$, $p-n=7$. (Without loss of generality we suppose that $|m-p| < |p-n|$.) Therefore the two options are $m=p+2$, $n=p+7$ and $m=p-2$, $n=p-7$. Simple calculations show that all triples (m, n, p) satisfying the conditions are $(3, 8, 1)$, $(4, 9, 2)$, $(6, 1, 8)$ and $(7, 2, 9)$. So the functions are $f(x) = x^2 - 11x + 17$, $f(x) = x^2 - 13x + 29$, $f(x) = x^2 - 7x - 1$ and $f(x) = x^2 - 9x + 7$.

Problem 9.2. Three points A_1, B_1 and C_1 lie on the sides BC, CA and AB of $\triangle ABC$ so that $AB_1 = C_1B_1$ and $BA_1 = C_1A_1$. Let D be the reflexion of C_1 in A_1B_1 ($D \neq C$). Prove that the line CD is perpendicular to the straight line through the circumcentres of $\triangle ABC$ and $\triangle A_1B_1C$.

Solution: It suffices to prove that D is the second intersecting point of the two circumcircles. We know that

$$\begin{aligned}\angle A_1DB_1 &= \angle A_1C_1B_1 = 180\deg - \angle BC_1A_1 - \angle AC_1B_1 \\ &= 180\deg - \angle C_1BA_1 - \angle C_1AB_1 = \angle A_1CB_1.\end{aligned}$$



On the other hand, $A_1D = A_1C_1 = A_1B$ and $B_1D = B_1C_1 = B_1A$, which shows that A_1 and B_1 are the circumcentres of $\triangle BC_1D$ and $\triangle AC_1D$. Therefore $\angle ADB = \angle ADC_1 + \angle BDC_1 = \frac{1}{2}\angle AB_1C_1 + \frac{1}{2}\angle BA_1C_1 = 90\deg - \angle C_1AB_1 + 90\deg - \angle C_1BA_1 = \angle ACB$. Since C and D lie in one and the same semiplane in regard to both A_1B_1 and AB, it follows from $\angle A_1DB_1 = \angle A_1CB_1$ and $\angle ADB = \angle ACB$ that D is the second intersecting point of the circumcircles of $\triangle A_1B_1C$ and $\triangle ABC$. This completes the proof.

Problem 9.3. All natural numbers from 1 to 1998 inclusive are written 9 times (so that there are 9 ones, 9 twos and so on) in the cells of a rectangular table with 9 rows and 1998 collumns, so that the difference between any two elements lying in one and the same column is no greater than 3. Find the maximum possible value of the smallest sum amongst all 1998 sums of the elements lying in one and the same column.

Solution: Consider the placement of the ones in the columns. If they are all in a single column, then the minimum sum of elements lying in one column is 9. Let all ones lie in exactly 2 columns. Therefore there are at least 5 ones in a single column and thus the minimal sum is no greater that $5 \cdot 1 + 4 \cdot 4 = 21$. If all ones are placed in exactly 3 columns, then the sum of all numbers in these three columns is at most $9 \cdot 1 + 9 \cdot 4 + 9 \cdot 3 = 72$. Hence the minimal sum is at most $72 : 3 = 24$. If all ones are placed in exactly 4 columns, then the sum of all numbers in these columns is at most $9 \cdot 1 + 9 \cdot 4 + 9 \cdot 3 + 9 \cdot 2 = 90$ and therefore the minimal sum is at most $90 : 4$, i. e., 22. It is impossible to have ones in more than 4 columns, because in that case the total number of 2s, 3s and 4s does not suffice to fill the remaining cells. Therefore the required sum is at most 24.

