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Complex Analysis and Applications '13
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Preface

ACADEMICIAN LJUBOMIR ILIEV AND THE DAY OF NATIONAL LEADERS

Virginia Kiryakova

Academician Ljubomir Iliev was born on April 20, 1913. *The centenary of his birthday* was celebrated partly by the Bulgarian mathematical community yet in April, in the frames of the traditional Spring Conference of the Union of Bulgarian Mathematicians. However, the section "Analysis, Geometry and Topology" as a successor of the "Complex Analysis" section founded by him, chose to host this memorial conference at the Institute (whose building of 1972 is considered to a great extent as his personal achievement) and to open it exactly on *November 1, the Bulgarian Day of National Leaders.*

The Day of National Leaders, called also *National Revival Day*, is a Bulgarian national holiday celebrated each year on *November 1.* It is to honor the leaders of the National Revival period, the Bulgarian educators, revolutionaries, spiritual mentors and scholars. A ritual raising of the national flag and official change of guards happen in front of the main entrance of the Presidential Administration, together with festival events, parades and torchlight processions organized by the universities, scientific institutions, religious and spiritual centers over the country. It is a festival of the historical memory and of our national self-confidence standing for year after year during the centuries of slavery, violence and people's suffering under foreign oppression, led and supported morally by these great men and women.

The Day of National Leaders arose in the difficult time of spiritual ruin after the First World War. The Bulgarian society collapsed the Renaissance ideals. For many, it was clear the real threat of disintegration of our national values. At that time Bulgarians chose the experience of their society, and stared at the great leaders of the Bulgarian spiritual past to find the way back to their equilibrium and stability as a nation. For the first time, this holiday was celebrated in the town of Plovdiv in 1909. Until then and nowadays, the date of November 1 is celebrated (in calendar's old style) as *the Day of St. Ivan (John) Rilski* (also, a commemoration of All Saints, or known as All Hallows' Day in many countries throughout the world). He is honored as the *patron-saint of the Bulgarian people and as one of the most important saints in the Bulgarian Orthodox Church.*

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In 1922, the National Assembly declared this holiday for all "deserving Bulgarians": "Let the day of St. John of Rila be a Day of National Leaders in celebration of the greatest Bulgarians, to awake in young people the good sense of existence and interest towards the figures of our past...". Since 1945 these celebrations have been temporarily interrupted and afterwards, revived by an act adopted by the 36th National Assembly in 1992, to resume the tradition of the feast. *But yet since 1991 the Union of Scientists in Bulgaria has adopted the National Leaders Day also as a Day of Bulgarian Science*! This Union is a co-organizer of our memorial Conference and we like to acknowledge their sponsorship.

Among the most popular Bulgarian national leaders are St. Ivan (John) of Rila, St. Paisius of Hilendar (Paisij Hilendarski), Sofronij Vrachanski, Gregory Tsamblak, Konstantin Kostenechki, Vladislav the Grammarian, Matthew the Grammarian, Neophyte Bozveli, brothers Dimitar and Konstantin Miladinovi, Georgi Sava Rakovsky, Vasil Levski, Hristo Botev, Stefan Karadza, Hadji Dimitar, Ljuben Karavelov, Dobri Chintulov, Ivan Vazov, and many others. *And among them, the name of Academician Ljubomir Iliev finds place without any doubt*!

He was born in the *town of Veliko Tarnovo* (with a meaning "Great" Tarnovo) referred also as the "City of the Tsars" (Emperors) and being the historical capital of the Second Bulgarian Empire (1185 – 1396). In the Middle Ages, the city was among the main European centres of culture and gave its name to the architecture of the Tarnovo Artistic School, painting of the Tarnovo Artistic School and literature; a quasi-cosmopolitan city, with many foreign merchants and envoys (incl. Armenians, Jewishes and Roman Catholics) besides a dominant Bulgarian population. May be, this town's origin was one of the reasons for *Iliev to feel himself not only as a Bulgarian* (proving by all his life and contributions to be a "deserving" one), *but also a citizen of the World.* He graduated the male secondary school there, then finished his mathematical education in 1936 at the Physics-Mathematics Department of Sofia University. L. Iliev had been a teacher in a Sofia secondary school, worked as Assistant Prof. (1941), Associate Prof. (1947), Full Prof. (1952) and head of "Advanced Analysis" Dept. (since 1952) at the University. Obtained his PhD degree in 1938, and Dr.Sc. - in 1958, became a Corresponding Member of Bulgarian Academy of Sciences (BAS) in 1958, and Member of Academy since 1967. He was the Director of our Institute (1964-1988); General scientific secretary for a long time and Vice-president of BAS (1968-1973); Vice-Rector of the Sofia University; long-term President of the Union of Scientists in Bulgaria and of the Union of Bulgarian Mathematicians; etc.

Among the topics of his scientific interests were: analytic and entire functions theory, zeroes of polynomials, univalent functions, analytical non-expendability of series, methodology of science and education, development of computer science, etc. He is author of a great number of mathematical papers, monographs and university textbooks, see List of his publications in this volume.

The scientific contributions and the impressive international activities of Acad. Iliev gave him a series of worldwide recognitions, among them – foreign memberships of Soviet (now Russian) Academy of Sciences, of the German Academy (then of German Democratic Republic), of the Hungarian Academy; Doctor Honoris Causa of Technical University - Drezden; President of the Council of the International Mathematical Center "St. Banach" - Warsaw (1974-1977); Chair of Balkan Mathematical Union (now inherited by MASSEE), Vice-President of International Federation for Information Processing; etc, etc.

But let us stress in this note on some of *his contributions to the Bulgarian culture, education and science as a human consciously being Bulgarian and belonging to Bulgaria*. Enormous are his activities and achievements in favor of the *modern education in Mathematics and Informatics in our country*, both in schools and universities. Just to mention the creation of the specialized mathematical high schools, the special attention arranged to the talented pupils, the introduction of the 3-cycles of higher education qualifications (Bachelor - Magister - Ph.D.) in Sofia University yet in 1970, long time before other European universities introduced it (as the so-called Bologna process, since 1999), etc.

It is common to speak about *"Iliev's era" in the development of Bulgarian mathematics*. Among many other ideas and achievements, he devoted much efforts to his goal to develop a wide range of research topics in the Institute, as wide as to cover almost all items in Mathematics Subject Classification. It was an era when this Institute (named either Mathematical Institute with Computing Centre, or Institute of Mathematics and Mechanics, or enlarged as United Centre for Mathematics and Mechanics including also the corresponding Faculty of Sofia University) incorporated several departments with more than 500 scholars in all areas of pure and applied mathematics, mechanics and computer sciences.

Iliev had his *leading role in introducing the Computer Science and Computational Technique in Bulgaria*, developed in frames of Mathematical Science. The first Bulgarian Computing Centre (1961), the first Bulgarian Computer "Vitosha" (1963), the Bulgarian calculator "Elka" (1965) - the 4th electronic calculator in the world (after the British, Italian and Japanese, and the first one (!) executing square root function) – are all projects initiated and established under his close guidance, sometimes with the risque he took on himself *to argue with the political officials*, as "Cybernetics" has been considered then as a wrong Western influence. Along with construction of the hardware, Acad.

Iliev's project included a well planned care to create the necessary scholars for the Informatics era, introducing courses in numerical mathematics and mathematical programming, arranging scholarships and PhD studies abroad for the "new people" necessary for the "new" era of Information Technologies.

Most of *Iliev's power was hidden in his organizational abilities and activities.* He was a real leader of the mathematical life in Bulgaria, and did this with a gift from nature and endless enthusiasm. Just to mention his policy *to organize in Bulgaria a series of big international mathematical congresses, followed by more specialized (topical) international conferences*, where in the time of the Iron curtain, mathematicians from both Eastern and Western countries could come to Bulgaria to meet and exchange ideas and experience. This was a real phenomenon, not commonly possible in that era. Thus, the Bulgarian mathematics, the country's culture, the old history and the state itself, were not only opened to the World but could serve as a *bridge* between its two parts.

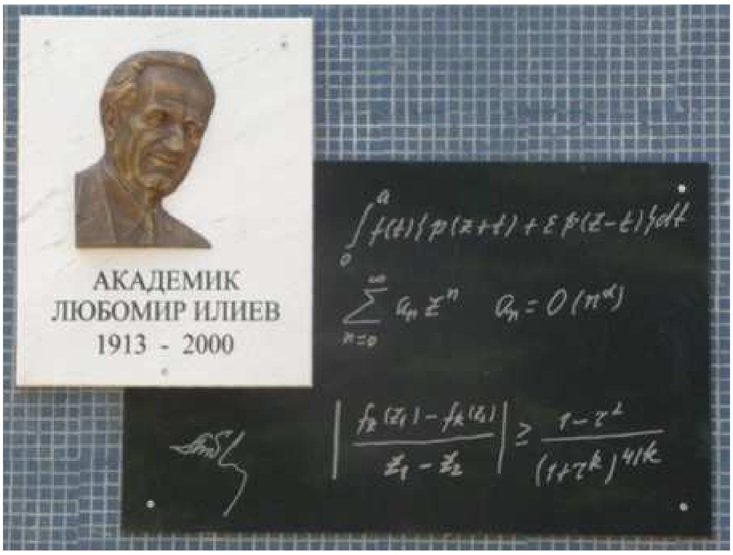
Among the *hobbies of Acad. Iliev related to Bulgarian history and culture*, it is worth to mention that he was a devoted numismatist. This was not only to collect coins, but to touch sources of authentic information on our history and culture, a possibility to develop his own theories and hypotheses. Another trend of dedication to the Bulgarian history and culture, was his initiative to name mathematical journals and books series founded by him, after purely Bulgarian words from history, having nothing in common (at first sight) with their mathematical contents. Such are the Bulgarian Mathematical Journal "Serdica", the Proceedings "Pliska" (both after old Bulgarian capitals), the series of mathematical monographs "Az Buki" (with the meaning: the first 2 letters of the Bulgarian alphabet), etc.

Finally, let us say few words of acknowledgements to Acad. L. Iliev of behalf of our section, as he founded it and was its first Head (during 1962-1988). *The "Complex Analysis" (CA) section* was one of the first departments to form the structure of Institute of Mathematics and Informatics (IMI) at Bulgarian Academy of Sciences (BAS) with clearly specialized subjects and serious scientific potential. It was among the departments that inherited the department "Advanced Analysis", directed by Acad. L. Tchakalov until 1962. In 1962 the departments "Complex Analysis" (with a head Corr. Member of BAS (then) L. Iliev), "Real and Functional Analysis" and "Differential Equations" were formed. From the beginning of its independent existence, the department "CA" achieved significant development both thematically and staff wise. Along with the traditional topics from the classical function theory - geometric function theory, distribution of zeroes of entire and meromorphic functions - new trends as several complex variables functions, complex geometry, special functions, integral transforms, fractional and operational calculi found place. Since 2010, Section "CA" joined with the former IMI sections "Real and Functional Analysis" and "Geometry and Topology", and is presently named as a new section "Analysis, Geometry and Topology" (AGT) – the Organizer of this memorial Conference. Several scientific groups work now in the following directions: Functions of One Complex Variable; Functions of Several Complex Variables and Complex Geometry; Transform Methods, Special Functions, Fractional and Operational Calculi; Geometry and Topology, etc. Currently, the section "AGT" consists of more than 20 members, being one of the most numerous departments at IMI.

Under the initiation and guidance of Academician L. Iliev, our department organized *the series of international conferences "Complex Analysis and Applications"* in the town of Varna (on Black Sea), held in 1981, 1983, 1985, 1987, 1991, where a great number of foreign and Bulgarian mathematicians took part. Thus the name of the present meeting has been chosen as a re-make of these conferences and to commemorate the role of their chairman. Here is to mention that Acad. Iliev had remarkable skills in organization of scientific events in local and international aspects. But he also left traditions and devotedly taught us and our colleagues how such congresses and conferences should be organized.

And if you have been satisfied by the organization of this "CAA '13" meeting, our aim has been to make it a worthy analytical continuation (although in smaller scale of participants) of the previous "CAA" conferences ...

Chair of International Program Committee of "CAA '13"
and on behalf of the "AGT" section –
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
"Acad. G. Bontchev" Str., Block 8, Sofia – 1113, BULGARIA



Haut-relief of Acad. Ljubomir Iliev
opened Nov. 1, 2013 on the entrance facade of Institute

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ACADEMICIAN LJUBOMIR ILIEV – LEADER OF THE BULGARIAN MATHEMATICAL COMMUNITY

(ON THE OCCASION OF HIS CENTENARY)

Blagovest Sendov

Abstract

As his student and close collaborator for many years, I have had the opportunity on many occasions to speak and write about the rich and fruitful activities of Academician Ljubomir Iliev, a leading Bulgarian mathematician and a leader of the Bulgarian mathematical community, for more details see for example [1] - [4].

On the occasion of his centenary, it is natural to try to evaluate his achievements in perspective. Among the Bulgarian mathematicians, active during the middle of the last century, Ljubomir Iliev takes the place after Nikola Obreshkoff and Ljubomir Tschakaloff. His results in complex analysis, namely on schlicht (univalent) functions, analytic inextensibility and overconvergence of series, distributions of the zeros of polynomials and entire functions, having integral representation, the inequality of Poul Turan and others, are part of the contemporary mathematics and are cited by today's researchers. All this is enough to call Academician Ljubomir Iliev one of the leading Bulgarian mathematicians in all history.

MSC 2010: 01A60, 01A70, 97-XX, 68-03, 30-XX

Key Words and Phrases: history of mathematics - 20th century; functions of a complex variable; secondary schools and university education in mathematics and computer science; development of mathematics, computer science, information processing, electronics and computer industry in Bulgaria; electronic calculator; first Bulgarian computer

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This is really true and a generally accepted fact, but Ljubomir Iliev was not only a leading Bulgarian mathematician, we may name among many others. What is very important and specific is, that Academician Ljubomir Iliev was an outstanding leader of the Bulgarian mathematical community. He had the vision and the ambition to work and organize, during his long active live, all

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avenues of the mathematical developments in Bulgaria. Today, after the radical political and economic changes in our country, not everything achieved in the past is evaluated without personal emotions. We shall try to present several things connected with the name of Ljubomir Iliev, which are not disputable.

His first important contribution is in the development of Bulgarian mathematical education at all levels. As a Vice Rector of the University of Sofia “St. Kliment Ohridski”, Ljubomir Iliev made a very important step for improving the quality of higher education in mathematics for preparing professional mathematicians and researchers. In 1950, in the Physics and Mathematics Faculty of the University were formed special groups for professional mathematicians, which were in fact a magistrature. In addition, with the decisive help of Ljubomir Iliev, 30 secondary schools specializing in mathematics were opened all over the country and one National mathematical gymnasium was opened in Sofia. Up to now, these specialized mathematical schools have played an extremely important role in maintaining the quality of secondary education. There have been many attempts to close these elite schools, or transform them but, they always manage to survive.

A very fruitful activity of Ljubomir Iliev is associated with the Bulgarian Mathematical Union. He helped, together with many other mathematicians and physicists, with the re-establishment of the Bulgarian Physics and Mathematics Society and the separation, afterwards into two unions. A unique characteristic of the Bulgarian Mathematical Union, inspired by the tradition and the leadership of Ljubomir Iliev is the unity of all Bulgarian mathematicians in a single professional organization, combining teachers in secondary schools and those working in the universities and in the Bulgarian Academy of Sciences. A demonstration of this unity is shown in the traditional Spring Mathematical Conferences, where students, teachers and researchers meet together every year in April.

Academician Ljubomir Iliev was the initiator and supporter of the development of informatics and computer science in Bulgaria. He started by proposing in 1959 the creation of the first course in numerical analysis, and after 1961, many other courses, which prepared the first specialists in applied mathematics and programming in Bulgaria.

One of the main projects of Ljubomir Iliev was the establishment of the First Bulgarian computer center, created in 1962 jointly under the Bulgarian Academy of Sciences and the University of Sofia “St. Kliment Ohridski”. It took a tremendous efforts and organizational talent to select a group of engineers, mathematicians an technicians to build the first Bulgarian digital electronic computer, which became operational in 1963. These pioneering efforts turned out to be the basis for the development of the Bulgarian electronics and computer industry. It is mostly forgotten today, that in the First Bulgarian computer center was designed and built one of the first electronic calculators in the world called “Elka”. This electronic calculator was advertised in The Financial Times and sold in Great Britain in 1968. “Elka” was the step to the popular in the former communist countries electronic computer “Pravetz”, produced in the electronic factories in the Bulgarian town Pravetz.

As a pioneer in the development of information processing, Ljubomir Iliev believed that eventually, Informatics will become a natural part of Mathematics. He was very active in international cooperation and became Vice President of the International Federation for Information Processing (IFIP). The Bulgarian membership in IFIP was a very good opportunity for many Bulgarian specialists to be in contact with the world leaders in the field.

As the General scientific secretary of the Bulgarian Academy of Sciences and for long time the Vice President of the Academy and Director of the Mathematical Institute (nowadays, Institute of Mathematics and Informatics - Bulg. Acad. Sci.), Ljubomir Iliev used all his influence for the benefit of the Bulgarian mathematical community. First of all, he introduced a concrete plan for the structure of the Mathematical Institute based on a theory for the structure of mathematical science itself. His ambition was to open opportunities in our country for the development of all branches of contemporary mathematics. Ljubomir Iliev defended the necessity to build a separate building for a big Institute of Mathematics and Informatics and succeeded in finishing this building. It is just a recognition of these efforts, that on the occasion of his centenary, a memorial relief of Ljubomir Iliev is on the front wall of this building.

Academician Ljubomir Iliev was a devoted patriot. In the center of the motivation for every one of his projects was the benefit for his country. He used every opportunity to show that Bulgaria is a country with a rich culture, long history, talented people and a prosperous future. He cared especially for the young mathematicians, who show capacity for research and leadership. One of his popular formulas was: “For the new fields of research - new young people”. Even in the period of the Cold War, Ljubomir Iliev was trying to fulfill his principle about the specialization of young scientists in Bulgaria: “Every young scientist has to have at least one specialization in the East and one in the West”. A leader is a leader, not because he is on top, but because he cares for the people he leads.

All the activities of Ljubomir Iliev as the leader of the Bulgarian mathematical community took place during the so-called totalitarian period. The big political and economic changes and the complete democratization of the country is accompanied by emotional criticism of almost everything created during the totalitarian regime. Nevertheless, everything done by Ljubomir Iliev is for the benefit of Bulgaria and its value is invariant under every political transformation.

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ACADEMICIAN LJUBOMIR ILIEV AND THE CLASSICAL COMPLEX ANALYSIS

Peter Rusev

Abstract

Academician Ljubomir Georgiev Iliev was born in 1913, 25 years after the establishment of the first Bulgarian institution of higher education and 10 years after its renaming as University. Next year the future academicians Kyril Popov, Ivan Tzenov and Ljubomir Tchakalov attained academic ranks as associate professors at the Faculty of Physics and Mathematics (of the Sofia University). They were successors and followers of the work of the pioneers of the higher education in mathematics in Bulgaria Emanuil Ivanov, Marin Batchevarov, Atanas Tinterov, Spiridon Ganev and Anton Shourek. Eight years later the future academician Nikola Obrechkov became also their colleague.

Popov and Tchakalov, who had defended their theses at European Universities, transferred the spirit of cultivating the mathematical science from the most prestigious scientific centers. The seminars under their guidance with the active participation of Obrechkov, being still a student, became "incubators" of young enthusiasts – future teachers at the Sofia University and at the arising Institutions of higher technical education. Among them, was Ljubomir Iliev, one of the most talented of their followers, one of the most brilliant from the third generation of Bulgarian mathematicians.

On the occasion of the 100th anniversary of the birth of Ljubomir Iliev and this current remake of the international conferences "Complex Analysis and Applications" (held in Varna, 1981, 1983, 1985, 1987) that were initiated and organized under his guidance, we try to present a short survey of some of his contributions to topics of the classical complex analysis.

MSC 2010: 30-XX; 30-03; 30B40, 30B50; 30C15; 30C45

Key Words and Phrases: functions of one complex variable; zeros of entire functions; analytical non-continuable power and Dirichlet series; classes of univalent functions

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1. Zeros of entire Fourier transforms

In 1737, L. Euler defined the function ζ by means of the equality

$$\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}}, \quad \sigma > 1,$$

and pointed out the validity of the representation

$$\zeta(\sigma) = \prod_{p \in P} \left(1 - \frac{1}{p^{\sigma}}\right)^{-1},$$

where P is the set of prime numbers.

In his memoir *Über die Anzahl der Primzahlen unter einer gegebenen Grösse, Monatsber. der Konigl. Preuss. Akad. der Wiss. zu Berlin aus dem Jahr* 1859 (1860), 671–680, B. Riemann extended Euler's definition by the equality

$$\zeta(s) = \sum_{n=1}^{\infty} \exp\,(-s\ \log n), \quad s = \sigma + it,\ \sigma > 1,\ t \in \mathbb{R}.$$

The functional equation

$$\pi^{-s/2}\,\Gamma(s)\zeta(s) = \pi^{(1-s)}\,\Gamma(1-s)\zeta(1-s), \tag{1}$$

bearing his name, realizes analytical continuation of this function in the whole complex plane as a meromorphic function with a single pole at the point $s = 1$. From (1) it follows, in particular, that the points $-2k, k \in \mathbb{N}$ are its simple zeros.

In the same memoir, Riemann stated the famous hypothesis which is neither proved nor rejected till now, namely that the function ζ, except this zeros named trivial, has infinitely many others and that all they are on the line Re $s = 1/2$. It is equivalent to the hypothesis that the introduced by him entire function

$$\xi(z) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \quad s = 1/2 + z,$$

has only real zeros.

At the end of the 19^{th} century but mostly in the first decades of 20^{th} one, the efforts of many mathematicians including first-class ones as Jensen, Pólya, Hardy and Titchmarsh turn to the problem of zero-distribution of entire functions defined as Fourier transforms of the kind

$$\int_a^b F(t)\exp(izt)\,dt, \quad -\infty \le a < b \le \infty.$$

Indisputable motive for their investigations is Riemann's representation of the function ξ in this form with an even function F on the interval $(-\infty, \infty)$. The first Bulgarian publications in this field, due to Tchakalov and Obrechkov, are influenced by the results of Pólya about the zero-distribution of the entire functions of the kind

$$\int_{-a}^{a} f(t)\exp(izt)\,dt, \quad 0 < a < \infty \tag{2}$$

and of their particular cases

$$\int_{0}^{a} f(t)\cos zt\,dt \tag{3}$$

and

$$\int_{0}^{a} f(t)\sin zt\,dt. \tag{4}$$

This direction becomes a field of intensive studies of academician Iliev.

An essential role in Pólya's investigations plays an algebraic statement, most frequently called theorem of Kakeya, saying that if $a_0 < a_1 < \cdots < a_n, n \in \mathbb{N}$, then the zeros of the polynomials

$$\sum_{k=0}^{n} a_k z^k$$

are in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$. Using it and applying the method of variation of the argument, Pólya obtains his famous result for reality and mutually interlacing of the zeros of the entire functions (3) and (4), provided the function f is positive and increasing in the interval $(0, a)$.

Another approach to the problem of zero-distribution of entire functions of the kind (3) and (4) is due to academician Iliev. It is based on his result that if the zeros of the algebraic polynomial P of degree $n \in \mathbb{N}$ are in the region $\{z \in \mathbb{C} : |z| > 1\}$ and P^* is the polynomial defined by $P^*(z) = z^n\overline{P(1/\overline{z})}$, then the zeros of the polynomial

$$P(z) + \gamma z^k P^*(z), \quad |\gamma| = 1,\ k \in \mathbb{N}_0,$$

are on the unit circle. This assertion, as well as the successful use of an algebraic result of N. Obrechkov, lead to one of the most essential achievements of academician Iliev. It says that if the function f is positive and increasing in the interval $(0, a), 0 < a < \infty$, and the zeros of the algebraic polynomial p are in the strip $\{z \in \mathbb{C} : \lambda \le \text{Re } z \le \mu\}$, then the zeros of the polynomial

$$\int_{0}^{a} f(t)\,\{p(z+t) + \gamma p(z-t)\}\,dt, \quad |\gamma| = 1, \tag{5}$$

are in the same strip too. The classical results of Pólya can be obtained by setting $p(z) = z^n$, $\gamma = \pm 1$, and letting n to go to infinity. Indeed, then the polynomials

$$P_n(f;z) = \int_0^a f(t) \left\{ \left(1 + \frac{izt}{n}\right)^n + \gamma \left(1 - \frac{izt}{n}\right)^n \right\} dt, \quad n \in \mathbb{N},$$

have only real zeros and, moreover,

$$\lim_{n\to\infty} P_n(f;z) = \int_0^a f(t) \left\{\exp(izt) + \gamma \exp(-izt)\right\} dt$$

uniformly on each bounded subset of $\mathbb{C}$.

A brilliant realization of one of the most fruitful ideas of academician Iliev concerns the class $E(a)$ of entire functions (3) having only real zeros. If $A(a)$, $0 < a < \infty$, denotes the set of the real functions $x(t), t \in \mathbb{R}$, such that $x(a) = 0$ and, moreover, $x'(it), t \in \mathbb{R}$, is a restriction to the real axes of a function of the Laguerre-Pólya class, i.e. it is either a real polynomial with only real zeros or an uniform limit of such polynomials. A witty algorithm ensures "reproduction" of this class. Its first application is that if $x(t) \in A(a), x(0) > 0$, and $\lambda > -1$, then the entire function

$$\int_0^a x^\lambda(t) \cos zt \, dt$$

has only real zeros. The particular case when $x(t) = 1 - t^{2q}, q \in \mathbb{N}$, leads to a result of Pólya saying that the entire function

$$\int_0^1 (1 - t^{2q}) \cos zt \, dt$$

has only real zeros.

The next application is one of the most significant achievements of academician Iliev which states that if $\varphi(t), t \in \mathbb{R}$, is a real, nonnegative, and even function, such that $\varphi'(it)$ is a restriction to the real axes of a function from the Laguerre-Pólya class, then the entire function

$$\int_0^\infty \exp(-\varphi(t)) \cos zt \, dt$$

has only real zeros. The particular case when $\varphi(t) = a \cosh t, a > 0$, is the well-known result of Pólya for the reality of the zeros of the entire function

$$\int_0^\infty \exp(-a \cosh t) \cos zt \, dt.$$

2. Analytically non-continuable power and Dirichlet series

To Weierstrass is due the first example of a convergent power series, namely

$$\sum_{n=0}^{\infty} a^n z^{b^n}, \quad a > 0,\ b \in \mathbb{N},\ b > 1,$$

which is non-continuable outside its circle of convergence, i.e. this circle is the domain of existence for the analytic function defined by its sum. This example became a starting point of a great number of studies on the singular points of functions defined by convergent power series and their analytical non-continuability.

The contributions of academician Iliev in this field are obtained mainly under the influence of works of such experts in the classical complex analysis, as Hadamard, Ostrowski, Fabry and Szegö.

Due to Szegö is the result that if each of the coefficients of the power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{6}$$

is equal to one for finitely many complex numbers $d_1, d_2, \ldots, d_s, d_j \neq d_k, j \neq k$, then it is either analytically non-continuable outside the unit disk, or the Maclorain series of a rational function of the kind

$$\frac{P(z)}{1 - z^m}, \quad m \in \mathbb{N},$$

where P is an algebraic polynomial, and this is possible if and only if the sequence of its coefficients is periodic after some subscript. Essential generalizations, extensions and various modifications of this result are obtained by academician Iliev. Typical one is the assertion for the series (6) with coefficients of the kind $a_n = \gamma_n c_n$, $n \in \mathbb{N}_0$. If the members of the sequence $\{\gamma_n\}_0^\infty$ accept finitely many values and for some $\alpha \in \mathbb{R}$ the sequence $\{c_n n^\alpha\}$ has a finite number of limit points all different from zero, then the requirement for non-periodicity after each subscript of the sequence $\{\gamma_n\}$ is sufficient for analytical non-continuability of the series (6).

It seems that academician Iliev was the first who obtained also Szegö's type theorems for Dirichlet series of the kind

$$\sum_{n=0}^{\infty} \gamma_n c_n \exp(-\lambda_n s).$$

3. Univalent functions

In the first decades of the past century, mainly after the works of P. Köbe and L. Bieberbach, a new branch of Geometric Function came into being. It is known now as Theory of the Univalent Functions. Its main object is the class S of functions f which are holomorphic and univalent in the unit disk and are normalized by the conditions $f(0) = 0, f'(0) = 1$, i.e. the functions with Maclorain expansion of the kind

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

in the unit disk, and such that $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$. Different subclasses of S, e.g. defined by additional requirements for convexity of the image $f(D)$ of the unit disk by means of a function $f \in S$, or by starlikeness of this image with respect to the zero point, are also studied. The central object of the efforts of a great number of investigators are theorems of deformation and coefficients estimates. The famous Bieberbach's conjecture that $|a_n| \leq n$, $n \geq 2$, which remained open till its confirmation by Lui de Brange, was one of the stimuli for publishing a great number of papers in prestigious journals and other publications. In this field, except Köbe and Bieberbach, many mathematicians as Littlewood, Hayman, Lewner, Szegö, Golusin and others have remarkable contributions.

Academician Iliev did not remain indifferent to this direction of studies. The results, published in his papers devoted to the univalent functions, weer create during a very short period. The main attention of their author was directed to the class S_k of the k-symmetric functions f_k from the class S, i.e. to those of them having Maclorain expansion of the kind

$$f_k(z) = z + a_1^{(k)} z^{k+1} + a_2^{(k)} z^{2k+1} + \dots . \tag{7}$$

One of the first results of academician Iliev is influenced by a theorem of Szegö about the divided difference of the functions in the class S. The successful use of a similar theorem of Goluzin for the class Σ of the functions f meromorphic and univalent in the region $\{\mathbb{C} \setminus \overline{D}\} \cup \{\infty\}$ and normalized by $f(\infty) = \infty, f'(\infty) = 1$, leads him to the inequalities

$$\frac{1 - r^2}{(1 + r^2)^{4/k}} \leq \left| \frac{f_k(z_1) - f_k(z_2)}{z_1 - z_2} \right| \leq \frac{1}{(1 - r^2)(1 - r^k)^{4/k}} \tag{8}$$

for each function $f_k \in S_k$ provided that $0 < |z_j| \leq r < 1, j = 1, 2, z_1 \neq z_2$. Its application leads to the result that the exact radius of univalence of the partial sums

$$\sigma_n^{(2)} = z + a_3^{(2)} z^3 + \dots + a_{2n+1}^{(2)} z^{2n+1}, \quad n = 1, 2, 3, \dots \tag{9}$$

of the function from the class S_2, i.e. the class of odd functions in S, is equal to $1/\sqrt{3}$.

Another application is that the partial sum

$$\sigma_n(z) = z + a_2 z^2 + \dots + a_n z^n$$

of a function from the class S is univalent in the circle $|z| < 1 - 4\log n/n$ for each $n \geq 15$ which improves a result of V. Levin. Similar result for the partial sums (9) of the functions from the class S_2 is that they are univalent in the circle $|z| < \sqrt{1 - 3\log n/n}$ for $n \geq 12$.

The problem for the radius of univalence of the partial sums

$$\sigma_n^{(3)}(z) = z + a_1^{(3)} z^4 + \cdots + a_n^{(3)} z^{3n+1}$$

of the 3-symmetric functions is also treated by of academician Iliev. As a result it is obtained that it is $\sqrt[3]{3}/2$ for $n \neq 2$. The proof is based on coefficients estimates for the functions from the class S_3 as well as on the left inequality in (9). For $n = 2$ its exactness is proved directly by the method of Löwner.

The inequality

$$\left|\frac{f_k(z_1) - f_k(z_2)}{z_1 - z_2}\right| \geq \frac{1 - r^2}{(1 + r^2)^{2/k}}, \quad |z_j| \leq r,\ 0 < r < 1,\ z_1 \neq z_2$$

is obtained under the additional assumption that the function f_k is convex. It is exact for $k = 1, 2$, i.e. for the functions from the class S as well as for the odd functions in this class. By its help the circle defined by the inequality $|z| < \{1 - (1 + 2/k)\log(n + 1)/(n + 1)\}^{1/k}$ is found, where the partial sum

$$z + a_1^{(k)} z^{k+1} + \cdots + a_n^{(k)} z^{nk+1}$$

of a function $f_k \in S_k$ is univalent for $n > \exp(k\sqrt{2k}/(2 + k)) - 1$.

An inequality for the divided difference of bounded functions in the class S_k is obtained, i.e. for the functions $f_k \in S_k$ such that $f_k(D)$ is a bounded domain.

The already mentioned contributions of academician Iliev and many others, e.g. for the inequality of Hamburger and Turan, for the problem of Pompeiu as well as for the numerical method based on the Newton iterations assigned him a merited position of one of the distinguished experts in actual areas of mathematical analysis where his effort has been directed during several decades in the past century.

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LIST OF PUBLICATIONS OF ACADEMICIAN LJUBOMIR ILIEV

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Application 1 – List of Scientific Publications of Acad. L. Iliev

1938

[1] Über die Nullstellen gewisser Integralausdrücke. Jahresber. Dtsch. Math.-Ver., 48 (1938) (9/12), 169-172 (German).

1939

[2] Several theorems on the distribution of zeros of polynomials. Jubilee proc. of physical-mathematical society, part II, 1939, 60-64 (Bulgarian).

[3] Über die Nullstellen einiger Klassen von Polynomen. Tohoku Math. J., 45 (1939), 259-264 (German).

1940

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1943

[7] Einige Probleme über nichtgleichmaessig gespannte ebene Membranen. Annuaire Univ. Sofia, Fac. Sci., 39 (1943), No 1, 251-286 (Bulgarian, German summary).

[8] Über das Gleichgewicht von elliptischen Membranen. Annuaire Univ. Sofia, Fac. Sci., 39 (1943), No 1, 409-426 (Bulgarian, German summary).

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1945

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1946

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1948

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1950

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1971

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1973

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1974

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1976

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Application 2 – List of Acad. L. Iliev's Publications on Education in Mathematics

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Other Popular Readings

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HYPERBOLIC BICOMPLEX VARIABLES

Lilia N. Apostolova

Abstract

The bicomplex numbers appeared in the work [8] of Corrado Segre for the goals of algebraic geometry. The hyperbolic bicomplex numbers are introduced and used later mainly for the goals of physics.

Matrix representation of the hyperbolic bicomplex numbers is given here. The determinant of the matrix representation is found. Invertible elements and idempotent elements are described. Two different representations of the hyperbolic bicomplex numbers by four idempotent elements are found.

Analogous results for the algebra of bicomplex numbers are given. Results of the analysis of functions of hyperbolic bicomplex variable, of bicomplex variable and of other generalized variables are given in [1], [2] [3], [4], [6], [7], [9].

MSC 2010: 32A30, 30G35

Key Words and Phrases: hyperbolic bicomplex number; bicomplex number; invertible element; idempotent element

1. Matrix representation of hyperbolic bicomplex number

Let us recall the definition of the hyperbolic bicomplex numbers (see [3], [4], [5]). They are defined as follows

$$\mathbf{R}(j_1, j_2) := \{x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3 : \; j_1^2 = j_2^2 = 1, \; j_1j_2 = j_2j_1\}, \qquad (1.1)$$

where $x_0, x_1, x_2, x_3 \in \mathbf{R}$ are real numbers.

The multiplicative table is the following one

	1	j_1	j_2	j_1j_2
1	1	j_1	j_2	j_1j_2
j_1	j_1	1	j_1j_2	j_2
j_2	j_2	j_1j_2	1	j_1
j_1j_2	j_1j_2	j_1	j_1	1

Table of multiplication of hyperbolic bicomplex units

 pp. 33–54

The addition and the multiplication by real scalar are defined component wise, and the multiplication of hyperbolic bicomplex numbers is defined by opening the brackets and using the identities for the units j_1 and j_2. The algebra $\mathbf{R}(j_1, j_2)$ is an associative, commutative algebra with zero divisors. So are for example the numbers $A(1 \pm j_1)$, $A(1 \pm j_2)$, and $A(1 \pm j_1 j_2)$, where A is a hyperbolic complex number. Indeed the product of such a number with $1 \mp j_1$, $1 \mp j_2$ or $1 \mp j_1 j_2$, respectively, is equal to zero. The distributive rule holds.

Let us consider the following three 4×4 matrices with real coefficients

$$J_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; J_2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; J_1 J_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is fulfilled $J_1^2 = E$, $J_2^2 = E$ and $J_1 J_2 = J_2 J_1$, $(J_1 J_2)^2 = E$, where E is the identity 4×4 matrix. The matrices E, J_1, J_2 and $J_1 J_2$ generate a commutative subalgebra $\mathfrak{M}$ of the full matrix algebra $M(4, \mathbf{R})$ of all 4×4 matrices with real elements, $\mathfrak{M} \hookrightarrow M(4, \mathbf{R})$.

We shall define a homomorphism F of the algebra $\mathbf{R}(j_1, j_2)$ of the hyperbolic bicomplex numbers in the algebra $\mathfrak{M}$ as follows

$$F : X = x_0 + j_1 x_1 + j_2 x_2 + j_1 j_2 x_3 \mapsto x_0 E + x_1 J_1 + x_2 J_2 + x_3 J_1 J_2 = M_X.$$

Then the image $F(X)$ of the hyperbolic bicomplex number X is the following matrix $M_X \in \mathfrak{M}$

$$M_X = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & x_3 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_3 & x_2 & x_1 & x_0 \end{pmatrix}. \tag{1.2}$$

This is a symmetric matrix with respect to the main diagonal and to the second diagonal.

The following identity is true

$$(E + J_1 J_2) \cdot (E - J_1 J_2) =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0,$$

i.e. the matrices $E + J_1 J_2$ and $E - J_1 J_2$ are zero divisors in the algebra $\mathfrak{M}$.

2. Determinant of the matrix representation of hyperbolic bicomplex number, coordinate hyperplanes, axes and planes

We obtain the following equalities for the determinant of the representing matrix M_X of the hyperbolic bicomplex number $X = x_0 + j_1 x_1 + j_2 x_2 + j_1 j_2 x_2$

$$D_-(x_0,x_1,x_2,x_3) = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & x_3 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_3 & x_2 & x_1 & x_0 \end{vmatrix} = x_0 \begin{vmatrix} x_0 & x_3 & x_2 \\ x_3 & x_0 & x_1 \\ x_2 & x_1 & x_0 \end{vmatrix} - \tag{2.1}$$

$$-x_1 \begin{vmatrix} x_1 & x_3 & x_2 \\ x_2 & x_0 & x_1 \\ x_3 & x_1 & x_0 \end{vmatrix} + x_2 \begin{vmatrix} x_1 & x_0 & x_2 \\ x_2 & x_3 & x_1 \\ x_3 & x_2 & x_0 \end{vmatrix} - x_3 \begin{vmatrix} x_1 & x_0 & x_3 \\ x_2 & x_3 & x_0 \\ x_3 & x_2 & x_1 \end{vmatrix} =$$

$$= x_0 \left(x_0 \begin{vmatrix} x_0 & x_1 \\ x_1 & x_0 \end{vmatrix} - x_3 \begin{vmatrix} x_3 & x_1 \\ x_2 & x_0 \end{vmatrix} + x_2 \begin{vmatrix} x_3 & x_0 \\ x_2 & x_1 \end{vmatrix} \right) -$$

$$-x_1 \left(x_1 \begin{vmatrix} x_0 & x_1 \\ x_1 & x_0 \end{vmatrix} - x_3 \begin{vmatrix} x_2 & x_1 \\ x_3 & x_0 \end{vmatrix} + x_2 \begin{vmatrix} x_2 & x_0 \\ x_3 & x_1 \end{vmatrix} \right) +$$

$$+x_2 \left(x_1 \begin{vmatrix} x_3 & x_1 \\ x_2 & x_0 \end{vmatrix} - x_0 \begin{vmatrix} x_2 & x_1 \\ x_3 & x_0 \end{vmatrix} + x_2 \begin{vmatrix} x_2 & x_3 \\ x_3 & x_2 \end{vmatrix} \right) -$$

$$-x_3 \left(x_1 \begin{vmatrix} x_3 & x_0 \\ x_2 & x_1 \end{vmatrix} - x_0 \begin{vmatrix} x_2 & x_0 \\ x_3 & x_1 \end{vmatrix} + x_3 \begin{vmatrix} x_2 & x_3 \\ x_3 & x_2 \end{vmatrix} \right) =$$

$$= (x_0^2 - x_1^2)^2 + (x_2^2 - x_3^2)^2 + 8x_0x_1x_2x_3 -$$

$$- \left(2x_0^2x_3^2 + 2x_0^2x_2^2 + 2x_1^2x_3^2 + 2x_1^2x_2^2\right) =$$

$$= \left(x_0^2 - x_1^2 + x_2^2 - x_3^2\right)^2 - \left(4x_0^2x_2^2 + 4x_1^2x_3^2 - 8x_0x_1x_2x_3\right) =$$

$$= (x_0^2 - x_1^2 + x_2^2 - x_3^2)^2 - 4(x_0x_2 - x_1x_3)^2 =$$

$$= (x_0^2 - x_1^2 + x_2^2 - x_3^2 - 2x_0x_2 + 2x_1x_3)\times$$

$$\times(x_0^2 - x_1^2 + x_2^2 - x_3^2 + 2x_0x_2 - 2x_1x_3) =$$

$$= \left((x_0 - x_2)^2 - (x_1 - x_3)^2\right)\left((x_0 + x_2)^2 - (x_1 + x_3)^2\right) =$$

$$= (x_0 - x_2 - x_1 + x_3)(x_0 - x_2 + x_1 - x_3)\times$$

$$\times(x_0 + x_2 - x_1 - x_3)(x_0 + x_2 + x_1 + x_3).$$

So we obtain the following

Theorem 2.1. *It is true that*

$$D_-(x_0, x_1, x_2, x_3) = (x_0 - x_2 - x_1 + x_3)(x_0 - x_2 + x_1 - x_3)\times$$
$$\times(x_0 + x_2 - x_1 - x_3)(x_0 + x_2 + x_1 + x_3) =$$
$$= ((x_0 - x_2)^2 - (x_1 - x_3)^2) \times ((x_0 + x_2)^2 - (x_1 + x_3)^2).$$

The determinant $D_-(x_0, x_1, x_2, x_3)$ is a positive real number in the cases

$$(x_0 - x_2)^2 > (x_1 - x_3)^2, \quad (x_0 + x_2)^2 > (x_1 + x_3)^2$$

and

$$(x_0 - x_2)^2 < (x_1 - x_3)^2, \quad (x_0 + x_2)^2 < (x_1 + x_3)^2.$$

In the cases

$$(x_0 - x_2)^2 > (x_1 - x_3)^2, \quad (x_0 + x_2)^2 < (x_1 + x_3)^2$$

and

$$(x_0 - x_2)^2 < (x_1 - x_3)^2, \quad (x_0 + x_2)^2 > (x_1 + x_3)^2$$

the determinant $D_-(x_0, x_1, x_2, x_3)$ is a negative real number.

Theorem 2.2. *The following four hyperplanes β_0, β_1, β_2, β_3 in $\mathbf{R}^4$*

$$\beta_0 : \ x_0 - x_2 - x_1 + x_3 = 0,$$
$$\beta_1 : \ x_0 - x_2 + x_1 - x_3 = 0,$$
$$\beta_2 : \ x_0 + x_2 - x_1 - x_3 = 0,$$
$$\beta_3 : \ x_0 + x_2 + x_1 + x_3 = 0,$$

which annihilate the determinant $D_-(x_0, x_1, x_2, x_3)$ of a hyperbolic bicomplex number $x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$ are in general position, i. e. the unique common point of these hyperplanes is the origin $(0, 0, 0, 0)$ in $\mathbf{R}^4$. In other words, the unique common point of the corresponding hyperplanes to β_0 β_1 β_2, β_3 in the algebra of hyperbolic bicomplex numbers is the hyperbolic bicomplex number 0.

Proof. Let us consider the determinant of the system of four linear equation in the theorem. This is the following real number, which we calculate straightforward

$$\begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 8.$$

This show that the system of linear equations has only zero solution (see for example [10], p. 92). $\square$

Theorem 2.3. *The four hyperplanes β_0, β_1, β_2, β_3 in $\mathbf{R}^4$ determine the following four lines l_0, l_1, l_2 and l_3,*

$$l_0 : \left| \begin{array}{l} x_0 = s, \\ x_1 = -s, \\ x_2 = -s, \\ x_3 = s, \end{array} \right. \qquad l_1 : \left| \begin{array}{l} x_0 = s, \\ x_1 = s, \\ x_2 = -s, \\ x_3 = -s, \end{array} \right.$$

$$l_2 : \left| \begin{array}{l} x_0 = s, \\ x_1 = -s, \\ x_2 = s, \\ x_3 = -s, \end{array} \right. \qquad l_3 : \left| \begin{array}{l} x_0 = s, \\ x_1 = s, \\ x_2 = s, \\ x_3 = s, \end{array} \right.$$

where the parameter s is a real number.

They form a system of coordinate axes, associated with the hyperbolic bicomplex number $x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$.

Proof. The axes l_0 is determined by the system of three linear homogeneous equations as follows

$$l_0 : \left| \begin{array}{l} \beta_1 : \ x_0 - x_2 + x_1 - x_3 = 0, \\ \beta_2 : \ x_0 + x_2 - x_1 - x_3 = 0, \\ \beta_3 : \ x_0 + x_2 + x_1 + x_3 = 0, \end{array} \right.$$

From the first and the second equations follows that $x_0 = x_3$. The first and the third equations implies $x_0 = -x_1$, and the second and the third equations gives $x_0 = -x_2$. So the line l_0 is the following one: $x_0 = -x_1 = -x_2 = x_3 = s$, $s \in \mathbf{R}$, i.e. this is the parametric line

$$l_0 : \left| \begin{array}{l} x_0 = s, \\ x_1 = -s, \\ x_2 = -s, \\ x_3 = s, \end{array} \right.$$

where the parameter s is a real number.

The axes l_1.

$$l_1 : \left| \begin{array}{l} \beta_0 : \ x_0 - x_2 - x_1 + x_3 = 0, \\ \beta_2 : \ x_0 + x_2 - x_1 - x_3 = 0, \\ \beta_3 : \ x_0 + x_2 + x_1 + x_3 = 0, \end{array} \right.$$

From the first and the second equations follows that $x_0 = x_1$. The first and the third equations implies $x_0 = -x_3$, and the second and the third equations gives $x_0 = -x_2$. So the line l_0 is the following one: $x_0 = x_1 = -x_2 = -x_3 = s$, $s \in \mathbf{R}$, i.e. this is the parametric line

$$l_1 : \left| \begin{array}{l} x_0 = s, \\ x_1 = s, \\ x_2 = -s, \\ x_3 = -s, \end{array} \right.$$

where the parameter s is a real number.

The axes l_2.

$$l_2 : \left| \begin{array}{l} \beta_0 : \; x_0 - x_2 - x_1 + x_3 = 0, \\ \beta_1 : \; x_0 - x_2 + x_1 - x_3 = 0, \\ \beta_3 : \; x_0 + x_2 + x_1 + x_3 = 0, \end{array} \right.$$

From the first and the second equations follows that $x_0 = x_2$. The first and the third equations implies $x_0 = -x_3$, and the second and the third equations gives $x_0 = -x_1$. So the line l_0 is the following one: $x_0 = -x_1 = x_2 = -x_3 = s$, $s \in \mathbf{R}$, i.e. this is the parametric line

$$l_2 : \left| \begin{array}{l} x_0 = s, \\ x_1 = -s, \\ x_2 = s, \\ x_3 = -s, \end{array} \right.$$

where the parameter s is a real number.

The axes l_3.

$$l_3 : \left| \begin{array}{l} \beta_0 : \; x_0 - x_2 - x_1 + x_3 = 0, \\ \beta_1 : \; x_0 - x_2 + x_1 - x_3 = 0, \\ \beta_2 : \; x_0 + x_2 - x_1 - x_3 = 0. \end{array} \right.$$

From the first and the second equations follows that $x_0 = x_2$. The first and the third equations implies $x_0 = x_1$, and the second and the third equations gives $x_0 = x_3$. So the line l_0 is the following one: $x_0 = x_1 = x_2 = x_3 = s$, $s \in \mathbf{R}$, i.e. this is the parametric line

$$l_3 : \left| \begin{array}{l} x_0 = s, \\ x_1 = s, \\ x_2 = s, \\ x_3 = s, \end{array} \right.$$

where the parameter s is a real number. □

Then we obtain the following theorem.

Theorem 2.4. *The four lines l_0, l_1, l_2 and l_3 in $\mathbf{R}^4$, determines six coordinate planes l_0l_1, l_0l_2, l_0l_3, l_1l_2, l_1l_3, and l_2l_3, associated with the hyperbolic* *bicomplex number* $x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$. *These are the following planes*

$$l_0l_1: \left| \begin{array}{l} x_0 = s+t, \\ x_1 = -s+t, \\ x_2 = -s-t, \\ x_3 = s-t, \end{array} \right. \qquad l_0l_2: \left| \begin{array}{l} x_0 = s+t, \\ x_1 = -s-t, \\ x_2 = -s+t, \\ x_3 = s-t, \end{array} \right.$$

$$l_0l_3: \left| \begin{array}{l} x_0 = s+t, \\ x_1 = -s+t, \\ x_2 = -s+t, \\ x_3 = s+t, \end{array} \right. \qquad l_1l_2: \left| \begin{array}{l} x_0 = s+t, \\ x_1 = s-t, \\ x_2 = -s+t, \\ x_3 = -s-t, \end{array} \right.$$

$$l_1l_3: \left| \begin{array}{l} x_0 = s+t, \\ x_1 = s+t, \\ x_2 = -s+t, \\ x_3 = -s+t, \end{array} \right. \qquad l_2l_3: \left| \begin{array}{l} x_0 = s+t, \\ x_1 = -s+t, \\ x_2 = s+t, \\ x_3 = -s+t, \end{array} \right.$$

where the parameters s *and* t *are real numbers.*

3. Invertible hyperbolic bicomples numbers

Definition 3.1. Invertible element in the commutative algebra A with an unit E is called an element $X \in A$, such that there exists solution $Y \in A$ of the equation $XY = E$.

Let us consider the matrix M_X of the hyperbolic bicomplex number X.

The hyperbolic bicomplex number $X = x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3 \neq 0$ is an invertible element in the algebra $\mathbf{R}(j_1, j_2)$ if and only if it satisfy the following matrix equation

$$M_X M_Y = E. \tag{3.1}$$

A necessary and sufficient condition for the hyperbolic bicomplex number X to be invertible, i.e. the matrix equation $M_X M_Y = E$ to be solvable is that the determinant of the matrix M_X not annihilate, i.e. this is the condition

$$D(x_0, x_1, x_2, x_3) := \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_0 & x_3 & x_2 \\ x_2 & x_3 & x_0 & x_1 \\ x_3 & x_2 & x_1 & x_0 \end{vmatrix} \neq 0.$$

Theorem 3.1. *The hyperbolic bicomplex number* X *is invertible, i.e. the matrix equation (*3.1*) has a solution* (y_0, y_1, y_2, y_3) *if and only if*

$$x_0 + x_2 \neq \pm(x_1 + x_3), \quad x_0 - x_2 \neq \pm(x_1 - x_3).$$

4. Idempotent hyperbolic bicomplex numbers

Using the commutative and the distributive rules for the hyperbolic bicomplex numbers $X = x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$ and $Y = y_0 + j_1y_1 + j_2y_2 + j_1j_2y_3$ we find that the product of these numbers is the following hyperbolic bicomplex number

$$XY = (x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3)(y_0 + j_1y_1 + j_2y_2 + j_1j_2y_3) =$$

$$= x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 + j_1(x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2)+$$

$$+j_2(x_0y_2 + x_1y_3 + x_2y_0 + x_3y_1) + j_1j_2(x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0).$$

Example 4.1. The square X^2 of the hyperbolic bicomplex number $X = x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$ is the following hyperbolic bicomplex number

$$\begin{aligned} X^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + 2j_1(x_0x_1 + x_2x_3) + \\ +2j_2(x_0x_2 + x_1x_3) + 2j_1j_2(x_0x_3 + x_1x_2). \end{aligned} \tag{4.1}$$

Definition 4.1. Idempotent element of an algebra A is called an element $X \in A$, such that it is fulfilled $X^2 = X$.

Theorem 4.1. *Idempotents in the algebra of the hyperbolic bicomplex numbers are the following sixteen hyperbolic bicomplex numbers:* $X = \frac{1}{2}(1 \pm j_1j_2)$, $X = \frac{1}{2}(1 \pm j_2)$, $X = \frac{1}{4}(1 + j_1 - j_2 - j_1j_2)$, $X = \frac{1}{4}(3 - j_1 - j_2 - j_1j_2)$, $X = \frac{1}{4}(1 + j_1 + j_2 + j_1j_2)$, $X = \frac{1}{4}(3 - j_1 + j_2 + j_1j_2)$, $X = \frac{1}{4}(1 - j_1 - j_2 + j_1j_2)$, $X = \frac{1}{4}(1 - j_1 + j_2 - j_1j_2)$, $X = \frac{1}{4}(3 + j_1 - j_2 + j_1j_2)$, $X = \frac{1}{4}(3 + j_1 + j_2 - j_1j_2)$, $X = 0$, $X = \frac{1}{2}(1 - j_1)$, $X = \frac{1}{2}(1 + j_1)$, *and* $X = 1$.

Proof. Using the formula (4.1) we see that the hyperbolic bicomplex number $X = x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$ is an idempotent element in the algebra of the hyperbolic bicomplex numbers if and only if it satisfy the following homogeneous system of four equations of second degree with real variables

$$\left| \begin{array}{l} x_0^2 + x_1^2 + x_2^2 + x_3^2 = x_0, \\ 2x_0x_1 + 2x_2x_3 = x_1, \\ 2x_0x_2 + 2x_1x_3 = x_2, \\ 2x_0x_3 + 2x_1x_2 = x_3, \end{array} \right. \tag{4.2}$$

.