The following example shows that this sum can be 24, consequently the answer is 24:

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 7 & 8 & \ldots & 1998 \\
1 & 1 & 1 & 2 & 2 & 2 & 7 & 8 & \ldots & 1998 \\
1 & 1 & 1 & 2 & 2 & 2 & 7 & 8 & \ldots & 1998 \\
3 & 3 & 3 & 5 & 5 & 5 & 7 & 8 & \ldots & 1998 \\
3 & 3 & 3 & 5 & 5 & 5 & 7 & 8 & \ldots & 1998 \\
3 & 3 & 3 & 5 & 5 & 5 & 7 & 8 & \ldots & 1998 \\
4 & 4 & 4 & 6 & 6 & 6 & 7 & 8 & \ldots & 1998 \\
4 & 4 & 4 & 6 & 6 & 6 & 7 & 8 & \ldots & 1998 \\
4 & 4 & 4 & 6 & 6 & 6 & 7 & 8 & \ldots & 1998
\end{array}$$

Problem 10.1. Find all values of the real parameter a for which the equation $x^3-3x^2+(a^2+2)x-a^2=0$ has three distinct roots x_1, x_2 and x_3 such that $\sin\left(\frac{2\pi}{3}x_1\right)$, $\sin\left(\frac{2\pi}{3}x_2\right)$ and $\sin\left(\frac{2\pi}{3}x_3\right)$ form (in some order) an aritmetic progression.

Solution: Since $x^3-3x^2+(a^2+2)x-a^2=(x-1)(x^2-2x+a^2)$, in order for there to be three distinct real roots it is necessary that $D=1-a^2>0$. Therefore $a^2<1$ and thus $1\geq\sqrt{1-a^2}>0$. The roots of our equation are $x_1=1$, $x_2=1+\sqrt{1-a^2}$, $x_3=1-\sqrt{1-a^2}$. It follows now that $x_2+x_3=2$ and $2\geq x_2>1$ and $1>x_3\geq 0$.

There are two cases to consider:

1. The second term of the progression is $\sin\left(\frac{2\pi}{3}x_1\right)$. Then

$$\begin{aligned}
\sin\left(\frac{2\pi}{3}x_2\right)+\sin\left(\frac{2\pi}{3}x_3\right) &= 2\sin\left(\frac{2\pi}{3}\right)\\
2\sin\left(\frac{2\pi}{3}\left(\frac{x_2+x_3}{2}\right)\right)\cos\left(\frac{2\pi}{3}\left(\frac{x_2-x_3}{2}\right)\right) &= 2\sin\left(\frac{2\pi}{3}\right)\\
\cos\left(\frac{\pi}{3}(x_2-x_3)\right) &= 1.
\end{aligned}$$

But $\frac{\pi}{3}|x_2-x_3|=\frac{2\pi}{3}\sqrt{1-a^2}\leq\frac{2\pi}{3}$, and hence $\frac{\pi}{3}(x_2-x_3)\in[-\frac{2\pi}{3},\frac{2\pi}{3}]$. Therefore $\cos\left(\frac{\pi}{3}(x_2-x_3)\right)=1$ when $x_2=x_3$, which is impossible, since the roots are distinct.

2. The first or the third term of the progression is $\sin\left(\frac{2\pi}{3}x_1\right)$. Then

$$\sin\left(\frac{2\pi}{3}\right)+\sin\left(\frac{2\pi}{3}x_i\right)=2\sin\left(\frac{2\pi}{3}(2-x_i)\right)$$

for $i = 2$ or 3. Hence

$$\sin\frac{2\pi}{3} + \sin\left(\frac{2\pi}{3}x_i\right) = 2\sin\frac{4\pi}{3}\cos\left(\frac{2\pi}{3}x_i\right) - 2\cos\frac{4\pi}{3}\sin\left(\frac{2\pi}{3}x_i\right).$$

After simple calculations we get $\cos\left(\frac{2\pi}{3}x_i\right) = -\frac{1}{2}$. From the restrictions for x_2 and x_3 we obtain $x_i = 1$ or $x_i = 2$. In the first case $a^2 = 1$, which is impossible, and in the second case $x_2 = 2, x_3 = 0$ and $a^2 = 0$.

Thus a has a unique value and it is $a = 0$.

Problem 10.2. A point C lies on the periphery of a circle. Two points A and B are chosen anticlockwise away from C such that if $\angle CAB = \alpha$ and $\angle CBA = \beta$, the following equality holds:

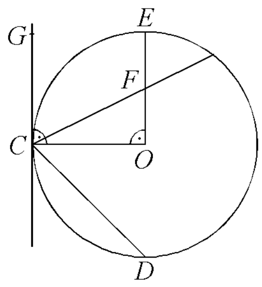
$$2\cos\left(\frac{\alpha}{2} + \beta\right) = \sin\left(\frac{\alpha}{2} - \beta\right).$$

Prove that the bisectors of $\angle CAB$ pass through a fixed point.