The system of four equations of second degree (4.2) can be rewritten in the following way

$$\left|\begin{array}{l}(x_0-\frac{1}{2})^2+x_1^2+x_2^2+x_3^2=\frac{1}{4},\\ 2(x_0-\frac{1}{2})x_1+2x_2x_3=0,\\ 2(x_0-\frac{1}{2})x_2+2x_1x_3=0,\\ 2(x_0-\frac{1}{2})x_3+2x_1x_2=0.\end{array}\right. \tag{4.3}$$

The following system of equations

$$\left|\begin{array}{l}(x_0-\frac{1}{2}+x_1)^2+(x_2+x_3)^2=\frac{1}{4},\\ (x_0-\frac{1}{2}-x_1)^2+(x_2-x_3)^2=\frac{1}{4},\\ 2(x_0-\frac{1}{2}+x_1)(x_2+x_3)=0,\\ 2(x_0-\frac{1}{2}-x_1)(x_2-x_3)=0,\end{array}\right. \tag{4.4}$$

is equivalent to the system (4.3). Here the first and the second equations in the system are obtained from the first and the fourth equations of the system (4.3), and the third and the fourth equations are obtained from the second and the third equations of the system (4.3).

The set of solutions of the systems of equations (4.3) and (4.4) coincide with the set of solutions of the following four systems of equations of first and second degree

$$\left|\begin{array}{l}(x_0-\frac{1}{2}+x_1)^2+(x_2+x_3)^2=\frac{1}{4},\\ (x_0-\frac{1}{2}-x_1)^2+(x_2-x_3)^2=\frac{1}{4},\\ 2x_0-1+2x_1=0,\\ 2x_0-1-2x_1=0,\end{array}\right. \tag{4.5}$$

$$\left|\begin{array}{l}(x_0-\frac{1}{2}+x_1)^2+(x_2+x_3)^2=\frac{1}{4},\\ (x_0-\frac{1}{2}-x_1)^2+(x_2-x_3)^2=\frac{1}{4},\\ 2x_0-1+2x_1=0,\\ x_2-x_3=0,\end{array}\right. \tag{4.6}$$

$$\left|\begin{array}{l}(x_0-\frac{1}{2}+x_1)^2+(x_2+x_3)^2=\frac{1}{4},\\ (x_0-\frac{1}{2}-x_1)^2+(x_2-x_3)^2=\frac{1}{4},\\ x_2+x_3=0,\\ 2x_0-1-2x_1=0\end{array}\right. \tag{4.7}$$

and

$$\left|\begin{array}{l}(x_0-\frac{1}{2}+x_1)^2+(x_2+x_3)^2=\frac{1}{4},\\ (x_0-\frac{1}{2}-x_1)^2+(x_2-x_3)^2=\frac{1}{4},\\ x_2+x_3=0,\\ x_2-x_3=0.\end{array}\right. \tag{4.8}$$

4.1. Solutions of the system (4.5)

The set of solutions of the system of equations (4.5) coincide with the set of solutions of the following system of equations

$$\left|\begin{array}{l} (x_2+x_3)^2=\frac{1}{4}, \\ (x_2-x_3)^2=\frac{1}{4}, \\ 2x_0-1+2x_1=0, \\ 2x_0-1-2x_1=0, \end{array}\right.$$

or, of the equivalent system of equations

$$\left|\begin{array}{l} 2x_2^2+2x_3^2=\frac{1}{2}, \\ 2x_2x_3=0, \\ 2x_0-1=0, \\ x_1=0. \end{array}\right.$$

The solutions of this system of equations are the 4-tuples of real numbers (x_0, x_1, x_2, x_3), where $x_0=\frac{1}{2}$, $x_1=x_2=0$, $x_3=\frac{1}{2}$, $x_0=\frac{1}{2}$, $x_1=x_2=0$, $x_3=-\frac{1}{2}$, $x_0=\frac{1}{2}$, $x_1=0$, $x_2=\frac{1}{2}$, $x_3=0$, $x_0=\frac{1}{2}$, $x_1=0$, $x_2=-\frac{1}{2}$, $x_3=0$, respectively. They give the following idempotent elements: $X=\frac{1}{2}(1\pm j_1j_2)$ and $X=\frac{1}{2}(1\pm j_2)$.

4.2. Solutions of the system (4.6)

The system (4.6) is equivalent to the system of equations of first and second degree

$$\left|\begin{array}{l} (x_2+x_3)^2=\frac{1}{4}, \\ (x_0-\frac{1}{2}-x_1)^2=\frac{1}{4}, \\ x_0-\frac{1}{2}+x_1=0, \\ x_2-x_3=0. \end{array}\right.$$

In this system the first and the second equations can be decompose into linear multiplier and it can be written as follows

$$\left|\begin{array}{l} (x_2+x_3+\frac{1}{2})(x_2+x_3-\frac{1}{2})=0, \\ (x_0-x_1)(x_0-x_1-1)=0, \\ x_0-\frac{1}{2}+x_1=0, \\ x_2-x_3=0. \end{array}\right.$$

Then there arises four system of linear equations, which gives all the solutions of the system (4.6). Namely these are the systems of equations

$$\left|\begin{array}{l} x_2+x_3+\frac{1}{2}=0, \\ x_0-x_1=0, \\ x_0-\frac{1}{2}+x_1=0, \\ x_2-x_3=0, \end{array}\right. \qquad \left|\begin{array}{l} x_2+x_3+\frac{1}{2}=0, \\ x_0-x_1-1=0, \\ x_0-\frac{1}{2}+x_1=0, \\ x_2-x_3=0, \end{array}\right.$$

and

$$\left| \begin{array}{l} x_2 + x_3 - \frac{1}{2} = 0, \\ x_0 - x_1 = 0, \\ x_0 - \frac{1}{2} + x_1 = 0, \\ x_2 - x_3 = 0 \end{array} \right. \qquad \left| \begin{array}{l} x_2 + x_3 - \frac{1}{2} = 0, \\ x_0 - x_1 - 1 = 0, \\ x_0 - \frac{1}{2} + x_1 = 0, \\ x_2 - x_3 = 0. \end{array} \right.$$

The solution of the first of system of equations is $x_0 = x_1 = \frac{1}{4}$, $x_2 = x_3 = -\frac{1}{4}$. The solution of the second system of equations is $x_0 = \frac{3}{4}$, $x_1 = -\frac{1}{4}$, $x_2 = x_3 = -\frac{1}{4}$. The solution of the third system of equations is $x_0 = x_1 = \frac{1}{4}$, $x_2 = x_3 = \frac{1}{4}$. The solution of the fourth system of equations is $x_0 = \frac{3}{4}$, $x_1 = -\frac{1}{4}$, $x_2 = x_3 = \frac{1}{4}$.

They determine the following four idempotent elements: $X = \frac{1}{4}(1 + j_1 - j_2 - j_1 j_2)$, $X = \frac{1}{4}(3 - j_1 - j_2 - j_1 j_2)$, $X = \frac{1}{4}(1 + j_1 + j_2 + j_1 j_2)$, and $X = \frac{1}{4}(3 - j_1 + j_2 + j_1 j_2)$.

4.3. Solutions of the system (4.7)

The system (4.7)

$$\left| \begin{array}{l} (x_0 - \frac{1}{2} + x_1)^2 + (x_2 + x_3)^2 = \frac{1}{4}, \\ (x_0 - \frac{1}{2} - x)^2 + (x_2 - x_3)^2 = \frac{1}{4}, \\ x_2 + x_3 = 0, \\ 2x_0 - 1 - 2x_1 = 0 \end{array} \right.$$

is equivalent to the system of equations

$$\left| \begin{array}{l} (x_0 - \frac{1}{2} + x_1)^2 - \frac{1}{4} = 0, \\ (x_2 - x_3)^2 - \frac{1}{4} = 0, \\ x_2 + x_3 = 0, \\ x_0 - \frac{1}{2} - x_1 = 0. \end{array} \right.$$

and this system of equations is equivalent to the following system of two equations of second degree and two linear equations

$$\left| \begin{array}{l} (x_0 - \frac{1}{2} + x_1 + \frac{1}{2})(x_0 - \frac{1}{2} + x_1 - \frac{1}{2}) = 0, \\ (x_2 - x_3 + \frac{1}{2})(x_2 - x_3 - \frac{1}{2}) = 0, \\ x_2 + x_3 = 0, \\ x_0 - \frac{1}{2} - x_1 = 0. \end{array} \right.$$

Then there arises four system of linear equations, which gives all the solutions of the system (4.7). Namely these are the systems of equations

$$\left| \begin{array}{l} x_0 - \frac{1}{2} + x_1 + \frac{1}{2} = 0, \\ x_2 - x_3 + \frac{1}{2} = 0, \\ x_2 + x_3 = 0, \\ x_0 - \frac{1}{2} - x_1 = 0, \end{array} \right. \qquad \left| \begin{array}{l} x_0 - \frac{1}{2} + x_1 + \frac{1}{2} = 0, \\ x_2 - x_3 - \frac{1}{2} = 0, \\ x_2 + x_3 = 0, \\ x_0 - \frac{1}{2} - x_1 = 0, \end{array} \right.$$

$$\left|\begin{array}{l} x_0-\frac{1}{2}+x_1-\frac{1}{2}=0,\\ x_2-x_3+\frac{1}{2}=0,\\ x_2+x_3=0,\\ x_0-\frac{1}{2}-x_1=0. \end{array}\right. \quad \text{and} \quad \left|\begin{array}{l} x_0-\frac{1}{2}+x_1-\frac{1}{2}=0,\\ x_2-x_3-\frac{1}{2}=0,\\ x_2+x_3=0,\\ x_0-\frac{1}{2}-x_1=0. \end{array}\right.$$

The solutions of these systems of equations are the 4-tuples (x_0, x_1, x_2, x_3) of real numbers as follows: $x_0 = \frac{1}{4}$, $x_1 = -\frac{1}{4}$, $x_2 = -\frac{1}{4}$, $x_3 = \frac{1}{4}$, $x_0 = \frac{1}{4}$, $x_1 = -\frac{1}{4}$, $x_2 = \frac{1}{4}$, $x_3 = -\frac{1}{2}$, $x_0 = \frac{3}{4}$, $x_1 = \frac{1}{4}$, $x_2 = -\frac{1}{4}$, $x_3 = \frac{1}{4}$ and $x_0 = \frac{3}{4}$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{4}$, $x_3 = -\frac{1}{4}$. They determine the following idempotent elements: $X = \frac{1}{4}(1 - j_1 - j_2 + j_1j_2)$, $X = \frac{1}{4}(1 - j_1 + j_2 - j_1j_2)$, $X = \frac{1}{4}(3 + j_1 - j_2 + j_1j_2)$ and $X = \frac{1}{4}(3 + j_1 + j_2 - j_1j_2)$.

4.4. Solutions of the system (4.8)

The system (4.8) is equivalent to the system of equations

$$\left|\begin{array}{l} (x_0-\frac{1}{2}+x_1)^2=\frac{1}{4},\\ (x_0-\frac{1}{2}-x_1)^2=\frac{1}{4},\\ x_2+x_3=0,\\ x_2-x_3=0. \end{array}\right.$$

and this system is equivalent to the following system of two secon degree and two linear equations

$$\left|\begin{array}{l} (x_0-\frac{1}{2}+x_1+\frac{1}{2})(x_0-\frac{1}{2}+x_1-\frac{1}{2})=0\\ (x_0-\frac{1}{2}-x_1+\frac{1}{2})(x_0-\frac{1}{2}-x_1-\frac{1}{2})=0,\\ x_2=0,\\ x_3=0, \end{array}\right.$$

There arise four systems each of them of four linear equations as follows

$$\left|\begin{array}{l} x_0-\frac{1}{2}+x_1+\frac{1}{2}=0\\ x_0-\frac{1}{2}-x_1+\frac{1}{2}=0,\\ x_2=0,\\ x_3=0, \end{array}\right. \qquad \left|\begin{array}{l} x_0-\frac{1}{2}+x_1+\frac{1}{2}=0\\ x_0-\frac{1}{2}-x_1-\frac{1}{2}=0,\\ x_2=0,\\ x_3=0, \end{array}\right.$$

$$\left|\begin{array}{l} x_0-\frac{1}{2}+x_1-\frac{1}{2}=0\\ x_0-\frac{1}{2}-x_1+\frac{1}{2}=0,\\ x_2=0,\\ x_3=0, \end{array}\right. \quad \text{and} \quad \left|\begin{array}{l} x_0-\frac{1}{2}+x_1-\frac{1}{2}=0\\ x_0-\frac{1}{2}-x_1-\frac{1}{2}=0,\\ x_2=0,\\ x_3=0, \end{array}\right.$$

The solutions of these four systems of equations are the following 4-tuples of real numbers $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_0 = \frac{1}{2}$, $x_1 = -\frac{1}{2}$, $x_2 = x_3 = -0$, $x_0 = \frac{1}{2}$, $x_1 = \frac{1}{2}$, $x_2 = 0 = x_3 = 0$, and $x_0 = 1$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$. They determine the following idempotent elements: $X = 0$, $X = \frac{1}{2}(1 - j_1)$, $X = \frac{1}{2}(1 + j_1)$, and $X = 1$. □

5. Decomposition of the algebra of hyperbolic bicomplex numbers and rule of multiplication in the base of morphisms

Let us transform the coordinate system of the underlying 4-dimensional vector space of the algebra of the hyperbolic bicomplex numbers as follows

$$\left| \begin{array}{l} u_0 = -x_0 - x_1 + x_2 + x_3, \\ u_1 = -x_0 - x_2 + x_1 + x_3, \\ u_2 = -x_0 - x_3 + x_1 + x_2, \\ u_3 = -x_0 - x_1 - x_2 - x_3. \end{array} \right. \tag{5.1}$$

The system of equations (5.1), solved with respect to x_0, x_1, x_2, x_3, is the following one

$$\left| \begin{array}{l} 4x_0 = -u_0 - u_1 - u_2 - u_3, \\ 4x_1 = -u_0 + u_1 + u_2 - u_3, \\ 4x_2 = u_0 - u_1 + u_2 - u_3, \\ 4x_3 = u_0 + u_1 - u_2 - u_3. \end{array} \right.$$

The Jacobi matrix of the transformation of the variables (x_0, x_1, x_2, x_3) with the variables (u_0, u_1, u_2, u_3) is the following one

$$\frac{\partial(u_0, u_1, u_2, u_3)}{\partial(x_0, x_1, x_2, x_3)} = \begin{pmatrix} -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

In this coordinate system the representation of the hyperbolic bicomplex number $X = x_0 + j_1 x_1 + j_2 x_2 + j_1 j_2 x_3$ will be the following one

$$4X = (-u_0 - u_1 - u_2 - u_3) + j_1(-u_0 + u_1 + u_2 - u_3) +$$

$$+ j_2(u_0 - u_1 + u_2 - u_3) + j_1 j_2(u_0 + u_1 - u_2 - u_3) =$$

$$= u_0(-1 - j_1 + j_2 + j_1 j_2) + u_1(-1 + j_1 - j_2 + j_1 j_2) +$$

$$+ u_2(-1 + j_1 + j_2 - j_1 j_2) + u_3(-1 - j_1 - j_2 - j_1 j_2) =$$

$$= -u_0(1 + j_1)(1 - j_2) - u_1(1 - j_1)(1 - j_2) - u_2(1 - j_1)(1 + j_2) - u_3(1 + j_1)(1 + j_2),$$

where $u_0, u_1, u_2, u_3 \in \mathbf{R}$ are real numbers.

Example 5.1. It is true that

$$1 = \frac{1}{4}((1 + j_1)(1 - j_2) + (1 - j_1)(1 - j_2) + (1 - j_1)(1 + j_2) + (1 + j_1)(1 + j_2)).$$

So we obtain the following

Theorem 5.1. *The following four idempotent elements of the algebra of hyperbolic bicomplex numbers*

$$\rho_0 = 1/4(1+j_1)(1-j_2), \quad \rho_1 = 1/4(1-j_1)(1-j_2),$$

$$\rho_2 = 1/4(1-j_1)(1+j_2), \quad \rho_3 = 1/4(1+j_1)(1+j_2)$$

decompose it as a sum of four one dimensional subalgebras as follows

$$\mathbf{R}(j_1,j_2) \approx \rho_0\mathbf{R}(j_1,j_2) \oplus \rho_1\mathbf{R}(j_1,j_2) \oplus \rho_2\mathbf{R}(j_1,j_2) \oplus \rho_3\mathbf{R}(j_1,j_2).$$

Moreover it is true that

$$\rho_0 \oplus \rho_1 = 1/2(1-j_2), \quad \rho_0 \oplus \rho_2 = 1/2(1-j_1j_2), \quad \rho_0 \oplus \rho_3 = 1/2(1+j_1),$$

$$\rho_1 \oplus \rho_2 = 1/2(1-j_1), \quad \rho_1 \oplus \rho_3 = 1/2(1+j_1j_2), \quad \rho_2 \oplus \rho_3 = 1/2(1+j_2).$$

and these morphisms are idempotent elements of the algebra of hyperbolic bicomplex numbers, too.

Let us consider two hyperbolic bicomplex numbers X, Y,

$$X = x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$$

and

$$Y = y_0 + j_1y_1 + j_2y_2 + j_1j_2y_3.$$

Let us represent it in the coordinate system above. We obtain

$$4X = -u_0(1+j_1)(1-j_2) - u_1(1-j_1)((1-j_2) - u_2(1-j_1)(1+j_2) - u_3(1+j_1)(1+j_2),$$

and

$$4Y = -v_0(1+j_1)(1-j_2) - v_1(1-j_1)((1-j_2) - v_2(1-j_1)(1+j_2) - v_3(1+j_1)(1+j_2).$$

Then their product will be the following hyperbolic bicomplex number

$$16XY =$$

$$= (-u_0(1+j_1)(1-j_2) - u_1(1-j_1)((1-j_2) - u_2(1-j_1)(1+j_2) - u_3(1+j_1)(1+j_2))\times$$

$$\times(-v_0(1+j_1)(1-j_2) - v_1(1-j_1)((1-j_2) - v_2(1-j_1)(1+j_2) - v_3(1+j_1)(1+j_2)) =$$

$$= 4u_0v_0(1+j_1)(1-j_2) + 4u_1v_1(1-j_1)(1-j_2) +$$

$$+4u_2v_2(1-j_1)(1+j_2) + 4u_3v_3(1+j_1)(1+j_2),$$

i.e.

$$4XY = u_0v_0(1+j_1)(1-j_2) + u_1v_1(1-j_1)(1-j_2) +$$

$$+u_2v_2(1-j_1)(1+j_2) + u_3v_3(1+j_1)(1+j_2),$$

where (u_0, u_1, u_2, u_3) and (v_0, v_1, v_2, v_3) are the coordinates of the hyperbolic bicomplex numbers X and Y, respectively in the considered coordinate system.

6. Other decomposition of the algebra of hyperbolic bicomplex numbers and rule of multiplication in the base of morphisms

Theorem 6.1. *The idempotent elements* $A = \frac{1}{4}(3 - j_1 - j_2 - j_1 j_2)$, $B = \frac{1}{4}(3 - j_1 + j_2 + j_1 j_2)$, $C = \frac{1}{4}(3 + j_1 - j_2 + j_1 j_2)$ *and* $D = \frac{1}{4}(3 + j_1 + j_2 - j_1 j_2)$ *are barycentric coordinates of the tetrahedron* $ABCD$ *in* $\mathbf{R}(j_1, j_2)$.

It is true that $AB = \frac{1}{2}(1 - j_1)$, $AC = \frac{1}{2}(1 - j_2)$, $AD = \frac{1}{2}(1 - j_1 j_2)$, $BC = \frac{1}{2}(1 + j_1 j_2)$, $BD = \frac{1}{2}(1 + j_2)$, $CD = \frac{1}{2}(1 + j_1)$.

The following identities holds $\frac{1}{2}(AB + AC + BD + CD) = 1$, $\frac{1}{2}(AB + CD + AD + BC) = 1$, $\frac{1}{2}(AC + BD + AD + BC) = 1$.

We omit the proof of this theorem.

Let us transform the coordinate system of the underlying 4-dimensional vector space of the algebra of the hyperbolic bicomplex numbers as follows

$$\left| \begin{array}{l} u_0 = -3x_0 + x_1 + x_2 + x_3, \\ u_1 = -3x_0 + x_1 - x_2 - x_3, \\ u_2 = -3x_0 - x_1 + x_2 - x_3, \\ u_3 = -3x_0 - x_1 - x_2 + x_3. \end{array} \right. \tag{6.1}$$

The system of equations (6.1), solved with respect to x_0, x_1, x_2, x_3, is the following one

$$\left| \begin{array}{l} 12x_0 = -u_0 - u_1 - u_2 - u_3, \\ 4x_1 = u_0 + u_1 - u_2 - u_3, \\ 4x_2 = u_0 - u_1 + u_2 - u_3, \\ 4x_3 = u_0 - u_1 - u_2 + u_3. \end{array} \right.$$

The Jacobi matrix of the transformation of the variables (x_0, x_1, x_2, x_3) to the variables (u_0, u_1, u_2, u_3) is the following one

$$\frac{\partial(u_0, u_1, u_2, u_3)}{\partial(x_0, x_1, x_2, x_3)} = \begin{pmatrix} -3 & 1 & 1 & 1 \\ -3 & 1 & -1 & -1 \\ -3 & -1 & 1 & -1 \\ -3 & -1 & -1 & 1 \end{pmatrix}.$$

In this coordinate system the representation of the hyperbolic bicomplex number $X = x_0 + j_1 x_1 + j_2 x_2 + j_1 j_2 x_3$ will be the following one

$$4X = 3(-u_0 - u_1 - u_2 - u_3) + j_1(u_0 + u_1 - u_2 - u_3) +$$

$$+ j_2(u_0 - u_1 + u_2 - u_3) + j_1 j_2(u_0 - u_1 - u_2 + u_3) =$$

$$= u_0(-3 + j_1 + j_2 + j_1 j_2) + u_1(-3 + j_1 - j_2 - j_1 j_2) +$$

$$+ u_2(-3 - j_1 + j_2 - j_1 j_2) + u_3(-3 - j_1 - j_2 + j_1 j_2),$$

where $u_0, u_1, u_2, u_3 \in \mathbf{R}$ are real numbers.

Example 6.1. It is true that

$$1 = -\frac{1}{12}((-3 + j_1 + j_2 + j_1 j_2) + (-3 + j_1 - j_2 - j_1 j_2)+$$
$$+(-3 - j_1 + j_2 - j_1 j_2) + (-3 - j_1 - j_2 + j_1 j_2)).$$

So we obtain the following

Theorem 6.2. *The following four mappings*

$$r_0 = -\frac{1}{12}(-3 + j_1 + j_2 + j_1 j_2), \quad r_1 = -\frac{1}{12}(-3 + j_1 - j_2 - j_1 j_2)$$
$$r_3 = -\frac{1}{12}(-3 - j_1 + j_2 - j_1 j_2), \quad r_3 = -\frac{1}{12}(-3 - j_1 - j_2 + j_1 j_2)$$

decompose the algebra of hyperbolic bicomplex numbers as a sum of one dimensional subalgebras as follows

$$\mathbf{R}(j_1, j_2) \approx r_0\mathbf{R}(j_1, j_2) \oplus r_1\mathbf{R}(j_1, j_2) \oplus r_2\mathbf{R}(j_1, j_2) \oplus r_3\mathbf{R}(j_1, j_2).$$

Moreover it is true that

$$r_0 \oplus r_1 = 1/2(1 - j_2), \quad r_0 \oplus r_2 = 1/2(1 - j_1 j_2), \quad r_0 \oplus r_3 = 1/2(1 + j_1),$$
$$r_1 \oplus r_2 = 1/2(1 - j_1), \quad r_1 \oplus r_3 = 1/2(1 + j_1 j_2), \quad r_2 \oplus r_3 = 1/2(1 + j_2),$$

Ten morphisms in this theorem and the morphisms ρ_0, ρ_1, ρ_2, ρ_4 from Theorem **5.1***, 0 and 1 give all idempotent elements of the algebra of hyperbolic bicomplex numbers.*

Let us consider two hyperbolic bicomplex numbers X, Y,

$$X = x_0 + j_1 x_1 + j_2 x_2 + j_1 j_2 x_3$$

and

$$Y = y_0 + j_1 y_1 + j_2 y_2 + j_1 j_2 y_3.$$

Let us represent it in the considered in this point coordinate system. We obtain

$$4X = u_0(-3 + j_1 + j_2 + j_1 j_2) + u_1(-3 + j_1 - j_2 - j_1 j_2)+$$
$$+u_2(-3 - j_1 + j_2 - j_1 j_2) + u_3(-3 - j_1 - j_2 + j_1 j_2),$$

and

$$4Y = v_0(-3 + j_1 + j_2 + j_1 j_2) + v_1(-3 + j_1 - j_2 - j_1 j_2)+$$
$$+v_2(-3 - j_1 + j_2 - j_1 j_2) + v_3(-3 - j_1 - j_2 + j_1 j_2).$$

Let us complute the product

$$16XY = (u_0(-3 + j_1 + j_2 + j_1 j_2) + u_1(-3 + j_1 - j_2 - j_1 j_2)+$$
$$+u_2(-3 - j_1 + j_2 - j_1 j_2) + u_3(-3 - j_1 - j_2 + j_1 j_2))\times$$
$$\times(v_0(-3 + j_1 + j_2 + j_1 j_2) + v_1(-3 + j_1 - j_2 - j_1 j_2)+$$

$$+v_2(-3-j_1+j_2-j_1j_2)+v_3(-3-j_1-j_2+j_1j_2)) =$$
$$= 4u_0v_0(-3+j_1+j_2+j_1j_2)+4u_1v_1(-3+j_1-j_2-j_1j_2)+$$
$$+4u_2v_2(-3-j_1+j_2-j_1j_2)+4u_3v_3(-3-j_1-j_2+j_1j_2),$$

i.e.

$$4XY = u_0v_0(-3+j_1+j_2+j_1j_2)+u_1v_1(-3+j_1-j_2-j_1j_2)+$$
$$+u_2v_2(-3-j_1+j_2-j_1j_2)+u_3v_3(-3-j_1-j_2+j_1j_2),$$

where (u_0, u_1, u_2, u_3) and (v_0, v_1, v_2, v_3) are the coordinates of the hyperbolic bicomplex numbers X and Y, respectively in the considered coordinate system in $\mathbf{R}(j_1, j_2)$.

7. Matrix representation of bicomplex number

Let us recall the definition of the algebra of bicomplex numbers $\mathbf{R}(j_1, j_2)$, $j_1^2 = j_2^2 = -1$ (see [8], [9]). It is defined as follows

$$\mathbf{R}(j_1, j_2) = \{x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3 : j_1^2 = j_2^2 = -1, j_1j_2 = j_2j_1\}, \qquad (7.1)$$

where $x_0, x_1, x_2, x_3 \in \mathbf{R}$ are real numbers.

Remark 7.1. Here, for the sake of simplicity, we use the same notation $\mathbf{R}(j_1, j_2)$ for the algebras of hyperbolic complex and the algebra of bicomplex numbers, but the equations for the units j_1 and j_2 are different ones in these two cases. It can be used also the notations are $\mathbf{R}(j_1, j_2; j_1^2 = j_2^2 = 1)$ and $\mathbf{R}(j_1, j_2; j_1^2 = j_2^2 = -1)$, respectively.

The addition and the multiplication by real scalar are defined componentwise, and the multiplication of elements of the algebra is defined by opening the brackets and using the identities of the units j_1 and j_2. The algebra $\mathbf{R}(j_1, j_2)$ is an associative, commutative algebra with zero divisors. So is for example the number $1 + j_1j_2$. Indeed the product of such a number with $1 - j_1j_2$ is equal to zero. The disributive rule holds.

The multiplicative table is the following

	1	j_1	j_2	j_1j_2
1	1	j_1	j_2	j_1j_2
j_1	j_1	-1	j_1j_2	$-j_2$
j_2	j_2	j_2j_1	-1	$-j_1$
j_1j_2	j_1j_2	$-j_2$	$-j_1$	1

Table of multiplication of bicomplex units

Let us consider the following three 4×4 matrices with real coefficients

$$J_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \; J_2 := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$J_1 J_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is fulfilled $J_1^2 = -E$, $J_2^2 = -E$ and $J_1 J_2 = J_2 J_1$, $(J_1 J_2)^2 = E$, where E is the identity 4×4 matrix. The matrices E, J_1, J_2 and $J_1 J_2$ generate a commutative subalgebra $\mathfrak{M}$ of the full matrix algebra $M(4, \mathbf{R})$ of all 4×4 matrices with real elements, $\mathfrak{M} \hookrightarrow M(4, \mathbf{R})$.

We shall define a homomorphism of the algebra $\mathbf{R}(j_1, j_2)$ of the bicomplex numbers in the algebra $\mathfrak{M}$ as follows

$$X = x_0 + j_1 x_1 + j_2 x_2 + j_1 j_2 x_3 \mapsto x_0 E + x_1 J_1 + x_2 J_2 + x_3 J_1 J_2 = M_X.$$

Then the obtained image of the bicomplex number X is the following matrix $M_X \in \mathfrak{M}$

$$M_X = \begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & -x_3 & x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ x_3 & -x_2 & -x_1 & x_0 \end{pmatrix}.$$

This is a matrix, which is symmetric with respect to the second diagonal. The following identity is true

$$(E + J_1 J_2) \cdot (E - J_1 J_2) =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0,$$

i.e. the matrices $E + J_1 J_2$ and $E - J_1 J_2$ are zero divisors in $\mathfrak{M}$.

8. Determinant of the matrix representation of bicomplex number

We obtain the following equalities for the determinant of the representing matrix M_X of the bicomplex number $X = x_0 + j_1 x_1 + j_2 x_2 + j_1 j_2 x_2$

$$D_+(x_0, x_1, x_2, x_3) = \begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & -x_3 & x_2 \\ -x_2 & -x_3 & x_0 & x_1 \\ x_3 & -x_2 & -x_1 & x_0 \end{vmatrix} = \qquad (8.1)$$

$$= x_0 \begin{vmatrix} x_0 & -x_3 & x_2 \\ -x_3 & x_0 & x_1 \\ -x_2 & -x_1 & x_0 \end{vmatrix} - x_1 \begin{vmatrix} -x_1 & -x_3 & x_2 \\ -x_2 & x_0 & x_1 \\ x_3 & -x_1 & x_0 \end{vmatrix} +$$

$$+x_2 \begin{vmatrix} -x_1 & x_0 & x_2 \\ -x_2 & -x_3 & x_1 \\ x_3 & -x_2 & x_0 \end{vmatrix} - x_3 \begin{vmatrix} -x_1 & x_0 & -x_3 \\ -x_2 & -x_3 & x_0 \\ x_3 & -x_2 & -x_1 \end{vmatrix} =$$

$$= (x_0^2 + x_1^2)^2 - 2x_0^2x_3^2 + 2x_2^2x_0^2 +$$

$$+2x_1^2x_3^2 - 2x_1^2x_2^2 + (x_2^2 + x_3^2)^2 + 8x_3x_0x_2x_1 =$$

$$= (x_0^2 + x_1^2)^2 + 2(x_0^2 - x_1^2)(x_2^2 - x_3^2) + (x_2^2 + x_3^2)^2 + 8x_3x_0x_2x_1 =$$

$$= (x_0^2 - x_1^2)^2 + 2(x_0^2 - x_1^2)(x_2^2 - x_3^2) + (x_2^2 - x_3^2)^2 +$$

$$+4x_0^2x_1^2 + 4x_2^2x_3^2 + 8x_3x_0x_2x_1 =$$

$$= (x_0^2 - x_1^2 + x_2^2 - x_3^2)^2 + 4x_0^2x_1^2 + 4x_2^2x_3^2 + 8x_3x_0x_2x_1 =$$

$$= (x_0^2 - x_1^2 + x_2^2 - x_3^2)^2 + 4(x_0x_1 + x_2x_3).$$

This is the determinant of a bicomplex number. This can be considered as an elliptic case of the four dimensional commutative non-division algebras.

Let us decompose the determinant $D_+(x_0, x_1, x_2, x_3)$ in a product of four bicomplex numbers as follows

$$D_+(x_0, x_1, x_2, x_3) = (x_0^2 - x_1^2 + x_2^2 - x_3^2)^2 + 4(x_0x_1 + 4x_2x_3)^2 =$$

$$= (x_0^2 - x_1^2 + x_2^2 - x_3^2)^2 - (2j_1x_0x_1 + 2j_1x_2x_3)^2 =$$

$$= (x_0^2 + j_1^2x_1^2 + x_2^2 + j_1^2x_3^2 + 2j_1x_0x_1 + 2j_1x_2x_3)\times$$

$$\times(x_0^2 + j_1^2x_1^2 + x_2^2 + j_1^2x_3^2 - 2j_1x_0x_1 - 2j_1x_2x_3) =$$

$$= ((x_0 + j_1x_1)^2 - j_2^2(x_2 + j_1x_3)^2)((x_0 - j_1x_1)^2 - j_2^2(x_2 - j_1x_3)^2) =$$

$$= (x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3)(x_0 + j_1x_1 - j_2x_2 - j_1j_2x_3)\times$$

$$\times(x_0 - j_1x_1 + j_2x_2 - j_1j_2x_3)(x_0 - j_1x_1 - j_2x_2 + j_1j_2x_3).$$

Definition 8.1. The numbers $x_0 + j_1x_1 - j_2x_2 - j_1j_2x_3$, $x_0 - j_1x_1 + j_2x_2 - j_1j_2x_3$ and $x_0 - j_1x_1 - j_2x_2 + j_1j_2x_3$ are called conjugate numbers of the bicomplex number $X = x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$.

9. Invertible bicomplex numbers, idempotent bicomplex numbers, base of four idempotent elements in the algebra of bicomplex numbers and multiplication rule of bicomplex numbers in this base

In this section we shall delivered some properties of the bicomplex numbers and their algebra.

9.1. Invertible bicomplex numbers

Theorem 9.1. *The determinant $D_+(x_0, x_1, x_2, x_3)$ annihilate on the set in $\mathbf{R}^4$ as follows: this is the union $\alpha_0 \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$ of the planes, where*

$$\alpha_0 : \left| \begin{array}{l} x_0 = s, \\ x_1 = -s, \\ x_2 = t, \\ x_3 = t, \end{array} \right. \quad \alpha_1 : \left| \begin{array}{l} x_0 = s, \\ x_1 = t, \\ x_2 = -t, \\ x_3 = s, \end{array} \right. \quad \alpha_2 : \left| \begin{array}{l} x_0 = s, \\ x_1 = t, \\ x_2 = t, \\ x_3 = -s. \end{array} \right. \quad \alpha_3 : \left| \begin{array}{l} x_0 = s, \\ x_1 = s, \\ x_2 = t, \\ x_3 = t. \end{array} \right.$$

The invertible elements of the algebra of bicomplex numbers are these bicomplex numbers $X = x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$, which have the corresponding point (x_0, x_1, x_2, x_3) not belonging to $\alpha_0 \cup \alpha_1 \cup \alpha_2 \cup \alpha_3$.

9.2. Idempotent bicomplex numbers

Using the commutative and the distributive rules for the multiplication of the bicomplex numbers $X = x_0+j_1x_1+j_2x_2+j_1j_2x_3$ and $Y = y_0+j_1y_1+j_2y_2+j_1j_2y_3$ we find that

$$X \cdot Y = (x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3)(y_0 + j_1y_1 + j_2y_2 + j_1j_2y_3) =$$
$$= x_0y_0 - x_1y_1 - x_2y_2 + x_3y_3 + j_1(x_0y_1 + x_1y_0 - x_2y_3 - x_3y_2)+$$
$$+j_2(x_0y_2 - x_1y_3 + x_2y_0 - x_3y_1) + j_1j_2(x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0).$$

This equality gives a bilinear form

$$X \cdot Y : \mathbf{R}(j_1, j_2) \times \mathbf{R}(j_1, j_2) \to \mathbf{R}(j_1, j_2); j_1^2 = j_2^2 = -1.$$

Example 9.1. The square X^2 of the bicomplex number $X = x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$ is the following bicomplex number

$$X^2 = x_0^2 - x_1^2 - x_2^2 + x_3^2 + 2j_1(x_0x_1 - x_2x_3)+$$
$$+2j_2(x_0x_2 - x_1x_3) + 2j_1j_2(x_0x_3 + x_1x_2).$$

Theorem 9.2. *The idempotent elements of the algebra of bicomplex numbers are the bicomplex numbers 0, 1, $\frac{1}{2}(1 + j_1j_2)$, $\frac{1}{2}(1 - j_1j_2)$.*

9.3. Decomposition of the algebra of bicomplex numbers and multiplication rule of bicomplex numbers in the base of morphisms

Example 9.2. It is true that

$$1 = \frac{1}{4}((1 + j_1)(1 - j_2) + (1 - j_1)(1 - j_2) + (1 - j_1)(1 + j_2) + (1 + j_1)(1 + j_2)).$$

It is true the following theorem

Theorem 9.3. *The following four mappings*

$$\pi_0 = 1/4(1 + j_1)(1 - j_2), \quad \pi_1 = 1/4(1 - j_1)(1 - j_2),$$

$$\pi_2 = 1/4(1 - j_1)(1 + j_2), \quad \pi_3 = 1/4(1 + j_1)(1 + j_2)$$

decompose the algebra of bicomplex numbers to four one dimensional real vector subspaces of the algebra

$$\mathbf{R}(j_1, j_2) \approx \pi_0\mathbf{R}(j_1, j_2) \oplus \pi_2\mathbf{R}(j_1, j_2) \oplus \pi_1\mathbf{R}(j_1, j_2) \oplus \pi_3\mathbf{R}(j_1, j_2).$$

Moreover it is true that

$$\pi_0 \oplus \pi_2 = 1/2(1 - j_1j_2), \quad \pi_1 \oplus \pi_3 = 1/2(1 + j_1j_2),$$

i.e. these sums, 0 and 1 are the idempotent elements of the algebra of bicomplex numbers.

Let us consider two hyperbolic bicomplex numbers $X, Y \in \mathbf{R}(j_1, j_2)$,

$$X = x_0 + j_1x_1 + j_2x_2 + j_1j_2x_3$$

and

$$Y = y_0 + j_1y_1 + j_2y_2 + j_1j_2y_3.$$

Let us consider their representation in the coordinate system above. Then

$$4X = -u_0(1+j_1)(1-j_2)-u_1(1-j_1)((1-j_2)-u_2(1-j_1)(1+j_2)-u_3(1+j_1)(1+j_2),$$

and

$$4Y = -v_0(1+j_1)(1-j_2)-v_1(1-j_1)((1-j_2)-v_2(1-j_1)(1+j_2)-v_3(1+j_1)(1+j_2).$$

Let us compute the product

$$16XY =$$

$$= (-u_0(1+j_1)(1-j_2)-u_1(1-j_1)((1-j_2)-u_2(1-j_1)(1+j_2)-u_3(1+j_1)(1+j_2))\times$$

$$\times(-v_0(1+j_1)(1-j_2)-v_1(1-j_1)((1-j_2)-v_2(1-j_1)(1+j_2)-v_3(1+j_1)(1+j_2)) =$$

$$= 4u_0v_0(1 + j_1)(1 - j_2) + 4u_1v_1(1 - j_1)(1 - j_2)+$$

$$+4u_2v_2(1 - j_1)(1 + j_2) + 4u_3v_3(1 + j_1)(1 + j_2),$$

i.e.

$$4XY = u_0v_0(1 + j_1)(1 - j_2) + u_1v_1(1 - j_1)(1 - j_2)+$$

$$+u_2v_2(1 - j_1)(1 + j_2) + u_3v_3(1 + j_1)(1 + j_2).$$

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PROPERTIES OF THE FUNDAMENTAL AND THE IMPULSE-RESPONSE SOLUTIONS OF MULTI-TERM FRACTIONAL DIFFERENTIAL EQUATIONS

Emilia Bazhlekova

Abstract

We study the multi-term fractional differential equation

$$(D_*^{\alpha} u)(t) + \sum_{j=1}^{m} \lambda_j (D_*^{\alpha_j} u)(t) + \lambda u(t) = f(t), \quad t > 0; \quad u(0) = c_0;$$

where D_*^{α} is the fractional derivative operator in the Caputo sense,

$$0 < \alpha_m < ... < \alpha_1 < \alpha \leq 1, \quad \lambda, \lambda_j > 0, \ j = 1, ..., m, \ m \in \mathbb{N} \cup 0.$$

This equation is a generalization of the classical relaxation equation, obtained for $m = 0, \alpha = 1$, and governs some fractional relaxation processes. Applying Laplace transform method, we find the fundamental and the impulse-response solutions of the equation, corresponding to $f(t) \equiv 0, c_0 = 1$, and $f(t) \equiv \delta(t), c_0 = 0$, respectively, where $\delta(t)$ is the Dirac delta function. The properties of the solutions are derived directly from their representations as Laplace inverse integrals. We prove that the fundamental and the impulse-response solutions are completely monotone functions and find their asymptotic expressions for small and large times. It appears that the asymptotic behaviour of the solutions for $t \to 0$ is determined by the largest order of fractional differentiation α and for $t \to \infty$ by the smallest order α_m. In all cases an algebraic decay is observed for $t \to \infty$. Some useful estimates for the solutions are also obtained. In the limiting case $m = 0$, in which the solutions can be expressed in terms of the Mittag-Leffler functions, some well-known properties of these functions are recovered from our results.

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Key Words and Phrases: Fractional calculus; fractional relaxation; Caputo derivative; Laplace transform; Mittag-Leffler function; completely monotone function.

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1. Introduction

We study the multi-term ordinary fractional differential equation

$$(D_*^{\alpha} u)(t) + \sum_{j=1}^{m} \lambda_j (D_*^{\alpha_j} u)(t) + \lambda u(t) = f(t), \quad t > 0, \tag{1.1}$$

subject to the initial condition $u(0) = c_0$, where D_*^{α} denotes the fractional derivative operator in the Caputo sense,

$$0 < \alpha_m < ... < \alpha_1 < \alpha \leq 1, \quad \lambda, \lambda_j > 0, \; j = 1, ..., m, \; m \in \mathbb{N} \cup 0, \tag{1.2}$$

$f(t)$ is a given sufficiently well-behaved function and $c_0 \in \mathbb{R}$ is a given constant. If $m = 0$ we set as usual $\sum_{j=1}^{0} \equiv 0$.

Let $u_0(t)$ and $u_\delta(t)$ be the fundamental and the impulse-response solutions of equation (1.1), i.e. corresponding to the data $f(t) \equiv 0, c_0 = 1$, and $f(t) \equiv \delta(t), c_0 = 0$, respectively, where $\delta(t)$ is the Dirac delta function.

Equation (1.1) is a generalization of the classical exponential relaxation equation $u'(t) + \lambda u(t) = 0$. Generalizations of this equation from the viewpoint of Fractional Calculus are discussed in [9] and equation (1.1) is in fact a particular case of the distributed order fractional relaxation equation in the Caputo sense, considered in [9]. The solutions of (1.1), represented in terms of the so-called multivariate Mittag-Leffler functions, is obtained in [8]. It seems however that the properties of the functions $u_0(t)$ and $u_\delta(t)$ have not been studied in detail in the literature.

On the other hand, except as solutions of the relaxation equation (1.1), the functions $u_0(t)$ and $u_\delta(t)$ appear as time-dependent components in the eigenfunction expansion of the solution of multi-term time-fractional diffusion equations on finite space domain, considered e.g. in [7, 2, 4, 6]. Thus, the knowledge of the behaviour of these functions is essential in the proof of the convergence of the series representing the solution and in obtaining regularity estimates for the solution.

In this paper we study the behaviour of the functions $u_0(t)$ and $u_\delta(t)$, based on their representations as Laplace inverse integrals.

2. Preliminaries

Here we formulate some definitions and basic facts from Fractional Calculus (see e.g. [1] or [5] for more details).

Let $\gamma \in (0, 1]$. Denote by J^γ the fractional Riemann-Liouville integral:

$$(J^\gamma f)(t) := \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^t (t-\tau)^{\gamma-1} f(\tau)\, d\tau, & 0 < \gamma < 1, \\ f(t), & \gamma = 0, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function. Denote by D^γ and D_*^γ the Riemann-Liouville and the Caputo fractional derivatives of order γ, defined by

$$(D^\gamma f)(t) := (J^{1-\gamma} f)'(t), \quad (D_*^\gamma f)(t) := (J^{1-\gamma} f')(t).$$

Note that for $\gamma = 1$ it holds $D^1 = D_*^1 = d/dt$.

The Laplace transform of a function $f(t)$ is denoted by $\mathcal{L}\{f(t)\}(s)$ or $\widehat{f}(s)$. The Laplace transforms of the operators of fractional integration and differentiation are given by

$$\mathcal{L}\{J^\gamma f\}(s) = s^{-\gamma} \widehat{f}(s), \tag{2.1}$$

$$\mathcal{L}\{D^\gamma f\}(s) = s^\gamma \widehat{f}(s) - (J^{1-\gamma} f)(0), \tag{2.2}$$

$$\mathcal{L}\{D_*^\gamma f\}(s) = s^\gamma \widehat{f}(s) - s^{\gamma-1} f(0). \tag{2.3}$$

Recall the definition of the Mittag-Leffler function:

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad E_\alpha(z) := E_{\alpha,1}(z), \quad \alpha, \beta, z \in \mathbb{C}, \ \Re\alpha > 0. \tag{2.4}$$

Its asymptotic expansion is given by

$$E_{\alpha,\beta}(-t) = -\sum_{k=1}^{N-1} \frac{(-t)^{-k}}{\Gamma(\beta - \alpha k)} + O(t^{-N}), \quad t \to +\infty. \tag{2.5}$$

The following identity for the Laplace transform of the Mittag-Leffler function holds true

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(-\mu t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha + \mu}. \tag{2.6}$$

A function $f(t)$ is said to be completely monotone for $t \geq 0$ iff

$$(-1)^n f^{(n)}(t) \geq 0, \text{ for all } n = 0, 1, ..., t \geq 0.$$

3. Properties of the fundamental and the impulse-response solutions

To solve equation (1.1) we apply Laplace transform and use (2.3). In this way, for the Laplace transform $\widehat{u}(s)$ of the solution $u(t)$ one gets

$$\widehat{u}(s) = c_0 \frac{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j}}{s(s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda)} + \frac{1}{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda} \widehat{f}(s). \tag{3.1}$$

Therefore the solution of equation (1.1) is given by

$$u(t) = c_0 u_0(t) + \int_0^t u_\delta(t - \tau) f(\tau)\, d\tau. \tag{3.2}$$

Here $u_0(t)$ is the fundamental solution (corresponding to $f(t) \equiv 0, c_0 = 1$) and $u_\delta(t)$ is the impulse-response solution (corresponding to $f(t) \equiv \delta(t), c_0 = 0$) and they are defined by their Laplace transforms as follows:

$$\widehat{u}_0(s) = \frac{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j}}{s(s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda)}, \quad \widehat{u}_\delta(s) = \frac{1}{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda}. \tag{3.3}$$

Next we find representations of $u_0(t)$ and $u_\delta(t)$ and study their properties. We work following the method proposed in [3]. Note that in the limiting case $m = 0$ (simple fractional relaxation) (3.3) and (2.6) imply that $u_0(t)$ and $u_\delta(t)$ are expressed explicitly in terms of the Mittag-Leffler functions:

$$u_0(t) = E_\alpha(-\lambda t^\alpha), \quad u_\delta(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha). \tag{3.4}$$

If moreover $\alpha = 1$ the solutions reduce to $u_0(t) = u_\delta(t) = exp(-\lambda t)$ (classical exponential relaxation). Therefore, our results generalize some well-known properties of the Mittag-Leffler and the exponential functions. For example, the functions u_0 and u_δ preserve the complete monotonicity property of the exponential function. In order to prove this, first we find appropriate representations of these functions.

Theorem 3.1. *The functions $u_0(t)$ and $u_\delta(t)$ have the representations:*

$$u_0(t) = \int_0^\infty e^{-rt} K_0(r)\, dr, \ u_\delta(t) = \int_0^\infty e^{-rt} K_\delta(r)\, dr, \ \textit{where} \tag{3.5}$$

$$K_0(r) = \frac{\lambda}{\pi r} \frac{B(r)}{(A(r)+\lambda)^2 + (B(r))^2}, \ K_\delta(r) = \frac{1}{\pi} \frac{B(r)}{(A(r)+\lambda)^2 + (B(r))^2} \tag{3.6}$$

$$A(r) = r^\alpha \cos \alpha\pi + \sum_{j=1}^m \lambda_j r^{\alpha_j} \cos \alpha_j \pi, \ B(r) = r^\alpha \sin \alpha\pi + \sum_{j=1}^m \lambda_j r^{\alpha_j} \sin \alpha_j \pi,$$

Proof. The function $u_\delta(t)$ is the inverse Laplace integral of $\widehat{u}_\delta(s)$, i.e.

$$u_\delta(t) = \frac{1}{2\pi i} \int_{Br} e^{st} \frac{1}{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda}\, ds, \tag{3.7}$$

where $Br = \{s;\ \mathrm{Re}\, s = \sigma,\ \sigma > 0\}$ is the Bromwich path. The function $\widehat{u}_\delta(s)$ has a branch point 0, so we cut off the negative part of the real axis. Note that the function $s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda$ has no zero in the main sheet of the Riemann surface including its boundaries on the cut. Indeed, if $s = \varrho e^{i\theta}$, with $\varrho > 0$, $\theta \in (-\pi, \pi)$, then

$$\mathrm{Im}\, \{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda\} = \varrho^\alpha \sin \alpha\theta + \sum_{j=1}^m \lambda_j \varrho^{\alpha_j} \sin \alpha_j \theta \neq 0,$$

since $\sin\alpha\theta$ and $\sin\alpha_j\theta$ have the same sign and $\lambda_j > 0$. Therefore, $u_\delta(t)$ can be found by bending the Bromwich path into the Hankel path $Ha(\varepsilon)$, which starts from $-\infty$ along the lower side of the negative real axis, encircles the disc $|s| = \varepsilon$ counterclockwise and ends at $-\infty$ along the upper side of the negative real axis:

$$u_\delta(t) = \frac{1}{2\pi i}\int_{Ha(\varepsilon)} e^{st}\frac{1}{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda}\, ds. \tag{3.8}$$

Taking $\varepsilon \to 0$ in (3.8) we obtain

$$u_\delta(t) = \int_0^\infty e^{-rt} K_\delta(r)\, dr, \tag{3.9}$$

where

$$K_\delta(r) = -\frac{1}{\pi}\mathrm{Im}\left\{ \left.\frac{1}{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda}\right|_{s=re^{i\pi}} \right\}. \tag{3.10}$$

From (3.10) we obtain the representation of $K_\delta(r)$ in (3.6).

Applying the same argument, we find the representation for $u_0(t)$. By the use of the relationship

$$\frac{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j}}{s(s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda)} = \frac{1}{s}\left(1 - \frac{\lambda}{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda}\right) \tag{3.11}$$

we get

$$K_0(r) = -\frac{1}{\pi}\mathrm{Im}\left\{ \left.\frac{s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j}}{s(s^\alpha + \sum_{j=1}^m \lambda_j s^{\alpha_j} + \lambda)}\right|_{s=re^{i\pi}} \right\} = \frac{\lambda}{r}K_\delta(r). \tag{3.12}$$

□

Remark 3.1. Representations (3.5) are appropriate for numerical computation of the solutions.

Remark 3.2. Note that $K_0(r)$ is a probability density function, since it is positive and $\int_0^\infty K_0(r)\, dr = 1$ in virtue of (3.5) and (3.18).

Remark 3.3. Representations (3.5) do not hold only in the limiting case of exponential relaxation $\alpha = 1, m = 0$, in which $B(r) \equiv 0$.

To find the asymptotic expansions of $u_0(t)$ and $u_\delta(t)$ as $t \to \infty$ we apply the following lemma to the representations in (3.5).

Lemma 3.1. *(Watson's lemma) Let $K \in L^1(\mathbb{R}^+)$ and assume*

$$K(r) \sim r^\beta, \ \text{as } r \to 0^+,$$

with $\operatorname{Re}\{\beta\} > -1$. *Then*

$$\int_0^\infty e^{-rt} K(r)\, dr \sim \Gamma(\beta+1) t^{-\beta-1}, \ t \to +\infty. \tag{3.13}$$

Theorem 3.2. *The functions $u_0(t)$ and $u_\delta(t)$ have the following properties:*

$u_0(t)$ *and* $u_\delta(t)$ *are completely monotone functions for* $t \geq 0$, (3.14)

$$u_0'(t) = -\lambda u_\delta(t), \quad t > 0, \tag{3.15}$$

$$u_0(t) \sim 1 - \lambda \frac{t^\alpha}{\Gamma(\alpha+1)}, \quad u_\delta(t) \sim \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t \to 0, \tag{3.16}$$

$$u_0(t) \sim \frac{\lambda_m t^{-\alpha_m}}{\lambda \Gamma(1-\alpha_m)}, \quad u_\delta(t) \sim -\frac{\lambda_m t^{-\alpha_m - 1}}{\lambda^2 \Gamma(-\alpha_m)}, \quad t \to +\infty. \tag{3.17}$$

Proof. Because of the assumptions (1.2) $K_0(r) > 0$ and $K_\delta(r) > 0$ for all $r > 0$. This together with the representations (3.5) implies that $u_0(t)$ and $u_\delta(t)$ are completely monotone functions.

By the property of the Laplace transform

$$f(0) = \lim_{s \to +\infty} s \widehat{f}(s),$$

we obtain from (3.3)

$$u_0(0) = 1. \tag{3.18}$$

Therefore, (2.3) with $\gamma = 1$ and (3.11) imply

$$\mathcal{L}\{u_0'(t)\}(s) = s\widehat{u}_0(s) - u_0(0) = s\widehat{u}_0(s) - 1 = -\lambda \widehat{u}_\delta(s)$$

and, taking the inverse Laplace transform, we obtain identity (3.15).