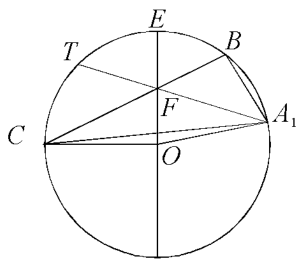
Solution: It is easy to see that $\alpha = 90\,\deg$, $\beta = 45\,\deg$ and $\tan\frac{\alpha}{2} = \frac{1}{2}$, $\beta = 90\,\deg$ satisfy the condition for α and β. Therefore the required point is the midpoint of OE–F (fig. 1).

From the premises of the problem we obtain

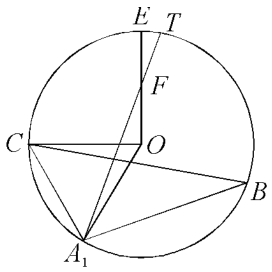
$$2\cos\frac{\alpha}{2}\cos\beta - 2\sin\frac{\alpha}{2}\sin\beta = \sin\frac{\alpha}{2}\cos\beta - \cos\frac{\alpha}{2}\sin\beta$$



черт. 1



черт. 2



черт. 3

and after dividing by $\cos\dfrac{\alpha}{2}\cos\beta$ ($\cos\dfrac{\alpha}{2} \neq 0$ (why?), and if $\cos\beta = 0$, i. e., $\beta = 90°$, we have one of the two cases already considered) we obtain

$$\tan\frac{\alpha}{2}\,(1 + 2\tan\beta) = 2 + \tan\beta$$

It follows in particular that if α is fixed, then β is uniquely determined.

Suppose that $\tan\dfrac{\alpha}{2} > 2$. Thus $\dfrac{\alpha}{2} > 45°$ and therefore $\alpha > 90°$. If $\tan\beta < 0$, we get $\beta > 90°$, which is impossible, since $\alpha + \beta < 180°$. If $\tan\beta > 0$, we get $2 + \tan\beta > (1 + 2\tan\beta)\,2$ i. e., $\tan\beta < 0$, which is a contradiction.

Therefore $\tan\dfrac{\alpha}{2} \leq 2$ and B lies on CED where $\angle GCF = \angle FCD$ and $\tan\angle GCF = 2$ (fig. 1).

Fix the point B such that $\alpha < 90°$. Let T be the midpoint of the arc CB and let A_1 be the intersecting point of TF and the circle. We shall show that $A_1 \equiv A$. We obtain $\angle OA_1F = \dfrac{\alpha}{2} - (90° - \beta) = \beta + \dfrac{\alpha}{2} - 90°$ and $\angle A_1FO = 45° - \dfrac{\alpha}{2} + \beta - 45° = \beta - \dfrac{\alpha}{2}$. It follows from the Sine Theorem for $\triangle A_1FO$ that $\dfrac{\sin\left(\beta-\frac{\alpha}{2}\right)}{\sin\left(\beta+\frac{\alpha}{2}-90^\circ\right)}=2$, which is equivalent to $2\cos\left(\frac{\alpha}{2}+\beta\right)=\sin\left(\frac{\alpha}{2}-\beta\right)$. Therefore $A_1\equiv A$.

The case of $\alpha>90^\circ$ can be dealt with by analogy. The condition for B to lie on CED shows that A_1 lies between C and B (fig. 3).

Problem 10.3. Let n be a natural number. Find the number of sequences $a_1a_2\dots a_{2n}$, where $a_i=+1$ or $a_i=-1$ for $i=1,2\dots,2n$, such that

$$\left|\sum_{i=2k-1}^{2l} a_i\right|\leq 2$$

for all k and l for which $1\leq k\leq l\leq n$.