The behaviour of the solutions as $t \to 0$ can be determined from the behaviour of their Laplace transforms as $\operatorname{Re}\{s\} \to +\infty$, as it is well known from the Tauberian theory for the Laplace transform. Then (3.16) follow from the expansions for $\operatorname{Re}\{s\} \to +\infty$

$$\widehat{u}_\delta(s) = \frac{1}{s^\alpha (1 + \sum_{j=1}^m \lambda_j s^{\alpha_j - \alpha} + \lambda s^{-\alpha})} = s^{-\alpha} + O(|s|^{\alpha_1 - 2\alpha}),$$

$$\widehat{u}_0(s) = \frac{1}{s} \left(1 - \frac{\lambda}{s^\alpha (1 + \sum_{j=1}^m \lambda_j s^{\alpha_j - \alpha} + \lambda s^{-\alpha})} \right) = s^{-1} - \lambda s^{-\alpha-1} + O(|s|^{\alpha_1 - 2\alpha - 1}),$$

and

$$\mathcal{L}^{-1}\{s^{-\gamma}\} = t^{\gamma-1}/\Gamma(\gamma), \ \gamma > 0.$$

The asymptotic estimates (3.17) are obtained applying the Watson's lemma to the representations (3.5). Since the dominant terms in the expansions of

$K_0(r)$ and $K_\delta(r)$ for small r are

$$K_0(r) \sim \frac{1}{\pi\lambda}\lambda_m r^{\alpha_m - 1}\sin\alpha_m\pi, \;\; K_\delta(r) \sim \frac{1}{\pi\lambda^2}\lambda_m r^{\alpha_m}\sin\alpha_m\pi, \;\; r \to 0^+$$

applying Lemma **3.1** to (3.5) we obtain (3.17) by the use of the identities (3.13) and

$$\frac{\sin\gamma\pi}{\pi} = \frac{1}{\Gamma(\gamma)\Gamma(1-\gamma)}.$$

□

Remark 3.4. We see that the leading terms of the asymptotic expansions for $t \to 0$ are the same as in the case $m = 0$, see (3.4) and (2.4). In contrast, for $t \to \infty$, the leading terms depend on α_m. Again, if $m = 0$ then for the asymptotic expansions for $t \to \infty$ we should take $\alpha_m = \alpha, \lambda_m = 1$ in (3.17), and obtain

$$u_0(t) \sim \frac{t^{-\alpha}}{\lambda\Gamma(1-\alpha)}, \quad u_\delta(t) \sim -\frac{t^{-\alpha-1}}{\lambda^2\Gamma(-\alpha)}, \quad t \to +\infty.$$

The same result could be obtained applying (2.5) to (3.4).

Theorem **3.2** implies that the functions $u_0(t)$ and $u_\delta(t)$ are positive and strictly decreasing towards 0 as t runs from 0 to $+\infty$. Some useful estimates implied by Theorem **3.2** are formulated in the next

Corollary 3.1. *The following estimates hold true:*

$$0 < u_0(t) < 1, \;\; u_\delta(t) > 0, \quad t > 0, \tag{3.19}$$

$$\int_0^T u_\delta(t)\,dt < \frac{1}{\lambda}, \quad T > 0, \tag{3.20}$$

$$u_0(t) \le \frac{M}{1+\lambda t^{\alpha_m}}, \quad t \ge 0, \tag{3.21}$$

where the constant M does not depend on t.

Proof. Inequalities (3.19) follow from the complete monotonicity of u_0 and u_δ and (3.18).

Applying (3.15) and (3.19) we get

$$\int_0^T u_\delta(t)\,dt = -\frac{1}{\lambda}\int_0^T u_0'(t)\,dt = \frac{1}{\lambda}(1 - u_0(T)) < \frac{1}{\lambda}.$$

The asymptotic expansion of $u_0(t)$ in (3.17) implies for sufficiently large t the estimate

$$u_n(t) \le \frac{C}{\lambda t^{\alpha_m}},$$

where the constant C does not depend on t. This together with the fact that $u_0(t)$ is continuous and monotonicaly decreasing and (3.18) gives (3.21). □

Next we give some additional properties of the solutions u_0 and u_δ, generalizing the properties of the Mittag-Leffler functions (3.4). They are obtained from (3.3) using (2.1)-(2.3) and other well-known properties of the Laplace transform.

The impulse response solution u_δ satisfies the equation with the Riemann-Liouville fractional derivative

$$D^\alpha u_\delta + \sum_{j=1}^{m} \lambda_j D^{\alpha_j} u_\delta + \lambda u_\delta = 0,$$

subject to the initial condition

$$\left(J^{1-\alpha} u_\delta + \sum_{j=1}^{m} \lambda_j J^{1-\alpha_j} u_\delta \right)\Big|_{t=0} = 1.$$

More precisely,

$$(J^{1-\alpha} u_\delta)\big|_{t=0} = 1, \ \ (J^{1-\alpha_j} u_\delta)\big|_{t=0} = 0.$$

This is a generalization of the well-known property that the function $t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$ satisfies the equation $D^\alpha u = \lambda u$ with initial condition $(J^{1-\alpha}u)\big|_{t=0} = 1$.

In addition, the following identities hold true

$$D_*^\alpha u_0 = -\lambda J^{1-\alpha} u_\delta, \quad D_*^{\alpha_j} u_0 = -\lambda J^{1-\alpha_j} u_\delta. \tag{3.22}$$

Since by the definition of the fundamental solution u_0

$$D_*^\alpha u_0 + \sum_{j=1}^{m} \lambda_j D_*^{\alpha_j} u_0 + \lambda u_0 = 0,$$

we obtain from (3.22)

$$u_0 = J^{1-\alpha} u_\delta + \sum_{j=1}^{m} \lambda_j J^{1-\alpha_j} u_\delta,$$

which is a generalization of the identity

$$E_\alpha(-\lambda t^\alpha) = J^{1-\alpha}(t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)).$$

4. Conclusion

In this paper, applying Laplace transform method, we find the fundamental and the impulse-response solutions of the multi-term fractional relaxation equation with the Caputo fractional derivatives. The properties of the solutions are then derived directly from their representations as Laplace inverse integrals. We prove that the fundamental and the impulse-response solutions are completely monotone functions, i.e. the complete monotonicity characteristic for the classical exponential relaxation is preserved. Asymptotic expressions for small and large times are also presented. It appears that the asymptotic behaviour of the solutions for $t \to 0$ is determined by the largest order of fractional differentiation α and for $t \to +\infty$ by the smallest order α_m, where an algebraic decay is observed. Based on the complete monotonicity and the asymptotic estimates as $t \to +\infty$ some estimates for the solutions are obtained. Some additional properties involving fractional integrals and derivatives of the solutions are also presented. All properties show that the fundamental and the impulse-response solutions of the considered problem are generalizations of the Mittag-Leffler functions $E_\alpha(-\lambda t^\alpha)$ and $t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$, respectively, obtained in the case of simple fractional relaxation.

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ON LOCAL STABILITY OF SOLUTIONS TO THE BELTRAMI EQUATION WITH DEGENERATION

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Abstract

In this paper we consider local stability and regularity properties such as conformality, asymptotic conformality, asymptotic homogeneity, and weak conformality at a point for sense-preserving homeomorphic solutions to the Beltrami equation, $f_{\bar{z}} = \mu f_z$, with degeneration. The coefficient μ is a complex-valued measurable function defined in a neighborhood of that point such that $|\mu| < 1$ a.e. Using analytic results, some developed in the author's thesis from 1988, we discuss sufficient conditions for local stability involving the complex dilatation and obtain a result on asymptotic homogeneity.

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Key Words and Phrases: Beltrami equation with degeneration, quasiconformal mappings, mappings of finite distortion, $\mu-$homeomorphisms, local stability, weak conformality, asymptotic homogeneity, asymptotic conformality, conformality.

1. Introduction

Let $f : \Omega \to f(\Omega)$ be an *ACL sense-preserving homeomorphism* in a domain $\Omega \subset \mathbb{C}$. Then f is differentiable a.e., has complex partial derivatives f_z and $f_{\bar{z}}$ a.e., and a Jacobian $J_f = |f_z|^2 - |f_{\bar{z}}|^2 \geq 0$ a.e. One says that f is *regular* at a point, if f is differentiable and $J_f > 0$ at that point.

Definition 1.1. Let μ be a Lebesgue-measurable complex-valued function in $\Omega \subset \mathbb{C}$ with $|\mu| < 1$ a.e. An ACL sense-preserving homeomorphism $f : \Omega \to \mathbb{C}$ satisfying

$$f_{\bar{z}} = \mu f_z \quad a.e. \tag{1.1}$$

is called a μ-homeomorphism.

(1.1) with $||\mu||_\infty = 1$ is called Beltrami equation with degeneration. Solutions to (1.1) include mappings of finite distortion, for which most of the theory

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of existence, uniqueness, regularity, compactness, etc. has been developed in the last 15 years or so, see [14]. If $||\mu||_\infty < 1$, (1.1) is the well-known Beltrami equation and its solutions are $K-$quasiconformal mappings. If $\mu = 0$ a.e. in Ω, the solution is a conformal mapping.

If f is a solution to (1.1) then at a regular point the complex dilatation μ_f of f is defined as $\mu_f = \dfrac{f_{\bar z}}{f_z}$ and $\mu_f = 0$ otherwise.

The local stability properties for solutions to (1.1), discussed in this paper, are conformality, asymptotic conformality, (asymptotic) homogeneity, and weak conformality at a point. We assume that the solutions are locally absolutely continuous, have $J_f > 0$ a.e. and satisfy (3.1). These assumptions will hold throughout the paper. The results apply to mappings of finite distortion and $K-$quasiconfomal mappings. For K-quasiconformal mappings local stability properties have been studied by Belinskii, Gutlyanskii, Lehto, Martio, Ryazanov, Reich, Teichmüller, Wittich and, for the more general class of $\mu-$homeomorphisms, by Jenkins, the author and others, see [2, 3, 18, 19, 16, 15, 6, 7, 13, 12, 5, 6, 17, 7, 8, 9, 10, 4].

We normalize the settings so that f is defined in a neighborhood of the origin $U = \{z : |z| < 1\}$ and $f(0) = 0$.

One says that f is **conformal at the origin** if

$$\lim_{r\to 0} \frac{f(z)}{z} = C \neq 0. \tag{1.2}$$

Conditions that assure conformality at the origin usually measure how close μ_f is to 0 in some integral sense.

f is called **asymptotically conformal** at the point 0 if $\mu_f \to 0$, as $z \to 0$. Asymptotic conformality at a point does not necessarily imply conformality at a point and vica versa.

Example 1.1. The function $f(z) = z(1 + \log |z|)$ has complex dilatation $\mu_f(z) = \dfrac{e^{2i\arg z}}{3 + 2\log |z|}$. Clearly, $\mu_f \to 0$, as $|z| \to 0$ but f is not conformal at 0.

An example constructed in [5, 8] considers f conformal at the origin for which $\mu_f = 1/3$.

Belinskii [2] observed that asymptotically conformal mappings at a point satisfy the following more general than differentiability property

$$f(z) = A(|z|)(z + o(|z|) \quad \text{as } z \to 0, \tag{1.3}$$

where $\lim_{\rho\to 0}\dfrac{A(t\rho)}{A(\rho)} = 1$, locally uniformly with respect to t (after possibly an appropriate normalization with pre/post composition with Möbius transformation).

One says that f is **(asymptotically) homogeneous** at the origin if

$$\lim_{z\to 0}\frac{f(\zeta z)}{f(z)} = \zeta, \tag{1.4}$$

locally uniformly with respect to ζ.

Such mappings were studied in depth in the works of Ryazanov [17] and Gutlyanskii and Ryazanov [13], see also the recent monograph [4].

A mapping is **weakly conformal** at a point if it preserve circles (is circle-like), i.e.

$$\lim_{r\to 0}\frac{\max\limits_{|z|=r}|f(z)|}{\min\limits_{|z|=r}|f(z)|} = 1, \tag{1.5}$$

and angles between rays emanating from the origin, i.e. for an appropriate choice of a branch of the argument

$$\lim_{r\to 0}\left[\arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1})\right] = \theta_2 - \theta_1, \tag{1.6}$$

uniformly in θ_1 and θ_2. Asymptotically homogeneous maps are weakly conformal and, in addition, preserve moduli of infinitesimal annuli, i.e.

$$\lim_{|z|\to 0}\frac{|f(tz)|}{|f(z)|} = t, \tag{1.7}$$

locally uniformly with respect to $t, t \in \mathbb{R}$. These properties fully characterize asymptotically homogeneous maps.

Below are a few examples of weakly conformal mappings at the origin.

Example 1.2. $f(z) = ze^{i\log\log\frac{e}{|z|}}$, [12], has complex dilatation $\mu_f(z) = \dfrac{ie^{2i\arg z}}{1+2i\log(e/|z|)}$. It is asymptotically conformal and homogeneous at 0, but not conformal at 0.

Example 1.3. The homeomorphism $f(z) = ze^{i\sqrt{-\log|z|}}$ has dilatation $\mu_f(z) = \dfrac{-ie^{2i\arg z}}{4\sqrt{-\log|z|}-1}$. It is asymptotically conformal and homogeneous at 0, but not conformal at 0.

Example 1.4. $f(z) = ze^{i(-\ln|z|)}$ has complex dilatation $\mu_f(z) = -\dfrac{ie^{2i\arg z}}{2-i}$. It is neither asymptotically conformal nor homogeneous, nor conformal at the origin.

Example 1.5. $f(z) = z|z|$ has complex dilatation $\mu_f(z) = \dfrac{e^{2i\arg z}}{3}$. It is a weakly conformal homeomorphism at the origin, which is neither asymptotically conformal nor homogeneous, nor conformal at the origin.

The above examples suggest that, as $z \to 0$, for a mapping homogeneous at a point, neither $|f|$ nor $\arg f$, can grow too fast. [17, 10].

2. History of results

In this section we provide a historical overview of some results concerning sufficient conditions for conformaliy, weak conformality and homogeneity at a point.

Let z be a regular point and α be a real number. Then $f_\alpha = f_z + e^{-2i\alpha} f_{\bar z}$ is the directional derivative of f in direction $\alpha, \alpha \in \mathbb{R}$. We define the *directional dilatation* $D_{f,\alpha}$ of f *in direction* α to be

$$D_{f,\alpha} = \frac{|f_\alpha|^2}{J_f}. \tag{2.1}$$

This notion was successfully used in some of the works of Andreian-Cazacu, see e.g. [1]. Since

$$D_{f,\alpha} = \frac{|f_\alpha|^2}{J_f} = \frac{|f_z + f_{\bar z} e^{-2i\alpha}|^2}{|f_z|^2 - |f_{\bar z}|^2} = \frac{1 + 2Re(\mu_f e^{-2i\alpha}) + |\mu_f|^2}{1 - |\mu_f|^2},$$

$$D_{f,\alpha} - 1 = 2\frac{Re(\mu_f e^{-2i\alpha}) + |\mu_f|^2}{1 - |\mu_f|^2}. \tag{2.2}$$

Since, for any α, one has that $-|\mu_f| \le Re(\mu_f e^{-2i\alpha}) \le |\mu_f|$,

$$\frac{1 - 2|\mu_f| + |\mu_f|^2}{1 - |\mu_f|^2} \le D_{f,\alpha} \le \frac{1 + 2|\mu_f| + |\mu_f|^2}{1 - |\mu_f|^2}.$$

Thus

$$\frac{1 - |\mu_f|}{1 + |\mu_f|} \le D_{f,\alpha} \le \frac{1 + |\mu_f|}{1 - |\mu_f|},$$

which is equivalent to $\dfrac{1}{D_f} \le D_{f,\alpha} \le D_f$, where $D_f = \dfrac{1 + |\mu_f|}{1 - |\mu_f|}$ is the *real dilatation* of f. Clearly, $D_f \ge 1$ a.e.

Certain deviations of the real dilatation D_f from 1 in a neighborhood of a point, equivalently of μ_f from 0, imply local stability of the solution f. A well-established result in this direction, in the theory of $K-$quasiconformal mappings, is the Teichmüller-Wittich-Belinskii theorem.

Theorem 2.1. [3, 18, 19] (Teichmüller-Wittich-Belinskii) *Let f be a quasiconformal mapping in U such that $f(0) = 0$. If*

$$\iint\limits_U \frac{|\mu_f|}{1-|\mu_f|}\frac{dA_z}{|z|^2} < \infty, \tag{2.3}$$

then f is conformal at 0.

Condition (2.3) was originally used in the equivalent form:

$$\iint\limits_U (D_f - 1)\,\frac{dA_z}{|z|^2} < \infty.$$

The precursor of Theorem (**2.1**) is the Teichmüller-Wittich theorem which states that (2.3) implies $\lim\limits_{|z|\to 0}\dfrac{|f(z)|}{|z|} = A \neq 0$. In 1988 the author [6], (see also [5]) obtained the following extension.

Theorem 2.2. *Let $f(z)$ be a differentiable μ-homeomorphism, $f : U \to U$, $f(0) = 0$, with complex dilatation μ_f. If*

$$\iint\limits_{|z|<1} \frac{|\mu_f|^2}{1-|\mu_f|^2}\frac{dA_z}{|z|^2} < \infty, \tag{2.4}$$

then f preserves circles, i.e. (1.5) *holds. In addition, there exists a constant $A > 0$, such that*

$$|f(z)| \sim A\exp\left[-\iint\limits_{|z|<|w|<1} D_{f,\theta}\frac{dA_w}{|w|^2}\right] \text{ as } z\to 0, \tag{2.5}$$

where $D_{f,\theta}$ is the directional derivative at $w = re^{i\theta}$, defined in (2.1).

As shown in in [5] and [6], if (2.3) holds then for some constant A_0,

$$\exp\left[-\iint\limits_{|z|<|w|<1} D_{f,\theta}\frac{dA_w}{|w|^2}\right] \sim A_0|z|, \quad \text{as } z\to 0. \tag{2.6}$$

Thus Theorem **2.2** implies the Teichmüller-Wittich theorem in a more general setting of differentiable $\mu-$homeomorphisms. Indeed, from (2.2) and (2.4) follows the convergence of the integral $\iint\limits_{|z|<|w|<1} D_{f,\theta}-1\frac{dA_w}{|w|^2}$, as $z\to 0$, which implies (2.6). The latter together with (2.5) implies the existence of a constant A such that $\lim\limits_{|z|\to 0}\frac{|f(z)|}{|z|}=A\neq 0$.

In 1992 Ryazanov [17], and in 1995 Gutlyanskii and Ryzanov [13], studied properties of asymptotically homogeneous/conformal $K-$ quasiconformal maps at a point and their applications to the study of symmetric homeomorphisms of the real line, a notion important in Teichmüller theory [11].

Theorem 2.3. [17] *For a quasiconformal map f of the extended complex plane onto itself, normalized by $f(0)=0$, $f(\infty)=\infty$ the following are equivalent*

1. f is differentiable in the sense of Belinskii at the origin, namely (1.3) holds.

2. $\mu_f(tz)\to 0$, as $t\to 0$ in the sense of dilatations.

3. There exists the limit

$$\lim_{z\to 0}\{\frac{f(z')}{f(z)}-\frac{z'}{z}\}=0,$$

as $|z'|\leq\delta|z|$, for any δ.

4. For all $\zeta\in\mathbb{C}$

$$\lim_{z\to 0}\frac{f(z\zeta)}{f(z)}=\zeta,$$

where the convergence is locally uniform in ζ.

Ryazanov [17] also showed that the module of an asymptotically homogeneous mapping can not grow too fast or too slow as $z\to 0$, namely

$$\lim_{z\to 0}\frac{\ln|f(z)|}{\ln|z|}=1,$$

which can be observed in Examples **1.1-1.5**.

In 1994 J. A. Jenkins and the author [7] obtained results for weak conformality and conformality at a point in the more general case of μ-homeomorphisms.

Theorem 2.4. [7]. *Assume that f is a μ-homeomorphism in U, $f(0)=0$. Let $\theta=\arg z$ and $0<r_2<r_1<1$. Assume that $D_{f,\theta+\alpha}\in L^1(A(r_2,r_1))$, for $\alpha=0$ and $\alpha=\frac{\pi}{2}$, where $D_{f,\alpha}$ is defined in (2.1) and $A(r_2,r_1)=\{z:r_2<|z|<r_1\}$.*

If

$$\iint\limits_U \frac{|\mu_f|^2 + |\Re e^{-2i\theta}\mu_f|}{1-|\mu_f|^2}\frac{dA_z}{|z|^2} < \infty, \tag{2.7}$$

then $\lim\limits_{|z|\to 0}\dfrac{|f(z)|}{|z|} = A \neq 0$ *holds and* f *is asymptotically a rotation on circles.*

Using extremal length techniques, introduced in [16], for estimating $\arg f$, we can show that if f satisfies the conditions of Theorem **2.4** then $\arg f = o(\sqrt{\log(1/r)})$, as $r \to 0$. Thus a map like the one from Example **1.3**, $f(z) = ze^{i\sqrt{-\log|z|}}$, rotates "too fast" and can not satisfy the conditions of Theorem **2.4**.

From Lemma 6.1 in [7] follows the next result.

Theorem 2.5. *Assume that* f *is a* μ*-homeomorphism in* U*,* $f(0) = 0$*. Let* $\theta = \arg z$ *and* $0 < r_2 < r_1 < 1$*. Assume that* $D_{f,\theta+\alpha} \in L^1(A(r_2, r_1))$*, for* $\alpha = 0$ *and* $\alpha = \frac{\pi}{2}$*, where* $D_{f,\alpha}$ *is defined in* (2.1) *and* $A(r_2, r_1) = \{z : r_2 < |z| < r_1\}$*. If*

$$\iint\limits_U \frac{|\mu_f|^2 + |\Re e^{-2i\theta}\mu_f|}{1-|\mu_f|^2}\frac{dA_z}{|z|^2} < \infty \tag{2.8}$$

and

$$\iint\limits_U \frac{\left||\mu_f|^2 - \Im e^{-2i\theta}\mu_f\right|}{1-|\mu_f|^2}\frac{dA_z}{|z|^2} < \infty, \tag{2.9}$$

then f *is conformal at* 0*, namely* (1.2) *holds.*

In 2003 V. Gutlyanskii and O. Martio [12] obtained the following two results on weak conformality and conformality of a $K-$quasiconformal map at a point.

Theorem 2.6. [12] *Let* f *be a* K*-quasiconformal homeomorphism in* U *such that* $f(0) = 0$*. If*

$$\iint\limits_U \frac{|\mu_f|^2}{|z|^2} dA_z < \infty, \tag{2.10}$$

then f *is weakly conformal at the origin.*

Theorem 2.7. [12] *Let f be a K-quasiconformal homeomorphism in U such that $f(0) = 0$. If (2.10) holds and if the singular integral*

$$\iint\limits_{U} \frac{\mu_f}{z^2} dA_z \tag{2.11}$$

exists in the sense of principal value, then f is conformal at $z = 0$.

The proofs of Theorem **2.6** and **2.7** rely on properties of K-quasiconformal mappings such as sequential compactness and boundedness of the dilatation.

In 2009 and 2010 [8, 9] the author obtained geometric sufficient and necessary conditions for weak conformality and conformality at the origin, as well as analytic sufficient conditions. These results extend Theorems **2.6** and **2.7** to the class of $\mu-$homeomorphisms. The proofs rely on analytic estimates of extremal lengths of curves separating or connecting the boundaries of images under f of annuli and the opposite sides of images of quadrilaterals formed by arcs of concentric circles and segments of radii.

Theorem 2.8. *Assume that f is a μ-homeomorphism in U, $f(0) = 0$ and that*

$$\lim_{r\to 0} \iint\limits_{r<|z|<1} D_{f,\theta+\alpha} - 1 \frac{dA}{|z|^2} \tag{2.12}$$

is finite for $\alpha = 0, \pi/2$. Then f is weakly conformal at the origin.

Theorem 2.9. *Assume that f is a μ-homeomorphism in U, $f(0) = 0$, and that*

$$\lim_{r\to 0} \iint\limits_{r<|z|<1} D_{f,\theta+\alpha} - 1 \frac{dA}{|z|^2}$$

is finite for $\alpha = 0, \pi/2$ and $\alpha = \alpha_0 \neq k\pi/2$, k any integer. Then f is conformal at the origin.

3. Asymptotic homogeneity

Let f be an a.e. regular, *locally absolutely continuous* solution to (1.1) in $U = \{z : |z| < 1\}, f(0) = 0$, with complex dilatation μ_f is not too close to 1 on a big set in a sense that there exists a constant $C_0 = C_0(t)$ such that

$$\limsup_{r\to 0} \iint\limits_{r<|z|<tr} \frac{1}{1 - |\mu_f|^2} \frac{A_z}{|z|^2} < C_0(t) < \infty. \tag{3.1}$$

In this section we show that if f satisfies (2.4) then it is asymptotically homogeneous at the origin. We use approaches and results from [5, 6, 8, 9], in particular Theorems **2.2** and **2.8**.

Theorem 3.1. *If*

$$\iint\limits_U \frac{|\mu_f|^2}{|z|^2} dA_z < \infty. \tag{3.2}$$

then f is homogeneous at the origin, namely $\lim_{z\to 0} \frac{f(\zeta z)}{f(z)} = \zeta$, *locally uniformly with respect to ζ.*

An equivalent statement is the following

Theorem 3.2. *If*

$$\iint\limits_U \frac{|\mu_f|^2}{|z|^2} dA_z < \infty,$$

then f is homogeneous at the origin, namely

$$f(z) = A(|z|)(z + o(|z|)), \quad \text{as } z \to 0,$$

where $\lim\limits_{\rho\to 0} \dfrac{A(t\rho)}{A(\rho)} = 1$, *locally uniformly with respect to t.*

Proof. As it was pointed out earlier, homogeneity is equivalent to weak conformality at a point and preservation of annuli (1.7), namely

$$\lim_{|z|\to 0} \frac{|f(tz)|}{|f(z)|} = t. \tag{3.3}$$

Weak conformality at a point requires preservation of circles and angles. Preservation of circles follows from Theorem **2.2** [6] which proof, under the conditions we consider here, is almost identical to the proof in [6]. One can show that f preserves angles between rays following the idea behind the proof of Theorem **2.8**. We show that (3.3) holds.

Indeed, according to (2.2) we have that

$$\left| \iint\limits_{|z|<|w|<t|z|} (D_{f,\theta} - 1) \frac{dA_w}{|w|^2} \right| \le 2 \iint\limits_{|z|<|w|<t|z|} \frac{|\mu_f|^2 + |Re(e^{-2i\theta}\mu_f)|}{1 - |\mu_f|^2} \frac{dA_w}{|w|^2}.$$

Using Cauchy-Schwarz inequality and (3.1) one has

$$\left(\iint\limits_{|z|<|w|<t|z|} \frac{|Re(e^{-2i\theta}\mu_f)|}{1-|\mu_f|^2}\frac{dA_w}{|w|^2}\right)^2 \leq C_0(t) \iint\limits_{|z|<|w|<t|z|} \frac{|\mu_f|^2}{1-|\mu_f|^2}\frac{dA_w}{|w|^2}.$$

(3.2) implies that locally uniformly with respect to t,

$$\iint\limits_{|z|<|w|<t|z|} \frac{|\mu_f|^2}{1-|\mu_f|^2}\frac{dA_w}{|w|^2} = o(1) \quad \text{as} \quad |z| \to 0.$$

Therefore

$$\left|\iint\limits_{|z|<|w|<t|z|} (D_{f,\theta} - 1)\frac{dA_w}{|w|^2}\right| = o(1)$$

as $|z| \to 0$, locally uniformly with respect to t.

In addition, (2.5) implies

$$|f(z)| \sim A \exp\left[-\iint\limits_{|z|<|w|<1} D_{f,\theta}\frac{dA_w}{|w|^2}\right],$$

as $|z| \to 0$. It follows that

$$\frac{|f(tz)|}{|f(z)|} \sim \exp\left[\iint\limits_{|z|<|w|<t|z|} D_{f,\theta}\frac{dA_w}{|w|^2}\right] \sim \exp\left[\iint\limits_{|z|<|w|<t|z|} D_{f,\theta} - 1 + 1\frac{dA_w}{|w|^2}\right],$$

as $|z| \to 0$, locally uniformly with respect to t and therefore (3.3) holds. Since f is weakly conformal at the origin, the asymptotic homogeneity of f at the origin, as stated in Theorem **3.1**, follows.

□

In [10] we show, among other things, that if f is asymptotically homogeneous map at the origin

$$\arg f(re^{i\theta}) = o(\log r) \text{ as } r \to 0 \text{ uniformly in } \theta.$$

A hint to this property could be observed in the earlier Examples **1.2** – **1.4**.

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COMMUTANT OF A CHEREDNIK TYPE OPERATOR ON THE REAL LINE

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Abstract

We characterize the commutant of the following Cherednik type singular differential-difference operator on the real line due to Mourou

$$\Lambda f(x) = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{y(x)-y(-x)}{2}\right) - \rho f(-x),$$

where $A(x) = |x|^{2\alpha+1}B(x)$, $\alpha > -\frac{1}{2}$, B being a positive C^∞ function on $\mathbb{R}$, and $\rho \geq 0$. This operator generalizes both Dunkl and Cherednik operators.

MSC 2010: 47B38, 47B39, 47B99

Key Words and Phrases: commutant of linear operator, convolution of functions

1. Introduction

The following first-order singular differential-difference operator on the real line was considered by Mourou in [6]:

$$\Lambda f(x) = \frac{df}{dx}(x) + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right) - \rho f(-x), \tag{1.1}$$

where $A(x) = |x|^{2\alpha+1}B(x)$, $\alpha > -\frac{1}{2}$, B being a positive C^∞ function on $\mathbb{R}$, and $\rho \geq 0$

A particular case is the operator

$$\begin{aligned} D_{\alpha,\beta}f(x) \quad = \quad & \frac{df}{dx}(x) + [(2\alpha+1)\coth x + (2\beta+1)\tanh x]\,\frac{f(x)-f(-x)}{2} \\ & -(\alpha+\beta+1)f(-x), \end{aligned} \tag{1.2}$$

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known as Jacobi-Cherednik operator [5], where

$$A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}; \quad \alpha \geq \beta > -\frac{1}{2}; \quad \rho = \alpha + \beta + 1.$$

The Cherednik operator [2] is a particular case of $D_{\alpha,\beta}$.

Mourou shows in [6], Theorem 3.6, that there exists a unique invertible transform V satisfying

$$\begin{cases} V\dfrac{d}{dx}f = \Lambda V f \\ Vf(0) = f(0) \end{cases} \tag{1.3}$$

called intertwining operator, which transforms the differentiation operator $D = \dfrac{d}{dx}$ into Λ. He finds also the inverse transform V^{-1}.

Using these transforms, Mourou defines generalized translation operators corresponding to the differential-difference operator Λ:

Definition 1.1. ([6]) The generalized translation operators T_x, $x \in \mathbb{R}$, associated with the operator Λ, are defined by

$$T^x f(y) = V_x V_y [V^{-1} f(x+y)], \quad y \in \mathbb{R}. \tag{1.4}$$

Some of the main properties of the translations T^x mentioned in [6] are:

(i) $T^0 = I$ - the identity, $\quad T^x T^y = T^y T^x, \quad \Lambda T^x = T^x \Lambda$.

(ii) $T^x f(y) = T^y f(x)$.

(iii) For any function f satisfying (1.3) the function $u(x,y) = T^x f(y)$ is the unique solution of the problem

$$\begin{cases} \Lambda_x u(x,y) = \Lambda_y u(x,y) \\ u(0,y) = f(y) \end{cases}.$$

Let us note that (iii) could be used as a definition of translation operators associated with Λ.

2. Right inverse operators of Λ and their Taylor expansions

Let L be an arbitrary right inverse operator of Λ in $C^1(\mathbb{R})$, i.e. if $f \in C^1(\mathbb{R})$, then $Lf(x) = y(x)$ is the solution of the equation

$$\Lambda y(x) = f(x), \quad \Phi(y) = 0, \tag{2.5}$$

where Φ is a given linear functional on $C^1(\mathbb{R})$ with $\Phi\{1\} = 1$.

In the general case it is not possible to give an explicit solution of this equation like it was made by the authors in [4] for the Dunkl operator, a particular case of Λ. Here we will find at least the differential equations to be solved. It is well known that such types of equations have solutions.

Let us represent y by its even and odd parts, namely $y = y_e + y_o$, where $y_e(x) = \dfrac{y(x)+y(-x)}{2}$ and $y_o(x) = \dfrac{y(x)-y(-x)}{2}$, and make the same with $f = f_e + f_o$. Substituting in (2.5), and equating separately the even and the odd parts, we get the system

$$y_o'(x) + \frac{A'(x)}{A(x)} y_o - \rho y_e(x) = f_e x \tag{2.6}$$

$$y_e'(x) + \rho y_o(x) = f_o(x) \tag{2.7}$$

for y_e and y_o. Substituting y_o from (2.7) into (2.6) we obtain a second order differential equation for y_e:

$$y_e'' + \frac{A'}{A} y_e' + \rho^2 y_e = f_o' + \frac{A'}{A} f_o - \rho f_e. \tag{2.8}$$

Similarly we have the following equation for y_o:

$$y_o'' + \frac{A'}{A} y_o' + \left(\frac{A''A - (A')^2}{A^2} + \rho \right) y_o = f_e' + \rho f_o. \tag{2.9}$$

It is well known that such second order differential equations have non-trivial solutions and in the sequel we will use the sum of some fixed solutions $y = y_e + y_o$ as one of the right inverse operators of the operator Λ, which will be denoted by $L = L(f)$. Then all such operators have to be of the form $L = L(f) + \Psi\{f\}$, where Ψ is a linear functional on $C^1(\mathbb{R})$.

In the general theory of right invertible operators (see Bittner [1], Przeworska-Rolewicz [7]) an important characteristic of L is its "initial projector"

$$Ff(x) = f(x) - L\Lambda f(x) = \Phi\{f\}. \tag{2.10}$$

It maps $C^1(\mathbb{R})$ onto $\ker \Lambda = \mathbb{R}$, i.e. it is a linear functional Φ on $C^1(\mathbb{R})$. Expressing Φ by Ψ, we obtain

$$\Phi\{f\} = f(0) - \Psi\{\Lambda f\}.$$

Let us note that $\Phi\{1\} = 1$ which expresses the projector property of F. The simplest case of an right inverse operator is when Φ is the Dirac functional $\Phi\{f\} = f(0)$.

Definition 2.1. The Appell type functions $\{A_n(x)\}_{n=0}^{\infty}$ associated with the operator Λ are introduced by the recurrences

$$A_0(x) \equiv 1, \quad \text{and} \quad \Lambda A_{n+1}(x) = A_n(x), \quad \Phi\{A_{n+1}\} = 0, \quad n \geq 0. \tag{2.11}$$

Lemma 2.1. *The Appell type functions have the representation*

$$A_n(x) = L^n\{1\}(x),$$

where L is the right inverse of the operator Λ.

Proof. By induction: If $n = 1$, then $\Lambda A_1(x) = A_0(x) \equiv 1 \equiv L^0\{1\}(x)$ and therefore $A_1(x) = L\{1\}(x)$. Now, suppose that the assertion is true for arbitrary $n \geq 0$. Then

$$\Lambda A_{n+1}(x) = A_n(x) = L^n\{1\}(x), \Phi\{A_{n+1}\} = 0,$$

hence $A_{n+1}(x) = LA_n(x) = LL^n\{1\}(x) = L^{n+1}\{1\}(x)$, which proves the lemma. □

Lemma 2.2. (Taylor formula with remainder term) *If $f \in C^n(\mathbb{R})$, then*

$$f(x) = \sum_{j=0}^{n-1} \Phi\{\Lambda^j f\}A_j(x) + L^n(\Lambda^n f)(x) \qquad \textit{and} \tag{2.12}$$

$$T^y f(x) = T^x f(y) = \sum_{j=0}^{n-1} \Phi\{T^x \Lambda^j f\}A_(y) + L^n(T^x \Lambda^n f)(y), \tag{2.13}$$

where $A_j(y) = L^j\{1\}(y)$ are the Appell type functions (2.11)*, related to the functional Φ.*

Proof. Delsarte [3], Bittner [1], and Przeworska-Rolewicz [7] proposed variants of the Taylor formula for right invertible operators in linear spaces. In our case the general Taylor formula is the obvious operator identity

$$I = \sum_{j=0}^{n-1} L^j F \Lambda^j + L^n \Lambda^n,$$

where I is the identity operator and $F = I - L\Lambda$. In functional form the above identity takes the form

$$f(x) = \sum_{j=0}^{n-1} L^j F \Lambda^j f(x) + L^n \Lambda^n f(x),$$

where the initial projector F of L (2.10) is a linear functional Φ:

$$Ff(x) = f(x) - L\Lambda f(x) = \Phi\{f\}.$$

F projects the space $C(\mathbb{R})$ into the space $\mathbb{R}$ of the constants. Hence the Taylor formula with remainder term for the operator Λ is

$$f(x) = \sum_{j=0}^{n-1} \Phi\{\Lambda^j f\} L^j\{1\}(x) + L^n \Lambda^n f(x), \tag{2.14}$$

which gives the result. (2.13) follows from (2.12) if we substitute $f(x)$ by $T_k^y f(x)$.

□

Lemma 2.3. *The span of the Appell functions* $\{A_n\}_{n=0}^{\infty}$ *is dense in* $C(\mathbb{R})$.

Corollary 2.1. *If* $f \in \operatorname{span}\{A_n\}_{n=0}^{\infty}$, *then*

$$f(x) = \sum_{j=0}^{\infty} \Phi\{\Lambda^j f\} A_j(x) \qquad \textit{and} \tag{2.15}$$

$$T^y f(x) = T^x f(y) = \sum_{j=0}^{\infty} \Phi\{T^x \Lambda^j f\} A_j(y), \tag{2.16}$$

where $A_j = L^j\{1\}$ *are the Appell functions.*

Further, we will use only the special case of the last formula, when $\Phi\{f\} = f(0)$. Then it takes the form

$$T^y f(x) = T^x f(y) = \sum_{j=0}^{\infty} \Lambda^j f(x) A_j y^j, \tag{2.17}$$

where A_j are constants.

3. General commutant of the operator Λ

Here we will prove the following theorem:

Theorem 3.1. *Let* $M : C(\mathbb{R}) \to C(\mathbb{R})$ *be a continuous linear operator with* $M : C^1(\mathbb{R}) \to C^1(\mathbb{R})$. *Then the following assertions are equivalent:*

(i) M *commutes with the Cherednik type operator* Λ, *defined by (1.1), in* $C^1(\mathbb{R})$, *i.e.* $M\Lambda = \Lambda M$;

(ii) M *commutes with all generalized translations, i.e.* $MT^y = T^y M$ *for every* $y \in \mathbb{R}$;

(iii) M *admits a representation of the form*

$$(Mf)(x) = \Phi_t\{T^t f(x)\} \tag{3.18}$$

with a continuous linear functional $\Phi : C(\mathbb{R}) \to \mathbb{C}$.

Proof. (i)⇒(ii)

Suppose that M commutes with the operator Λ in $C^1(\mathbb{R})$, i.e. $M\Lambda f = \Lambda M f$ for $f \in C^1(\mathbb{R})$. Then, for arbitrary $y \in \mathbb{R}$ and any function $f \in \operatorname{span}\{A_n\}_{n=0}^{\infty}$, Taylor formula (2.17) implies

$$(MT^y f)(x) = (MT^x f)(y) = M\left(\sum_{n=0}^{\infty}(\Lambda^n f)(x)a_n y^n\right)$$
$$= \sum_{n=0}^{\infty}(M\Lambda^n f)(x)a_n y^n = \sum_{n=0}^{\infty}(\Lambda^n Mf)(x)a_n y^n$$
$$= \sum_{n=0}^{\infty}\Lambda^n(Mf))(x)a_n y^n = (T^x Mf)(y) = (T^y Mf)(x).$$

Since $MT^y = T^y M$ is true for the Appell functions $\{A_n\}_{n=0}^{\infty}$, then this is true for arbitrary $f \in C^1(\mathbb{R})$. One should simply use approximation by functions of $\text{span}\{A_n\}_{n=0}^{\infty}$.

(ii)⇒(i)

Suppose $MT^t = T^t M$ for every $t \in \mathbb{R}$. For arbitrary polynomial $f(x)$ reverse the order in the above chain of equalities as follows:

$$\sum_{n=0}^{\infty}(M\Lambda^n f)(x)a_{k,n}y^n = (M(T^t f))(x)$$
$$= (T^t(Mf))(x) = \sum_{n=0}^{\infty}(\Lambda^n Mf)(x)a_{k,n}y^n.$$

The sums have to coincide for every x and hence the coefficients of y^n are equal for arbitrary n. For $n = 1$ it follows that

$$(M(\Lambda f))(x) = (\Lambda(Mf))(x). \tag{3.19}$$

Assuming that (3.19) is true for polynomials, it follows that it is true for arbitrary $f \in C^1(\mathbb{R})$ since f could be approximated by polynomials.

(ii)⇒(iii)

Let

$$MT^y f(x) = T^y Mf(x), \qquad \forall y \in \mathbb{R}. \tag{3.20}$$

The property $T^y f(x) = T^x f(y)$ applied to the right hand side of (3.20) gives

$$(M(T^y f))(x) = (T^x(Mf))(y). \tag{3.21}$$

Define the linear functional Φ as

$$\Phi\{f\} := (Mf)(0).$$

Then, substituting $y = 0$ in (3.21) and taking into account that T^0 is the identity operator, one has

$$(M(T^y f))(0) = (T^0(Mf))(y) = (Mf)(y). \tag{3.22}$$

The left hand side is the value of the functional Φ for the function $g(x) = (T^y f)(x)$, and hence

$$(Mf)(y) = \Phi_t\{(T^y f)(t)\} = \Phi_t\{(T^t f)(y)\}$$

using (3.21) and property (i) from Lemma 1. This in fact is the desired representation (3.18) of the commutant of D_k with y for x, and with the dumb variable t instead of y.

(iii)$\Rightarrow$(ii)

It is a matter of a direct check to show that the operators of the form (3.18) commute with T^y for every $y \in \mathbb{R}$:

$$\begin{aligned} MT^y f(x) &= \Phi_t\{(T^t T^y f)(x)\} = \Phi_t\{(T^y T^t f)(x)\} \\ &= T^y \Phi_t\{T^t f)(x)\} = T^y M f(x). \end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let the operators M and N commute with the operator Λ in the sense of Theorem* **3.1**. *Then*

$$MN = NM,$$

i.e. M and N commute between themselves.

Proof. According to (3.18)

$$(Mf)(x) = \Phi_t\{T^t f(x)\} \qquad \text{and} \qquad (Nf)(x) = \Phi_s\{T^s f(x)\}.$$

Then

$$MNf(x) = \Phi_t\{T^t \Phi_s\{T^s f(x)\}\} = \Phi_t\Phi_s\{T^t T^s f(x)\} = \Phi_t\Phi_s\{T^s T^t f(x)\}.$$

But using the property

$$T^t T^s = T^s T^t$$

from Definition **1.1** and the Fubini property

$$\Phi_t \Phi_s = \Phi_s \Phi_t$$

we get

$$MNf(x) = NMf(x).$$

Thus the proof is finished. □

Here we considered only the general commutant of the operator Λ, but as in [4] it is possible to consider also the commutant in an invariant hyperplane. This will be done in another publication.

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EXTENSION OF DUHAMEL PRINCIPLE FOR LINEAR NONLOCAL INITIAL-BOUNDARY VALUE PROBLEMS

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Abstract

Local and nonlocal boundary value problems (BVPs) for the classical equations of mathematical physics in rectangular domains often are solved by Fourier method or some of its extensions, intended for the nonlocal case.

We aim to make more effective the Fourier method for general classes of nonlocal BVPs for the heat and wave equations in a strip. To this end the method is combined with an extension of the Duhamel principle to the space variable. Nonclassical operational calculi, custom-tailored for the specific problems, are used. Thus explicit solutions of the considered problems are obtained.

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1. Introduction

The classical Heaviside-Mikusiński operational calculus [10] is intended mainly to solving of initial value problems both for ODEs and for PDEs. The authors [4] proposed an extension of the Heaviside-Mikusiński operational calculus for nonlocal Cauchy problems for ordinary differential equations with constant coefficients.

The attempt of Mikusiński [10] to encompass boundary value problems for linear PDEs with constant coefficients hardly could be considered as successful. One reason for this is the fact that Mikusiński used his direct operational calculus only in its one-dimensional variant.

M. Gutterman [7] proposed a multi-dimensional varianl of Mikusiński's calculus, intended to initial value problems for linear PDEs with constant coefficient. Nevertheless, as he acknowledges, "So far our method does not apply to some mixed problems (such applications would probably demand further improvements)".

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Here we intend on simple situations to outline a feasible direction in application of operational calculus approach to some initial-boundary value problems for classical equations of mathematical physics: the heat equation and the wave equation. Instead of exclusively local boundary value conditions, we include a general nonlocal boundary value condition, determined by an arbitrary linear functional. We will consider our equations in a finite space domain, a case which is more involved than the case of an infinite domain, where integral transforms are applicable.

In order to elucidate our approach at the simplest situation, we will consider the following two nonlocal boundary value problems.

1.1. Problem 1

Solve the heat equation

$$u_t = u_{xx} + F(x,t),\ 0 < x < 1,\, 0 < t \tag{1.1}$$

in the strip $G = \{(x,t) : 0 \le x \le 1, 0 \le t\}$ with the initial condition

$$u(x,0) = f(x) \tag{1.2}$$

and boundary value conditions

$$u(0,t) = 0,\ \Phi_\xi\{u(\xi,t)\} = 0, \tag{1.3}$$

where Φ is a given linear functional in $C^1[0,1]$.

Usually, in the literature the local case $\Phi\{f\} = f'(1) - hf(1)$ is considered.

In the general case Φ is determined by a Stieltjes integral representation

$$\Phi\{f\} = Af(1) + \int_0^1 f'(\xi)d\alpha(\xi), \tag{1.4}$$

where $A = const,\ \alpha$ is a function with bounded variation on $[0,1)$.

1.2. Problem 2

Solve the wave equation

$$u_{tt} = u_{xx} + F(x,t),\ 0 < x < 1,\, 0 < t$$

with initial conditions

$$u(x,0) = f(x),\ u_t(x,0) = g(x). \tag{1.5}$$

and BVCs

$$u(0,t) = 0,\ \Phi_\xi\{u(\xi,t) = 0 \tag{1.6}$$

with a linear functional Φ in $C^1[0,1]$

Here we propose a unified operational calculus approach intended to encompass both problems, in order to find their solutions in an explicit form.

The main features of our approach could be seen in the authors paper [4] and in more details in Dimovski [3]. Along with the classical Duhamel convolution

$$(\varphi \overset{t}{*} \psi) = \int\limits_0^t \varphi(t-\tau)\psi(\tau)d\tau, \tag{1.7}$$

we use the non-classical convolution (see Dimovski [2], p. 119)

$$(f \overset{x}{*} g)(x) = -\frac{1}{2}\,\Phi_\xi \left\{\int_0^\xi h(x,\eta)\,d\eta\right\} \tag{1.8}$$

with

$$h(x,\xi) = \int_x^\xi f(\xi{+}x{-}\eta)\,g(\eta)\,d\eta - \int_{-x}^\xi f(|\xi{-}x{-}\eta|)\,g(|\eta|)\,\mathrm{sgn}(\xi{-}x{-}\eta)\,\eta\,d\eta, \tag{1.9}$$

and the composed functional

$$\tilde{\Phi} = \Phi_\xi \circ \int_0^\xi . \tag{1.10}$$

In the operational calculus, we are briefly to outline, the basic roles are played by the integration operator

$$l_t\,u(x,t) = \int_0^t u(x,\tau)\,d\tau \tag{1.11}$$

and the right inverse operator

$$L_x\,u(x,t) = \int_0^x (x-\xi)\,u(\xi,t)\,d\xi - \frac{x}{\Phi(\xi)}\Phi_\xi\left\{\int_0^\xi (\xi-\eta)\,u(\eta,t)\,dt\right\} \tag{1.12}$$

of the operator $\dfrac{\partial^2}{\partial\,x^2}$ on the strip $G = \{0 \le x \le 1,\ 0 \le t\}$.

The operator L_x satisfies the BVCs $(L\,u)\,(0,t) = 0$ and $\Phi_\xi\{(L\,u)\,(\xi,t) = 0$ and it is uniquely determined by them and the requirement to be a right inverse operator of $\dfrac{\partial^2}{\partial\,x^2}$ in G.

The operator L_x exists under the condition $\Phi\{\xi\} \ne 0$. Some formal simplification occurs when we assume

$$\Phi\{\xi\} = 1 \tag{1.13}$$

Then

$$L_x u(x,\xi) = \int_0^x u(x-\xi)\,u(\xi,x)\,d\xi - x\,\Phi_\xi\left\{\int_0^\xi (\xi-\eta)\,u(\xi,x)\,dx.\right\} \tag{1.14}$$

Both l_t and L_x are convolution operators:

$$l_t\,u(x,t) = \{1\} \overset{t}{*} \{u(x,t)\} \tag{1.15}$$

and

$$L_x\, u(x,t) = \{x\} \overset{x}{*} \{u(x,t)\}. \tag{1.16}$$

Relations (1.15) and (1.16) will be used frequently in our operational calculus.

2. Two-dimensional operational calculus for the operators l_t and L_x

Instead of convolutions (1.7) and (1.8) we introduce the following combined two-dimensional convolution

$$\begin{gathered}\{u(x,t)\} * \{v(x,t)\} \\ = -\frac{1}{2}\Phi_\xi \circ \int_0^\xi \left[\int_x^\eta \int_0^t u(\eta + x - \zeta, t-\tau)\, v(\xi,\tau)\, d\tau\, d\zeta \right. \\ \left. - \int_{-x}^\eta \int_0^t u(|\eta - x - \zeta|)\, v(|\zeta|) sgn(\eta - x - \zeta)\, d\tau\, d\zeta \right] d\eta.\end{gathered} \tag{2.1}$$

It is designed in such a way that for $u(x,t) = f(x)\,\varphi(t)$ and $v(x,t) = g(x)\,\psi(t)$ to have

$$(u * v)(x,t) = (f \overset{x}{*} g)(\varphi \overset{t}{*} \psi). \tag{2.2}$$

Lemma 2.1. *If $u(x) \in C(G)$, then*

$$L_x u \ = \ \{x\} \overset{x}{*} \{u(x,t)\} \tag{2.3}$$

and

$$l_t u \ = \ \{1\} \overset{t}{*} \{u(x,t)\}. \tag{2.4}$$

The proof is a simple check.

Consider the convolution algebra $(C(G), *)$ and the linear operators, which map $C(G)$ into itself. We single out those of them which are multipliers of the convolution algebra $(C(G), *)$.

Definition 2.1. (Larsen, [9]) An operator $A : (C(G) \to C(G)$ is said to be a multiplier of the convolution algebra $C(G), *)$ if it holds the relation

$$A(u * v) = (A\, u) * v \tag{2.5}$$

for arbitrary $u, v \in C(G)$.

Lemma 2.2. *Let $f \in C[0,1]$ and $\varphi \in C[0,\infty)$. Then the convolution operators $f \overset{x}{*}$ and $\varphi \overset{t}{*}$ are multipliers of $(C(G), *)$.*

The assertion of this lemma becomes plausible, if we look at (2.2) and using an approximation argument.

Definition 2.2. Let $f \in C[0,1]$ and $\varphi \in C[0,\infty]$. The convolution multipliers $[f]_t := f \overset{x}{*}$ and $[\varphi]_x := \varphi \overset{t}{*}$ are called partially numerical multipliers with respect to t and x, respectively.

Important cases of multipliers ate the operators L_x and l_t, due to (2.3) and (2.4): $L_x = [x]_t$, $l_t = [1]_x$.

Lemma 2.3. *The multipliers of the convolution algebra $(C(G), *)$ form a commutative ring $\mathfrak{M}$ with a nonempty set $\mathfrak{N}$ of the non-zero non-divisors of zero.*

Non-divisors of zero are e.g. the identity operator l and the operators L_x and l_t.

The next step is to make localization of $\mathfrak{M}$ with respect to $\mathfrak{N}$, i.e. to form the ring of the multiplier fractions $\dfrac{M}{N}$ with $M \in \mathfrak{M}$ and $N \in \mathfrak{N}$.

Denote by $\mathcal{M}$ the ring of the multiplier fractions.

In $\mathcal{M}$ could be embedded $(C(G), *)$, $(C[0,1], \overset{x}{*})$, $\in (C[0,\infty], \overset{t}{*})$ and the number fields $\mathbb{R}$ and $\mathbb{C}$ (see Dimovski [2]).

We introduce the inverse elements of L_x and l_t in the ring $\mathcal{M}$:

$$S_x = \frac{1}{L_x}, \tag{2.6}$$

$$s_t = \frac{1}{l_t}. \tag{2.7}$$

The elements S_x and s_t are not identical with $\dfrac{\partial^2}{\partial x^2}$ and $\dfrac{\partial}{\partial t}$ but they are closely connected with them.

Theorem 2.1. *a) Let $u \in C^2(G)$. Then*

$$\frac{\partial^2 u}{\partial x^2} = S_x u - [\Phi_\xi\{u(\xi,t)\}]_x - S_x\{(1-x)u(0,t)\} \tag{2.8}$$

b) Let $u \in C^1(G)$. Then

$$\frac{\partial u}{\partial t} = su - [u(x,0)]_t. \tag{2.9}$$

Proof. It is easy to verify the identity
$L_x \frac{\partial^2 u}{\partial x^2} = u - x\Phi_\xi\{u(\xi,t)\} - (1-x)u(0,t)$
Multiplying with S_x, we get (2.8). The identity (2.9) follows from

$$l_t \frac{\partial u}{\partial t} = u - u(x,0)$$

by multiplying with s_t. □

Formulas (2.8) and (2.9) are basic for our operational calculus.

3. Algebraization of BVPs 1 and 2

Let us assume that Problem 1 has a classical solution $u(x,t) \in C^2(G)$. Using formulas (2.8) and (2.9) together with the conditions (1.2) and (1.3), from the equation

$$u_t = u_{xx} + F(x,t)$$

we get

$$s_t u - [f(x)]_t = S_x u + F.$$

Problem 1 obtains the algebraic form

$$(s_t - S_x)\, u = [f(x)]_t + F \tag{3.1}$$

which is a linear equation of the first degree for the unknown function u in $\mathcal{M}$.