Solution: It is clear that a sequence having $a_{2k-1}+a_{2k}=0$ for $1\leq k\leq n$ satisfies the condition of the problem, because any sum of the form $\sum_{i=2k-1}^{2l}a_i$ equals zero. There are 2^n such sequences. Let us determine the number of sequences such that there exists a k for which $a_{2k-1}+a_{2k}\neq 0$. Let $k_1,k_2,\dots,k_s$ be all k with the above property. It is easily seen that if $a_{2k_i-1}+a_{2k_i}=2$ (-2), then $a_{2k_{i+1}-1}+a_{2k_{i+1}}=-2$ (2). Therefore all sums $a_{2k_i-1}+a_{2k_i}$ (and so also $a_{2k_i-1}a_{2k_i}$) are uniquely determined by $a_{2k_1-1}+a_{2k_1}$ (there are two possibilities for $a_{2k_1-1}a_{2k_1}$). There are two possibilities for any of the remaining $n-s$ pairs (for which $a_{2t-1}+a_{2t}=0$). Therefore there are

$$2^n+2\cdot 2^{n-1}\binom{n}{1}+2\cdot 2^{n-2}\binom{n}{2}+\dots+2\cdot 2^{n-k}\binom{n}{k}+\dots+2\cdot 2\binom{n}{n-1}+2\cdot\binom{n}{n}.$$

sequences with the required property. By adding and subtracting 2^n to and from the above expression we get:

$$2.\left(2^n\binom{n}{0}+2^{n-1}\binom{n}{1}+2^{n-2}\binom{n}{2}+\cdots+2\binom{n}{n-1}+\binom{n}{n}\right)-2^n=2\cdot3^n-2^n$$

Thus there are $2\cdot 3^n - 2^n$ sequences.

Problem 11.1. Consider the function $f(x) = \sqrt{x} + \sqrt{x-4} - \sqrt{x-1} - \sqrt{x-3}$, $x \geq 4$.

a.) Find $\lim\limits_{x\to\infty} f(x)$.

b.) Prove that $f(x)$ is an increasing function.

c.) Find the number of real roots of the equation $f(x) = a\sqrt{\frac{x-3}{x}}$, where a is a real parameter.

Solution: a.) By grouping the first and third radicals and the second and fourth radicals and rationalising we get that when $x > 4$, $f(x) = \frac{1}{\sqrt{x}+\sqrt{x-1}} - \frac{1}{\sqrt{x-4}+\sqrt{x-3}}$. Therefore $\lim\limits_{x\to\infty} f(x) = 0$.

b.) When $x > 4$,

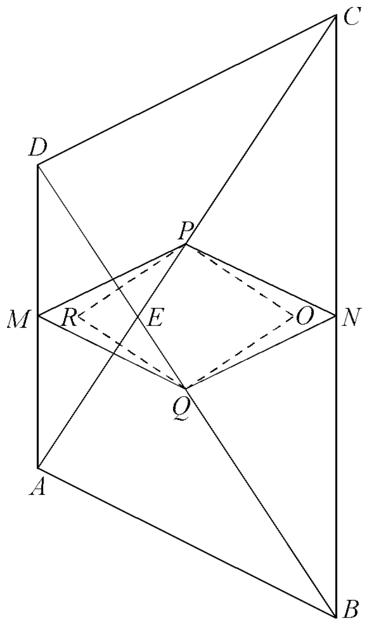
$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x-1}} - \frac{1}{2\sqrt{x-3}} + \frac{1}{2\sqrt{x-4}} \\ &= \frac{1}{2\sqrt{x-3}\sqrt{x-4}(\sqrt{x-3}+\sqrt{x-4})} + \\ &\quad - \frac{1}{2\sqrt{x}\sqrt{x-1}(\sqrt{x}+\sqrt{x-1})} > 0. \end{aligned}$$

Therefore $f(x)$ is an increasing function if $x \geq 4$.

c.) It follows from a.) and b.) that $f(x) < 0$ when $x \geq 4$, i. e., the equation could have a solution only if $a < 0$. Let $a < 0$. The function $g(x) = a\sqrt{\frac{x-3}{x}} = a\sqrt{1 - \frac{3}{x}}$ is decreasing and continuous and $\lim\limits_{x\to\infty} g(x) = a < 0$. Since $f(x)$ is an increasing and continuous function and $\lim\limits_{x\to\infty} f(x) = 0$, in accordance with the Bolzano–Weierstraß Theorem the equation $f(x) = g(x)$ has a solution (and it is a unique one) exactly when $f(4) \leq g(4)$, i. e., if $2(1 - \sqrt{3}) \leq a < 0$.