Its formal solution is

$$u = \frac{[f(x)]_t}{s_t - S_x} + \frac{1}{s_t - S_x}\, F, \tag{3.2}$$

provided $s_t - S_x$ is a nondivisor of zero in $\mathcal{M}$. It is clear, that this requirement is equivalent to an assertion for the uniqueness of the solution.

Assume that there exist a solution for Problem 1 with $F(x,t) \equiv 0$ and $f(x) = x$. Denote this solution by $\Omega(x,t)$. From (3.2) we get

$$\Omega(x,t) = \frac{1}{S_x\,(s_t - S_x)}. \tag{3.3}$$

Then

$$u(x,t) = \frac{\partial^2}{\partial x^2}\left\{\Omega(x,t) \overset{x}{*}) f(x)\right\} + \frac{\partial^3}{\partial x^2 \partial t}\left(\Omega(x,t) * F(x,t)\right) \tag{3.4}$$

is an explicit solution of Problem 1 for arbitrary $f \in C[0,1]$ and $F \in C(G)$, provided the denoted partial derivatives exist. Usually, the special solution $\Omega(x,t)$ may be obtained by the Fourier method in a form of a series.

Example 3.1. Let $\Phi\{f\} = \int_0^1 f(\xi)\, d\xi$, $F(x,t) \equiv 0$. The corresponding problem is studied in detail by Ionkin [8]. The solution in the form (3.4) is exibited in Dimovski [2], p. 173-174.

Next we consider Problem 2, assuming that it has a classical solution $u \in C^2(G)$. Applying twice formula (2.9), we get

$$u_{tt} = s_t^2\, u - s_t\, [g(x)]_t - [f(x)]_t.$$

From (2.8) we get

$$u_{xx} = S_x u$$

and Problem 2 takes the algebraic form

$$(s_t^2 - S_x)\, u = s_t[g(x)]_t + [f(x)]_t + F. \tag{3.5}$$

Assuming that $s_t^2 - S_{xx}$ is a nondivisor of 0, we get the formal solution

$$u = \frac{s_t}{s_t^2 - S_x}\,[g(x)]_t + \frac{1}{s_t^2 - S_x}[f(x)]_t + \frac{1}{s_t^2 - S_x}\,F. \tag{3.6}$$

Let us assume that Problem 2 has a solution $\Omega(x,t)$ for $f(x) = x$, $g(x) \equiv 0$, $F(x,t) \equiv 0$. From (3.4) we get

$$\Omega(x,t) = \frac{1}{S_x(s_t^2 - S_x)}. \tag{3.7}$$

Then (3.4) can be represented in the form

$$u(x,t) = s_t S_x \left(\Omega(x,t) \overset{x}{*} g(x)\right) + S_x \left(\Omega(x,t) \overset{x}{*} g(x)\right) + s_t S_x \left(\Omega(x,t) * F(x,t)\right).$$

Thus we obtain the explicit representation

$$\begin{aligned} u(x,t) = \frac{\partial^3}{\partial x^2 \partial t} \left(\Omega(x,t) \overset{x}{*} g(x)\right) + \frac{\partial^2}{\partial x^2} \left(\Omega(x,t) \overset{x}{*} f(x)\right) \\ + \frac{\partial^3}{\partial x^2 \partial t} \left(\Omega(x,t) * F(x,t)\right). \end{aligned} \tag{3.8}$$

The special solution $\Omega(x,t)$, usually can be obtained in a series form by the Fourier method.

Example 3.2. Problem 2 with $\Phi(f) = \int_0^1 f(\xi)d\xi, g(x) \equiv 0$ and $F(x,t) \equiv 0$ is considered in detail by S. Beilin [1]. Solution in the form (3.8) is realized in author's paper [5].

Representations (3.4) and (3.8) could be considered as extensions of the classical Duhamel principle, but with respect to the space variable.

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A HISTORICAL SURVEY ON THE PLACE AND ROLE OF TASKS IN MATHEMATICS TEACHING

Valentina Gogovska

Abstract: Mathematical tasks are a main tool to achieve educational, practical and instructional aims of mathematics teaching. In order to achieve long-term, as well as comprehensive adoption of the prescribed material, it is necessary to solve a significant number of tasks. Trying to emphasize the significance of mathematical tasks, it is sufficient to ask ourselves the following question: "Is mathematics teaching possible without mathematical tasks?" In the beginning we will try to point out the role of mathematical tasks through history.

Mathematical tasks were the basic tool for strengthening mathematical knowledge in pre-Greek period. However, over time, tasks were replaced first by theorems and than by concepts. Therefore, historically there is a certain dynamics between *the set of theorems and set of tasks.* Does this mean that with the change of the position of tasks through history their significance has been lost too?

MSC 2010: 97A30, 97D50

Key Words and Phrases: mathematical tasks, concepts and theorems, axioms, definitions, didactic tools

1. First section of the paper

The first mathematical texts are the Egyptian papyruses: Rhind and Moscow. They contain formulations and solutions to specific problems of everyday practice. In them, rules are determined on solving tasks that can be reduced to solving linear and quadratic equations and systems of equations as well. The Egyptians managed to calculate the approximate values of quadratic roots for some numbers. They learned the formula for the sum of the members of arithmetic progression, as well as the formula for the sum of squares of sequence of natural numbers. In Egypt, there is an improvement in approximating to the number pi, from the Babylonian value of 3 to 3.16. The formula for calculating the volume of a pyramid and the volume of a truncated pyramid with square as its base are determined. Even then, they used the theorem now known as the Pythagorean and its reverse theorem. However, mathematics still did not exist as a science. In this period, the Egyptians and Babylonians successfully answer the question 'HOW?'; how it is done (calculated) without answering why it is done.

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The answer to this very important question is obtained in ancient Greece. In ancient Greece, accurate was considered only that which could be reasonably proved. In ancient Greece, systems of concepts and statements regarding these concepts were formed. The idea for deductive structuring of knowledge played the main role in these systems of concepts and statements, especially in geometry. The following took place: grouping of objects with similar characteristics, limiting the scope of the concept, occurring of definitions, looking into general characteristics…suddenly theorems are formed… With the occurrence of theorems, they became sufficient. Right then, ancient Greece began to change.

Mathematics in ancient Greece was rich and comprehensive, more profound and extensive than any other intellectual activity created previously in Mesopotamia, ancient Egypt, India and China.

The answer, or at least the attempt to answer the question how would Western culture develop and look like if ancient Greek heritage was not included in its basis, would be considerably complicated and rather pointless. The influence that ancient Greek philosophy, science and art had on the West **is impossible to estimate** – even today many consider ‘classical’ philosophy as a synonym to ancient Greek philosophy, ‘classical’ drama as a synonym to ancient Greek drama, ‘classical’ sculpting as a synonym to ancient Greek sculpting. In fact, the achievements of those relatively small in numbers people, living in a relatively small area in the eastern Mediterranean, did not only spread towards the West, but also towards the East, following Alexander the Great’s expedition, through the so-called Silk Road which connected Europe to China even during the Roman Empire. It is important to mention that the influence of Greek culture on India, Middle Asia, even the Japanese islands was spread in other ways as well.

When the ancient Greeks assumed the leading role in science and culture, **mathematics** gained new stimulus and direction for further development. While mathematical regularities of the ancient Egyptians, Babylonians and Indus were obtained in empirical manner and adopted without proof, ancient Greeks implemented the principle according to which mathematical regularities should not be accepted as true until they were proved. In this period, the inductive method was abandoned, and the deductive method developed, which had an immeasurable significance in further development of mathematics and other fields. Parallel with the development of the deductive method, the need arose for systemization of some mathematical disciplines, especially geometry. This entire developmental success of ancient mathematics lasted for several centuries, from VI to II century BC.

Before the Greeks, mathematics was mainly in the hands of priests. They were the ‘knowledgeable’ ones who spread knowledge to whom they wanted and to the extent they wanted. They were the ‘masters’ in engineering and other activities where calculations were needed. It is important to mention here that they left their mark in mathematics.

Unlike them, the Greeks dispersed around the remote shores and islands developed as small states, and were ruled by wealthy citizens who obtained their wealth through trade. It is important to point out that the superiority of the priesthood was felt in lesser degree. In Greece, mathematics passed from the priests to the wealthy citizens. The first learned Greeks, known as the first philosophers, travelled through Egypt or Babylon. That was what Thales and Pythagoras did, as well as many others, including Plato. After these exhausting voyages, the future scientists could rest and contemplate. To some extent, their contemplation was conditioned by their status! On the one hand, they were independent from the ones who gave them knowledge i.e. educated them, thus enabled science to develop independently from religion; on the other hand, they were not guided by the needs of their surrounding, but by their own wishes and abilities, seeking answers they were interested in, but had no practical value, at least, for the immediate surroundings.

Among the first Greek mathematicians was Thales, a merchant from Miletus (624-547 BC). Thales was one of the Seven Wise Men of Greece, Phoenician by origin. He was the first to claim that the soul is immortal and predicted the solar eclipse of 28th May 585 BC. “Know thyself” and “Nothing in excess” are attributed to him. Travelling through Egypt, he familiarized himself with geometry and astronomy. In his old age, in his home he dedicated himself to science and his students. Thales is considered to be the founder of the so-called Ionian school that marked the beginning of a new age in history, the age of public schools. Thales was the first to adopt the principle for ‘school available to anyone’. He invited students from all quarters of the learned world, and told them:’ I will teach you everything I know myself’. The truth behind this statement was verbal, but nonetheless, Thales in defense of his claim often said: ‘It is like that’. Hence the famous dogmatic sentence ‘The teacher said so’ as an answer to the question ‘Why is that so?’ which, at that time, was often the only basis for proof.

He was the first to form the proposition about the equality of right angles, the proposition for equality of the base angles of an isosceles triangle and its converse. He also knew some propositions about similar right triangles. He used denotations for congruent triangles and similar triangles. Thales managed to solve some practical geometric tasks, for example, calculating the height of the Cheops pyramid and the distance of ships from the shore.

Thales determined the height of the Cheops pyramid by using a similarity with isosceles right triangles. He set up a stick on a flat sand surface and drew a circle around the base of the stick with a radius equal to the length of the stick. Thales waited for the moment when the shadow of the top of the stick was on the circle, that is, when the length of the visible part of the stick was equal to the length of the shadow. By measuring the stick’s shadow he determined the height of the pyramid.

Thales determined the distance of ships from the shore by using the similarity of some right triangles.

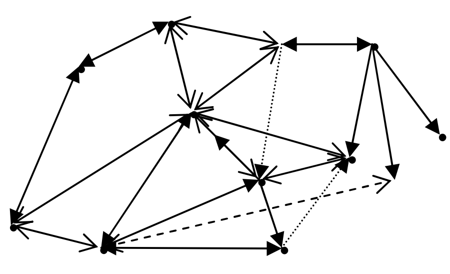
Starting from the first natural numbers and the simplest geometric figures, discussed as drawings of real-life objects, we can notice gradual accumulation of knowledge that, at a certain time incited leaps, presented by the birth of new methods, ideas, procedures, …, firstly superficially, and then thoroughly with formed mathematical concepts and statements for their properties.

In ancient India, development of mathematical knowledge was connected with the creation of transferable symbols in arithmetic and algebra, the decimal number system, and the application of basic technical procedures in mathematics.

Mathematicians in the Arab world inherited what was created by the Greek and Indian mathematicians; they introduced some new points but did not manage to find an applicable symbiosis between the Indian technical procedures and Greek deduction. The main merit for the development of mathematics lies in the preservation of the ancient Greek achievements, the heritage of ancient Indian people and the capabilities that emerged from these in Europe, All this begins to be taught in Western Europe in the 13^{th} century. Apart from these achievements on global level, mathematicians from Western Europe, with the help of the Arabs are responsible for the occurrence of new array of knowledge and ideas regarding different specific phenomena.

2. Second section of the paper

The tools through which mathematical knowledge can be strengthened are: concepts, axioms, definitions, proofs of theorems, algorithms, tasks and their solutions. The concepts are the focal point. The axioms, definitions and theorems serve to connect different concepts. Therefore, the concepts, axioms, definitions, and theorems form a structure. This structure can be graphically presented in the following manner:



In the graph, the dots represent mathematical concepts. The bidirectional continuous arrows represent axioms. They connect the primary concepts. The unidirectional continuous arrows represent definitions. They connect each defined concept to the concept used in the definition. The arrows are directed from the defining towards the defined concept. The bidirectional broken arrows represent theorems. They connect the defining concepts to the concepts we define or the primary concepts. Proofs can be modeled by ‘constructing’ bidirectional broken arrows representing theorems. In the graph, tasks, solutions to tasks and algorithms are not represented. Seeking the answer why is that so, we will look at the following:

Theorems in principle refer to the entire scope of the concept (contained in the definitions). Unlike theorems, tasks refer only to separate elements of the scopes of concepts or the real subsets of those scopes.

At that, these subsets are not formed as a scope to a certain concept. When this is done, the appropriate tasks become theorems.

Therefore, historically there is certain dynamics between *the set of theorems and set of tasks.*

Tasks were the basic tool for strengthening mathematical knowledge in pre-Greek period. Over time, tasks were replaced by theorems. Since then, tasks are a didactic tool wherever there is teaching.

Functions of tasks as didactic tool

With the occurrence of theorems, the belief that tasks and solving the same are a suitable tool to adopt concepts and theorems, consolidate knowledge and develop skills to reason with axioms, definitions and theorems and thus determine the level of this knowledge is gradually accepted. For these reasons solving tasks is an important didactic tool in mathematics teaching.

The content of some non-mathematical tasks, as well as solving tasks as an activity with appropriate organization, can influence the formulation of important personal qualities such as: activity or passivity, thoughtfulness or negligence, increased interest or aversion to knowledge, sense for beauty, diligence or laziness, etc.

One very important activity for mathematics teachers is seeking answers to the questions “How”, “Where” and “When” to solve tasks in the mathematics course.

Mathematical knowledge is used in different places, according to its role:

- to introduce new knowledge and skills (tasks as parts of a given theorem), to decrease the load of the proof of the theorem or complex procedures
- to detect new knowledge, i.e. solving tasks by introducing new knowledge
- consolidate new knowledge and skills
- to detect skills for solving non-mathematical tasks using mathematical tasks
- to control, asses and consolidate knowledge and skills during oral examinations and conducting different types of tests or quizzes
- to detect omissions in the students’ knowledge and skills

In favor of the previously stated - the usage of mathematical tasks to detect omissions in students' knowledge and skills, it is crucial that we mention the necessity of using elementary tasks i.e. tasks from the first type to eliminate the mistakes made by students. Mistakes most frequently occur because of: poor adoption of properties, false analogy with some equations, superficial adoption of theorems regarding equivalence, usually when students perform equivalent transformations and they are not asked to nominate the theorems they apply, obtained fixed representations from working with equations, and poor adoption of properties regarding numerical inequalities.

The best way to correct this type of mistakes is by proper adoption of the theory, repetition of the definitions, properties and theorems, proper application of the same, as well as usage of counter-examples.

For the needs of this survey a short questionnaire was prepared. The questionnaire was distributed to more than fifty successful university and school teachers, students from the university, and primary and high school students, and in continuation we will describe the conclusions we reached.

The short questionnaire consisted of 5 questions which are:

1. Which are the tools through which mathematical knowledge can be strengthened?
2. Which are the focal points?
3. What is a mathematical task?
4. Describe the meaning and importance of a mathematical task. How has the role of mathematical tasks changed through history?
5. How long do your tests last?

The questionnaire facilitated the detection of the situation giving a clear signal that in our conditions, the methodological knowledge is more declarative than realistic. To support this, we will mention the most important conclusions:

Almost everyone included in the survey, except two, believe that mathematical tasks are focal points, and only 50% wrote other tools such as definitions, theorems, algorithms, tasks, and nobody has given complete answers. Everyone knows to write some mathematical tasks, but only 50% try to give the definition and only 30% succeed. Everyone knows the importance of mathematical tasks but there is a big differences in the given answers.

Teachers give and students take tests which last an entire school hour, instead of short assessments, ignoring the fact of the extent and manner in which "the deviations from the path of cognition are significant for the students" , as well as of the process of acquiring cognitive and permanent knowledge.

The analysis of this questionarie raises many new questions, but the imposed conclusion is the absence of real methodical skills which are unfortunately average in almost all mentioned educational institutions. It is important to note that the changed place of mathematical tasks does not influence their significance.

Trying to emphasize the significance of mathematical tasks, we conclude that mathematics teaching is impossible without mathematical tasks.

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ON THE MATRIX APPROACH IN THE COMPLEX ANALYSIS AND ITS GEOMETRIC APPLICATIONS

Milen Hristov

Abstract

The main purpose of this article is to extend the theoretical background for the computer-aided geometric design by the essential use of the matrix unitary group. Based on the unitary-matrix representation of the field of complex numbers we consider unitary-matrix curves and unitary-matrix valued holomorphic functions of unitary-matrix argument. In this sense basic results are the unitary-matrix analogues of: the Frenet's formulas, signed curvature and turning angle (Theorem **1.2**), the complex derivative of a function of one complex variable (Theorem **1.3**) and Cauchy-Riemann equations (Corollary **1.2**). We apply these results to obtain the signed curvature of plane curve, defined by barycentric linear combination with respect to a fixed triangle (Theorem **2.1**) and as a consequence of a plane rational Bézier curve (Corollary **2.1**). Finally we describe in matrix sense the conformal image of a plane curve and get formula (2.4) for its signed curvature. As an application we consider the conformal images of a plane rational Bézier curve by Möbius and Zhukovski-type transformations. These ideas was motivated by the books [3], [2] and theoretic basics in [1] and [4]. This article continues author's works [5] and [6].

MSC 2010: 30C20, 30E05, 51B10, 53A04, 65D17

Key Words and Phrases: functions of one complex variable; unitary group; plane unitary matrix curves; conformal transformation; rational Bézier curves

1. Unitary-matrix representations of the basic objects of the complex analysis

1.1. Unitary-matrix representations of the field of complex numbers

We refer to the well known unitary-matrix representation of the field $\mathbb{C}$ of complex numbers, formulated in the following

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Theorem 1.1. *The field* $\mathbb{C} = \{z = a + bi : a, b \in \mathbb{R}, i = \sqrt{-1}\}$ *of complex numbers is isomorphic to the unitary matrix group*

$$\mathbb{U} = \left\{ Z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = aE + bJ : \, a, b \in \mathbb{R}, \; \mathbb{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \, \mathbb{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\mathbb{J}^2 \right\}.$$

The isomorphism arises from linear isomorphism between the two-dimensional real vector spaces $\mathbb{C} = \mathrm{span}\{1, i : i^2 = -1\}$ and $\mathbb{U} = \mathrm{span}\{\mathbb{E}, \mathbb{J} : \mathbb{J}^2 = -\mathbb{E}\}$. We give the correspondence between the algebraic operations in $\mathbb{C}$ and $\mathbb{U}$ in the following

Corollary 1.1. *Let the complex numbers*

$$z = a + bi, \quad z_1 = a_1 + b_1 i, \quad z_2 = a_2 + b_2 i$$

be represented by the $\mathbb{U}$*-matrices*

$$Z = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad Z_1 = \begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{pmatrix}$$

respectively. Then

$$\begin{array}{rcl} \mathrm{Re}\{z\} & \longleftrightarrow & \frac{1}{2}\mathrm{trace}\{Z\} \\ \mathrm{Im}\{z\} & \longleftrightarrow & -\frac{1}{2}\mathrm{trace}\{\mathbb{J}Z\} \\ & & \\ \overline{z} = a - bi & \longleftrightarrow & Z^T \; (\textit{the transposed of } Z) \\ |z| = \sqrt{a^2 + b^2} & \longleftrightarrow & \sqrt{\det Z} \\ z_1 \pm z_2 & \longleftrightarrow & Z_1 \pm Z_2 \\ z_1 z_2 & \longleftrightarrow & Z_1 Z_2 \\ & & \\ \dfrac{1}{z} & \longleftrightarrow & (Z)^{-1} = \dfrac{1}{\det Z}.Z^T \\ & & \\ \dfrac{z_1}{z_2} & \longleftrightarrow & Z_1(Z_2)^{-1} = \dfrac{1}{\det Z_2} Z_1.Z_2^T \end{array}$$

$$z = |z|(\cos\varphi + i \sin\varphi) = |z|\mathrm{e}^{i\varphi} \quad \longleftrightarrow \quad Z = \sqrt{\det Z}.E_\varphi, \; E_\varphi = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

(*trigonometric form of* z) (*trigonometric form of* Z).

We have $E_\varphi(E_\psi)^\epsilon = E_\varphi E_{\epsilon\psi} = E_{\varphi+\epsilon\psi}$, $\epsilon = \pm 1$. *The matrix analogue of:*

- *De Moivre's formula is*

$$z^n \longleftrightarrow Z^n = \sqrt{(\det Z)^n}.(E_\varphi)^n = \sqrt{(\det Z)^n}.E_{n\varphi}, \; n \in \mathbb{Z}.$$

- *the n-th root is*

$$(\sqrt[n]{z})_k \longleftrightarrow (\sqrt[n]{Z})_k = \sqrt[2n]{\det Z}.E_{\frac{\varphi+2k\pi}{n}},\ k=1,2,\ldots,n.$$

Proof. All the assertions are proved by direct matrix calculations. □

By means of Corollary **1.1** the algebraic properties and problems in $\mathbb{C}$ generate algebraic properties and problems in $\mathbb{U}$ and vise versa. Moreover $\mathbb{U}$ is endowed with two metrics. The standard euclidean metric ($\overset{(\varepsilon)}{g}$) and the spherical metric ($\overset{(\sigma)}{g}$) in $\mathbb{C}$ generate $\mathbb{U}$-metrics, denoted by the same letters and defined as follows:

$$\overset{(\varepsilon)}{g}(Z_1,Z_2) = \sqrt{\det(Z_1 - Z_2)}, \quad Z_1, Z_2 \in \mathbb{U}, \tag{1.1}$$

$$\overset{(\sigma)}{g}(Z_1,Z_2) = \frac{\sqrt{\det(Z_1 - Z_2)}}{\sqrt{1+\det Z_1}\sqrt{1+\det Z_2}}, \quad Z_1, Z_2 \in \mathbb{U}. \tag{1.2}$$

Both metrics are equivalent over any finite set of $\mathbb{U}$-matrices and are $\mathbb{J}$-invariant:

$$\overset{(\varepsilon)}{g}(Z_1,Z_2) = \overset{(\varepsilon)}{g}(\mathbb{J}Z_1,\mathbb{J}Z_2), \quad \overset{(\sigma)}{g}(Z_1,Z_2) = \overset{(\sigma)}{g}(\mathbb{J}Z_1,\mathbb{J}Z_2)$$

The metric (1.2) is used for infinite sets. Analogously to the complex extended plane $\mathbb{C}^* = \mathbb{C}\cup\infty$, we consider the extension $\mathbb{U}^*$ of $\mathbb{U}$ by adding the matrix at infinity Ω – the analogue of the complex point at infinity (∞). By means of the stereographic projection of the $\mathbb{C}$-plane over the Riemann sphere $\boldsymbol{\sigma}$: $x^2+y^2+z^2 = z$ from its north pole $N(0,0,1)$ and by using Theorem **1.1** one talks about stereographic projection of $\mathbb{U}$ over the Riemann sphere $\boldsymbol{\sigma}$. Thus the matrix at infinity $\Omega \equiv \mathbb{J}\Omega$ added to $\mathbb{U}$ corresponds to the north pole N of $\boldsymbol{\sigma}$ and

$$\overset{(\sigma)}{g}(Z_1,\Omega) = \frac{1}{\sqrt{1+\det Z_1}} = \overset{(\sigma)}{g}(\mathbb{J}Z_1,\mathbb{J}\Omega), \quad Z_1 \in \mathbb{U}.$$

In this way the topology of $\mathbb{C}$ ($\mathbb{C}^*$) induces a topology of $\mathbb{U}$ ($\mathbb{U}^*$). For local considerations the metric (1.1) usually is used. From geometric point of view $\mathbb{U}$ is an euclidean space with respect to the euclidean scalar product (we denote it by "$\bullet$") which arises from the same one for the linear space of vector-positions representing the complex numbers. Concretely if the vectors $\vec{r}_s = (x_s, y_s)$ represent $z_s = x_s + iy_s$, $s = 1,2$, then the geometric scalar product $\vec{r}_1.\vec{r}_2 = x_1x_2 + y_1y_2 = \mathrm{Re}\{z_1\overline{z}_2\}$.

By means of Corollary **1.1** for the euclidean scalar product in $\mathbb{U}$ we get

$$Z_1 \bullet Z_2 = \frac{1}{2}\mathrm{trace}\{Z_1Z_2^T\}, \quad Z_1, Z_2 \in \mathbb{U}. \tag{1.3}$$

This scalar product is compatible with the metric (1.1):

$Z_1 \bullet Z_1 = \det Z_1 = \overset{(\varepsilon)}{g}{}^2(Z_1, O) =: |Z_1|^2$, where O is the zero-(2×2)-matrix.

The angle between $Z_1, Z_2 \in \mathbb{U}$ is defined as usually by $\cos\angle(Z_1, Z_2) = \dfrac{Z_1 \bullet Z_2}{|Z_1||Z_2|}$.
We call the matrix

$$\Gamma(Z_1, Z_2) = \begin{pmatrix} Z_1 \bullet Z_1 & Z_1 \bullet Z_2 \\ Z_2 \bullet Z_1 & Z_2 \bullet Z_2 \end{pmatrix} \tag{1.4}$$

$\mathbb{U}$-Gram matrix for the ordered pair (Z_1, Z_2) of $\mathbb{U}$-matrices. Clearly for any real linear combination $p = p_1z_1 + p_2z_2$, $(p_s \in \mathbb{R},\ z_s \in \mathbb{C}, s = 1, 2)$ and its $\mathbb{U}$-matrix $P = p_1Z_1 + p_2Z_2$ we have

$$\det P = p\bar{p} = (p_1, p_2)\Gamma(Z_1, Z_2)(p_1, p_2)^T. \tag{1.5}$$

1.2. Unitary-matrix representations of the complex-valued functions

1.2.1. Complex-valued functions of real argument and its $\mathbb{U}$-matrix analogues. Let $J \subseteq \mathbb{R}$ be an open interval. We consider $\mathbb{C}$-valued function of real argument $t \in J$

$$\begin{array}{rcl} f : J & \longrightarrow & \mathbb{C} \\ t & \longmapsto & f(t) = u(t) + iv(t). \end{array}$$

From geometric and $\mathbb{U}$-matrix point of view we have the following one-to-one correspondences

$$f(t) = u(t) + iv(t) \overset{1-1}{\longleftrightarrow} \vec{r}(t) = (u(t), v(t)) \overset{1-1}{\longleftrightarrow} F(t) = \begin{pmatrix} u(t) & v(t) \\ -v(t) & u(t) \end{pmatrix}. \tag{1.6}$$

We call $F(t)$ in (1.6) $\mathbb{U}$-*matrix valued function of real argument*. In case $f(t)$ is continuous (i.e. $u(t), v(t)$ are continuous at once), then in $\mathbb{C}(z)$-plane there exists the curve

$$c : z = f(t) \iff c : \vec{r}(t) = (u(t), v(t)),\ t \in J\,,$$

and in the unitary group $\mathbb{U}$ – the matrix curve $C : Z = F(t)$, $t \in J$.