Problem 11.2. The convex quadrangle $ABCD$ is inscribed in a circle with centre O. Let E be the intersecting point of AC and BD. Prove that if the midpoints of AD, BC and OE lie on a straight line, then $AB = CD$ or $\angle AEB = 90$ deg.

Solution: It suffices to prove that if $\angle AEB \neq 90$ deg, then $AB = CD$. Let $\angle AEB \neq 90$ deg. If $O \equiv E$, then $ABCD$ is a rectangle and therefore $AB = CD$. Suppose $O \not\equiv E$. Let M, N, P, Q be the midpoints of AD, BC, AC, BD, respectively, and R be the intersecting point of the straight lines through P and Q perpendicular to BD and AC, respectively. It is clear that $MPNQ$ and $OPRQ$ are parallelograms. Therefore the midpoints of MN and OR coincide with the midpoint of PQ, and since the midpoint of OE lie on MN, we get that $RE \| MN$. On the other hand R is the orthocentre of $\triangle PQE$ and therefore $RE \perp PQ$. Hence $MN \perp PQ$, i. e., the parallelogram $MPNQ$ is a rhombus. It is easy to see now that $AB = 2PN = 2NQ = CD$, which solves the problem.



Note: The above solution shows that if O is the intersecting point of the axes of symmetry of AC and BD, then the assertion of the problem and its opposite are true for a quadrangle that is not inscribed in a circle. This could be demonstrated by using complex numbers or trigonometry.

Problem 11.3. Let $\{a_m\}_{m=1}^{\infty}$ be a sequence of integer numbers such that their decimal representations consist of even digits ($a_1 = 2$, $a_2 = 4$, $a_3 = 6, \ldots$). Find all integer numbers m such that $a_m = 12m$.

Solution: Let m be an integer number such that $m = b_0 + b_1 \cdot 5 + \cdots + b_n \cdot 5^n$. Denote $f(m) = 2b_0 + 2b_1 \cdot 10 + \cdots + 2b_n \cdot 10^n$. It is clear that $\{f(m) \mid m \in \mathbb{N}\}$ is the set of integer numbers with only even digits in their decimal representation. Since $f(m_1) < f(m_2) \iff m_1 < m_2$, it follows that $a_m = f(m)$ for any m. Therefore it suffices to find all m such that

$$12(b_0 + b_1 \cdot 5 + \cdots + b_n \cdot 5^n) = 2b_0 + 2b_1 \cdot 10 + \cdots + 2b_n \cdot 10^n,$$

i. e.,

$$(1) \qquad 6(b_0 + b_1 \cdot 5 + \cdots + b_n \cdot 5^n) = b_0 + b_1 \cdot 10 + \cdots + b_n \cdot 10^n.$$

Since $b_0+b_1\cdot 5+\cdots+b_n\cdot 5^n \leq 5^{n+1}-1$ and $b_0+b_1\cdot 10+\cdots+b_n\cdot 10^n \geq 10^n$, it follows from (1) that $6(5^{n+1}-1) \geq 10^n$, i. e., $6 \cdot 5^{n+1} > 10^n$. Thus $2^n < 30$ and therefore $n \leq 4$. If $n = 4$, we get from (1) that $b_0 + 4b_1 + 10b_2 = 50b_3 + 1250b_4 \geq 1250$, which is imposible. In the same way it is easy to show that $n \geq 3$, i. e., $n = 3$. In this case $b_0+4b_1+10b_2 = 50b_3$. Obviously $b_3 = 1$ and $b_0 = b_1$, because b_0-b_1 is divisible by 5. As a result we have the equation $b_0+2b_2 = 10$, and its solutions are $b_0 = 2, b_2 = 4$ and $b_0 = 4, b_2 = 3$. Therefore all integer numbers m with the required property are $m = 2+2\cdot 5+4\cdot 5^2+5^3 = 237$ and $m = 4 + 4 \cdot 5 + 3 \cdot 5^2 + 5^3 = 224$.