As an example to the centered at z_0 having radius R circle (open disc) $c : |z - z_0| = R \quad (\tilde{c} : |z - z_0| < R)$ corresponds $\mathbb{U}$-matrix circle (open disc)

$$C : \overset{(\varepsilon)}{g}(Z, Z_0) = \sqrt{\det(Z - Z_0)} = R \quad (\widetilde{C} : \overset{(\varepsilon)}{g}(Z, Z_0) = \sqrt{\det(Z - Z_0)} < R).$$

The $\mathbb{U}$-matrix parametric equations of the last ones are

$$C : Z = Z_0 + R.E_t,\ t \in [0, 2\pi), \qquad \widetilde{C} : Z = Z_0 + \rho.E_t,\ t \in [0, 2\pi),\ \rho \in [0, R).$$

Further we give the matrix analogues of the basic facts of the differential geometry of plane curves.

The *differentiation of* $\mathbb{U}$*-matrix valued function of real argument* is defined by means of (1.6):

$$f'(t) = u'(t) + iv'(t) \overset{1-1}{\longleftrightarrow} \vec{r}'(t) = (u'(t), v'(t)) \overset{1-1}{\longleftrightarrow} F'(t) = \begin{pmatrix} u'(t) & v'(t) \\ -v'(t) & u'(t) \end{pmatrix}.$$

The vector $\vec{t} = \vec{r}'(t) = (u'(t), v'(t))$ is tangent and $\vec{n} = (-v'(t), u'(t))$ is normal to the curve c. Thus for the matrix curve $C : Z = F(t)$, $t \in J$ we call the matrices

$$T(t) = \begin{pmatrix} u'(t) & v'(t) \\ -v'(t) & u'(t) \end{pmatrix} \qquad \text{and} \qquad N(t) = \begin{pmatrix} -v'(t) & u'(t) \\ -u'(t) & -v'(t) \end{pmatrix} = \mathbb{J}.T(t)$$

matrix tangent and *matrix normal* respectively. This two matrices are orthogonal with respect to the scalar product (1.3): $T(t) \bullet N(t) = 0$. We call the pair $\{T(t), N(t)\}$ *local* $\mathbb{U}$*-matrix Frenet frame.* C^r-smoothness is defined in a standard way. The *integration* for such matrix functions is defined likewise:

$$\int F(t)dt = \begin{pmatrix} \int u(t)dt & \int v(t)dt \\ -\int v(t)dt & \int u(t)dt \end{pmatrix} = \underbrace{\begin{pmatrix} U(t) & V(t) \\ -V(t) & U(t) \end{pmatrix}}_{=\Phi(t)} + \underbrace{\begin{pmatrix} C_1 & C_2 \\ -C_2 & C_1 \end{pmatrix}}_{=C=\text{const}}$$

and
$$\int_a^b F(t)dt = \Phi(b) - \Phi(a).$$

The length of the arc of the matrix curve $C : Z = F(t)$, $t \in [a, b]$ is

$$s(C) = \int_a^b \sqrt{\det F'(t)}dt, \qquad \text{the function} \qquad s(t) = \int_{t_0}^t \sqrt{\det F'(\lambda)}d\lambda$$

is said to be arc-length parametrization and the values $s = s(t)$ arc-length parameter. For any matrix curve $C : Z = F(s)$ parametrized by its arc-length s one gets $\det F'(s) = 1$ and its matrix tangent and matrix normal are of length 1:

$$T(s) \bullet T(s) = |T(s)|^2 = 1, \qquad N(s) \bullet N(s) = |N(s)|^2 = 1.$$

The last one means that there exists function $\theta(s)$ so that trigonometric forms

$$F'(s) = T(s) = E_{\theta(s)}\,, \quad N(s) = \mathbb{J}T(s) = E_{\frac{\pi}{2}} E_{\theta(s)} = E_{\theta(s)+\frac{\pi}{2}}$$

are valid. The geometric meaning of $\theta(s)$ (called turning angle) is well known – this is the measure of the oriented angle between the Re^+-axis and the tangent vector $\vec{t}$ to the curve $c : z = u(s) + iv(s)$.

We give $\mathbb{U}$-matrix variant of the well known Frenet formulas for plane curves in the following theorem.

Theorem 1.2. *Let* $C : Z = F(s) = \begin{pmatrix} u(s) & v(s) \\ -v(s) & u(s) \end{pmatrix}$ *be at least* C^2 *smooth regular* $\mathbb{U}$*-matrix curve, parametrized by its arc-length* $s \in J \subseteq \mathbb{R}$*. Then*

$$F''(s) = k(s)\mathbb{J}F'(s) \quad (\textit{Frenet's } \mathbb{U}\textit{-matrix formula}),$$

where the function $k(s)$ *(called signed curvature of* C*) is expressible as*

$$k(s) = \frac{1}{2}\mathrm{trace}\{F''(s).(\mathbb{J}F'(s))^T\} = -\frac{1}{2}\mathrm{trace}\{F'(s).(\mathbb{J}F''(s))^T\} = \begin{vmatrix} u'(s) & v'(s) \\ u''(s) & v''(s) \end{vmatrix}.$$

Moreover $k(s) = \theta'(s)$*, i.e.* $\theta(s) = \int_{s_0}^{s} k(\lambda)d\lambda$*, where* $\theta(s)$ *is the turning angle.*

Proof. For the orthonormal Frenet pair $\{T(s) = F'(s); N(s) = \mathbb{J}T(s)\}$ in the euclidean 2-space $(\mathbb{U}, \bullet)$ it is valid

$$\begin{cases} T'(s) = F''(s) = \ell(s)T(s) + k(s)N(s) \\ N'(s) = \mathbb{J}T'(s) = -k(s)T(s) + \ell(s)N(s). \end{cases}$$

Then $\ell(s) = T(s) \bullet T'(s) = \frac{1}{2}(T(s) \bullet T(s))' = 0$. Now

$$\begin{cases} T'(s) = k(s)N(s) = k(s)\mathbb{J}T(s) \\ N'(s) = -k(s)T(s) = k(s)\mathbb{J}N(s) \end{cases} \iff F''(s) = k(s)\mathbb{J}F'(s).$$

The last equality can be rewritten as $\mathbb{J}F''(s) = -k(s)F'(s)$. Thus for the signed curvature we get

$$k(s) = F''(s) \bullet (\mathbb{J}F'(s)) = -F'(s) \bullet (\mathbb{J}F''(s))$$

and applying (1.2) we obtain the expressions for $k(s)$ in Theorem **1.2**. To prove $k(s) = \theta'(s)$ we use the trigonometric form given just before and counting $F''(s) = (E_{\theta(s)})' = \theta'(s)\mathbb{J}F'(s)$ jointly with the Frenet $\mathbb{U}$-matrix formula.

□

It is easy to check by using the function of arc-length parametrization that the expression for the signed curvature for $\mathbb{U}$-matrix curve

$C : Z = F(t) = \begin{pmatrix} u(t) & v(t) \\ -v(t) & u(t) \end{pmatrix}$ where t is not the arc-length parameter is

$$k(t) = \frac{\mathrm{trace}\{F''.(\mathbb{J}F')^T\}}{2(\det F')^{\frac{3}{2}}} = -\frac{\mathrm{trace}\{F'.(\mathbb{J}F'')^T\}}{2(\det F')^{\frac{3}{2}}} = \frac{\begin{vmatrix} u' & v' \\ u'' & v'' \end{vmatrix}}{2(\det F')^{\frac{3}{2}}}. \tag{1.7}$$

Now we call the $\mathbb{U}$-matrix curve

$$E : E(t) = F(t) + \frac{1}{k(t)\sqrt{\det F'(t)}}N(t) = F(t) + \frac{1}{k(t)\sqrt{\det F'(t)}}\mathbb{J}F'(t)$$

$\mathbb{U}$-*matrix evolute* for the curve $C : Z = F(t)$. This $\mathbb{U}$-matrix curve is the envelope of the $\mathbb{U}$-matrix normals to the $\mathbb{U}$-matrix curve $C : Z = F(t)$.

1.2.2. Complex-valued functions of complex argument and its $\mathbb{U}$-matrix analogues. Let D be a domain in the complex plane $\mathbb{C}(z = x + iy)$ and let

$$\begin{array}{rcl} f : D & \longrightarrow & \mathbb{C} \\ z & \longmapsto & w = f(z) = u(x,y) + iv(x,y) \end{array}$$

be a function of the complex argument z taking its values in the complex plane $\mathbb{C}(w = u + iv)$. This situation by using Corollary **1.1** and the induced topology in $\mathbb{U}$ can be translated into $\mathbb{U}$-matrix form in a following manner.

Definition 1.1. Let D be a $\mathbb{U}$-matrix domain. **$\mathbb{U}$-matrix valued function of $\mathbb{U}$-matrix argument** is the map

$$\begin{array}{rcl} F : \quad D & \longrightarrow & \mathbb{U} \\ Z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} & \longmapsto & W = F(Z) = \begin{pmatrix} u(x,y) & v(x,y) \\ -v(x,y) & u(x,y) \end{pmatrix}. \end{array} \tag{1.8}$$

We denote by $\mathrm{F}\mathbb{U}(D)$ the set of all the such functions.

From geometric point of view $w = f(z)$ generates the map

$$\Phi : \left| \begin{array}{l} u = u(x,y) = \frac{1}{2}\mathrm{trace}\{F(Z)\} \\ v = v(x,y) = -\frac{1}{2}\mathrm{trace}\{\mathbb{J}F(Z)\} \end{array} \right. , \tag{1.9}$$

which acts over the points of the domain $D \subseteq \mathbb{R}^2(Oxy)$ into a point-set in $\mathbb{R}^2(Ouv)$. Here $Oxy = \{O, \vec{e}_1, \vec{e}_2\}$ and Ouv are orthonormal coordinate systems compatible with the complex planes $\mathbb{C}(z = x+iy)$ and $\mathbb{C}(w = u+iv)$ respectively.

When $u(x,y)$ and $v(x,y)$ are continuous, i.e. Φ is continuous, then $F(Z)$ is said to be continuous. In this case there exists the pair of surfaces

$$\{S_{\mathrm{Re}} : \zeta = u(x,y), S_{\mathrm{Im}} : \zeta = v(x,y)\} \tag{1.10}$$

in $\mathbb{R}^3$ with respect to the orthonormal coordinate system $Oxy\zeta$.

Conclusion. *The complex analysis of continuous functions of one complex variable is equivalent with each of the following:*

(i) *the study of continuous maps* (1.9),

(ii) *the study of the pairs of surfaces* (1.10),

(iii) *the study of continuous* $\mathbb{U}$-*matrix valued functions of one* $\mathbb{U}$-*matrix variable* (1.8).

When $u(x,y)$ and $v(x,y)$ are differentiable with respect to x and y then there exist the operators of partial differentiations ∂_x and ∂_y, so that for $F(Z)$

given by (1.8)

$$\partial_x F \overset{\text{def}}{=\!=} \begin{pmatrix} u'_x(x,y) & v'_x(x,y) \\ -v'_x(x,y) & u'_x(x,y) \end{pmatrix}, \quad \partial_y F \overset{\text{def}}{=\!=} \begin{pmatrix} u'_y(x,y) & v'_y(x,y) \\ -v'_y(x,y) & u'_y(x,y) \end{pmatrix}.$$

Moreover for a vector $\vec{h} = (h_1, h_2)$ in the plane $Oxy = \{O, \vec{e}_1, \vec{e}_2\}$ we have the derivative of $F(Z)$ in $\vec{h}$-direction at $Z \in D$, corresponding to $z = x + iy \in \mathbb{C}$

$$\partial_{\vec{h}} F_{\,|(x,y)} \overset{\text{def}}{=\!=} \begin{pmatrix} \dfrac{\partial u}{\partial \vec{h}}(x,y) & \dfrac{\partial v}{\partial \vec{h}}(x,y) \\ \\ -\dfrac{\partial v}{\partial \vec{h}}(x,y) & \dfrac{\partial u}{\partial \vec{h}}(x,y) \end{pmatrix}.$$

Thus $\partial_x F = \partial_{\vec{e}_1} F$ and $\partial_y F = \partial_{\vec{e}_2} F$. The derivatives of higher order $\partial^k_{x^s y^{k-s}} F$, $\partial^k_{\vec{p}^s \vec{q}^{k-s}} F$ are defined likewise.

We denote by:

- $\mathrm{C}^0\mathrm{F}\mathbb{U}(D)$ **the set of the continuous** $F(Z) \in \mathrm{F}\mathbb{U}(D)$**,**
- $\mathrm{C}^r\mathrm{F}\mathbb{U}(D)$ **the set of the** r**-times smooth** $F(Z) \in \mathrm{F}\mathbb{U}(D)$**, i.e.,** $F(Z) \in \mathrm{F}\mathbb{U}(D)$ **so that** $\partial^k_{x^s y^{k-s}} F \in \mathrm{C}^0\mathrm{F}\mathbb{U}(D)$, $k = 1, 2, \ldots, r$.

Further we give the $\mathbb{U}$-matrix analogue of the complex derivative. In order to do this we shall consider jointly any vector $\vec{h} = (h_1, h_2)$ as the complex number $h = h_1 + ih_2$ and as the corresponding $\mathbb{U}$-matrix $H = \begin{pmatrix} h_1 & h_2 \\ -h_2 & h_1 \end{pmatrix}$.

Definition 1.2. The function $F(Z) \in \mathrm{F}\mathbb{U}(D)$ is said to be $\mathbb{U}$-differentiable at $\mathbb{U}$-point (i.e. matrix) Z, if there exists the following finite limit

$$F'(Z) = \lim_{\mathbb{U} \ni H \to O} [F(Z + H) - F(Z)].H^{-1}.$$

We call the matrix $F'(Z)$ first $\mathbb{U}$-derivative of $F(Z)$ at Z. We call $F(Z)$ $\mathbb{U}$-holomorphic over D (at the $\mathbb{U}$-point Z_0 resp.) if $F'(Z)$ exists for all $Z \in D$ (Z in an open $\mathbb{U}$-matrix neighborhood of Z_0 respectively).

Theorem 1.3. *Let $F(Z) \in \mathrm{F}\mathbb{U}(D)$ be $\mathbb{U}$-differentiable at $\mathbb{U}$-point Z_0, corresponding to $z_0 = x_0 + iy_0$ and $\vec{z}_0 = (x_0, y_0)$. Then for an arbitrary fixed $\mathbb{U}$-matrix $H = \begin{pmatrix} h_1 & h_2 \\ -h_2 & h_1 \end{pmatrix}$ with $\det H = 1$ (i.e. $H \in SO(2, \mathbb{R})$) and its corresponding unit vector $\vec{h} = (h_1, h_2)$ the following formula is valid*

$$F'(Z_0) = H^T.\partial_{\vec{h}} F_{\,|(x_0,y_0)}.$$

Proof. We shall use the standard complex language and the obtain formula will be translated in a matrix form. So let $\vec{h} = (h_1, h_2)$ be an arbitrary fixed unit vector, corresponding to $h = h_1 + ih_2 : |h| = 1$.

$$f'(z_0) \overset{def}{=\!=} \lim_{t\to 0} \frac{f(z_0 + th) - f(z_0)}{th}$$

$$= \lim_{t\to 0} \frac{u(x_0 + th_1, y_0 + th_2) + iv(x_0 + th_1, y_0 + th_2) - u(x_0, y_0) - iv(x_0, y_0)}{t(h_1 + ih_2)}.$$

Since $|h| = 1$ by replacing $\frac{1}{h} = \overline{h} = h_1 - ih_2$ for $f'(z_0)$ we get

$$\overline{h}\Big[\underbrace{\lim_{t\to 0} \frac{u(x_0 + th_1, y_0 + th_2) - u(x_0, y_0)}{t}}_{\left(\frac{\partial u}{\partial x}\cdot h_1 + \frac{\partial u}{\partial y}\cdot h_2\right)_{|(x_0,y_0)}} + i\cdot \underbrace{\lim_{t\to 0} \frac{v(x_0 + th_1, y_0 + th_2) - v(x_0, y_0)}{t}}_{\left(\frac{\partial v}{\partial x}\cdot h_1 + \frac{\partial v}{\partial y}\cdot h_2\right)_{|(x_0,y_0)}}\Big].$$

Thus we obtain

$$f'(z_0) = \overline{h}.\left(\frac{\partial u}{\partial \vec{h}} + i\frac{\partial v}{\partial \vec{h}}\right)_{|(x_0,y_0)}.$$

Finally by using Corollary **1.1** we translate this result into $\mathbb{U}$-matrix form and get the formula for the matrix $F'(Z_0)$. □

Now we get the $\mathbb{U}$-matrix form of the well known Cauchy-Riemann (C-R) equations which are necessary and sufficient conditions for a function $F(Z) \in C^1F\mathbb{U}(D)$ to be $\mathbb{U}$-holomorphic. This follows immediately from Theorem **1.3** by equalizing the expressions for $F'(Z)$ obtained for the basic vectors $\vec{e}_1$ and $\vec{e}_2$ and their corresponding matrices $\mathbb{E}$ and $\mathbb{J}$. In such a way we get the following

Corollary 1.2 (The $\mathbb{U}$-matrix C-R equation). *The $\mathbb{U}$-matrix equation for $\mathbb{U}$-holomorphic $F(Z)$ given by (1.8), which is equivalent to the standard C-R equations:* $\left| \begin{array}{l} u'_x = v'_y \\ u'_y = -v'_x \end{array}\right.$ *for holomorphic $f(z) = u(x, y) + iv(x, y)$ is*

$$\mathbb{J}.\partial_{\vec{e}_1}F = \partial_{\vec{e}_2}F \iff \mathbb{J}.\partial_{\vec{e}_2}F = -\partial_{\vec{e}_1}F.$$

With an arbitrary $\mathbb{U}$-holomorphic $F(Z)$ in $\mathbb{U}$-domain D defined by (1.8) the following geometric objects are associated:

- the Jacobi matrix of the map (1.9): $J_\Phi(x, y) = \begin{pmatrix} u'_x & u'_y \\ v'_x & v'_y \end{pmatrix}$,
- the pair of normal vectors

$$\{\overrightarrow{N}_{\mathrm{Re}} = (u'_x, u'_y, -1), \quad \overrightarrow{N}_{\mathrm{Im}} = (v'_x, v'_y, -1)\}$$

to the pair of surfaces (1.10),

• the pair of gradient vectors

$$\{\nabla u = (u'_x, u'_y) = \Pi_{|\delta}(\overrightarrow{N}_{\mathrm{Re}}), \quad \nabla v = (v'_x, v'_y) = \Pi_{|\delta}(\overrightarrow{N}_{\mathrm{Im}})\},$$

orthogonal to the pairs of level curves for (1.10)

$$\{C_{\mathrm{Re}} = S_{\mathrm{Re}} \cap \delta, \quad C_{\mathrm{Im}} = S_{\mathrm{Im}} \cap \delta\}$$

where $\Pi_{|\delta}$ is the orthogonal projection of $\mathbb{R}^3$ over level plane $\delta : \zeta = C$ $(C \in \mathbb{R})$ with respect to the orthonormal coordinate system $Oxy\zeta$.

In the list below we summarize the geometric behavior of these objects, based on Corollary **1.2** and the additional condition: $F'(Z) \neq O$ for all $Z \in D$.

$\boxed{1}$ *The Jacobi matrix $J_\Phi(x, y)$ of the map (1.9) is invertible in D:*

$$\det J_\Phi = \det(\partial_{\vec{e}_s} F_{|(x,y)}) = \det F'(Z) = (u'_x)^2 + (u'_y)^2 = (v'_x)^2 + (v'_y)^2 > 0, \;\; s = 1, 2$$

and Φ is an orientation preserving conformal map ;

$\boxed{2}$ *The gradient equality $\nabla v = \nabla u.\mathbb{J}$ holds, i.e. $\{\nabla u, \nabla v\}$ is positively oriented orthogonal pair: $\nabla u.\nabla v = 0, \quad |\nabla u| = |\nabla v| = \sqrt{\det J_\Phi}$ with respect to the euclidean scalar product ;*

$\boxed{3}$ *The vector equality $\overrightarrow{N}_{\mathrm{Im}} = \overrightarrow{N}_{\mathrm{Re}}.(\mathbb{J}\times 1)$ holds, where $\mathbb{J}\times 1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,*

$$|\overrightarrow{N}_{\mathrm{Re}}| = |\overrightarrow{N}_{\mathrm{Im}}| = \sqrt{1 + \det J_\Phi}, \quad \cos\angle(\overrightarrow{N}_{\mathrm{Re}}, \overrightarrow{N}_{\mathrm{Im}}) = \frac{1}{1 + \det J_\Phi};$$

$\boxed{4}$ *The pair (1.10) consists of conjugate harmonic surfaces (the Laplace equations $\Delta u = \Delta v = 0$ hold) for which each pair of level curves $\{C_{\mathrm{Re}}, C_{\mathrm{Im}}\}$ is orthogonal.*

Finally, $\mathbb{U}$-matrix transcendental functions can be defined by the corresponding power series of $\mathbb{U}$-matrix argument.

2. Rational Bézier curves in the complex plane and its conformal images by Möbius and Zhukovski-type transformations

The Bézier curves are well known and useful tools in the field of computer-aided geometric design (CAGD) studying by many authors in this sense (e.g. see [2] and latest editions). From the view point of the barycentric analytic geometry of the real projective plane these curves have quite rich geometric behavior (e.g. see [5, 6]). We are going to consider such curves especially in the complex plane and its $\mathbb{U}$-matrix representation as in Subsection **1.2.1**.

2.1. Rational Bézier curves in the complex plane and its curvatures

The general construction is the following. Let $z_\ell = x_\ell + iy_\ell$, $\ell = 0, 1, 2, \ldots, n$ be $n+1$ complex points. Let $Z_\ell \in \mathbb{U}$ be the matrices, corresponding to the fixed z_ℓ. One takes the $(n+1)$ Bernstein's polynomials of degree n

$$B_n^\ell(t) = \binom{n}{\ell} t^\ell (1-t)^{n-\ell}, \quad \ell = 0, 1, 2, \ldots, n.$$

Obviously the barycentric identity $\sum_{\ell=0}^{n} B_n^\ell(t) = 1$ holds. It is well known fact that the sequence $\{B_n^\ell(t)\}_{n=1}^{\infty}$ is uniformly convergent over $[0, 1]$.

Remark. *From now on we use the Einstein's summation denotation:* $a_s b^s$.

The barycentric linear (shortly: b-linear) combination

$$c_\sigma : z(t) = B_n^\ell(t).z_\ell, \quad 0 \le t \le 1, \tag{2.1}$$

is known as standard Bézier curve of power n and basic polygon

$$\sigma = \{z_\ell = x_\ell + iy_\ell,\ \ell = 0, 1, 2, \ldots, n\}$$

(represented in Bernstein's polynomial basis). Likewise we call the b-linear $\mathbb{U}$-matrix combination

$$C_\sigma : Z(t) = B_n^\ell(t).Z_\ell = \begin{pmatrix} B_n^\ell(t).x_\ell & B_n^\ell(t).y_\ell \\ -B_n^\ell(t).y_\ell & B_n^\ell(t).x_\ell \end{pmatrix}, \quad 0 \le t \le 1,$$

standard $\mathbb{U}$-matrix Bézier curve of power n and basic $\mathbb{U}$-matrix polygon $\sigma = \{Z_\ell \in \mathbb{U},\ \ell = 0, 1, 2, \ldots, n\}$.

Further we consider only the case $n = 2$ and denote by Δ the basic triangle $\{z_0 z_1 z_2\}$ in $\mathbb{C}$ and the basic $\mathbb{U}$-matrix triangle $\{Z_0 Z_1 Z_2\}$ in $\mathbb{U}$. The curve

$$\overset{(u)}{c}_\Delta : z(t) = b^\ell(u, t) z_\ell, \quad t \in [0, 1],\ u > -1, \tag{2.2}$$

defined by the b-linear combination with functional coefficients

$$b^0(u, t) = \frac{B_2^0(t)}{B_2(u, t)} = \frac{(1-t)^2}{B_2(u, t)},$$

$$b^1(u, t) = \frac{u.B_2^1(t)}{B_2(u, t)} = \frac{u.2t(1-t)}{B_2(u, t)},$$

$$b^2(u, t) = \frac{B_2^2(t)}{B_2(u, t)} = \frac{t^2}{B_2(u, t)}$$

and normalizer $B_2(u, t) = B_2^0(t) + u.B_2^1(t) + B_2^2(t) = (1-t, t) \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix} \begin{pmatrix} 1-t \\ t \end{pmatrix}$

is called plane rational Bézier curve of order two with normal parametrization. By replacing z_ℓ with Z_ℓ in (2.2) we get its $\mathbb{U}$-matrix analog – the $\mathbb{U}$-matrix curve

$$\overset{(u)}{C}_{\Delta} : Z(t) = \begin{pmatrix} b^\ell(u,t)x_\ell & b^\ell(u,t)y_\ell \\ -b^\ell(u,t)y_\ell & b^\ell(u,t)x_\ell \end{pmatrix}, \quad t \in [0,1],\ u > -1. \tag{2.3}$$

The parameter u controls the geometric shape of the curve (2.2): for $u \in (-1,1)$ the curve is ellipse, for $u = 1$ the curve is (2.1) – the standard Bézier parabola, for $u \in (1,\infty)$ the curve is hyperbola. The geometric behavior of all such curves with respect to Δ is well familiar (see [2]). In [5] and [6] associated to these curves geometric objects and properties are obtained by barycentric calculus. The curvature of the standard Bézier parabola (2.1) in [5] is introduced. Our aim is to express the signed curvature of (2.2). Further the derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial^2 f}{\partial t^2}$, etc. will be denoted like $\dot{f}$, $\ddot{f}$, etc.. We give the following general result.

Theorem 2.1. *Let $c_{\Delta} : z = f(t) = \beta^\ell(t)z_\ell$ be C^2-smooth regular curve, defined by b-linear combination with C^2-coefficients $\beta^\ell(t) : \sum_{\ell=0}^{2}\beta^\ell(t) = 1$ with respect to basic triangle $\Delta = \{z_\ell = x_\ell + iy_\ell, \ell = 0,1,2\}$. Let $C_{\Delta} : Z = F(t) = \beta^\ell(t)Z_\ell$ be its corresponding $\mathbb{U}$-matrix curve by means of (2.3). Then the signed curvature of c_{Δ} is*

$$k_{c_{\Delta}}(t) = \pm S_{\Delta} . \frac{2\delta(t)}{(\det \dot{F}(t))^{\frac{3}{2}}},$$

where S_{Δ} is the area of the basic triangle, $\delta(t) = \det\begin{pmatrix} \dot\beta^\ell & \dot\beta^{\ell+1} \\ \ddot\beta^\ell & \ddot\beta^{\ell+1} \end{pmatrix}$ for an arbitrary $\ell \in \{0,1,2,\ (modulo\,3)\}$ and

$$\det \dot{F}(t) = (\dot\beta^0, \dot\beta^2) \overset{(0,2)}{\Gamma} (\dot\beta^0, \dot\beta^2)^T$$

with $\overset{(0,2)}{\Gamma} = \Gamma(Z_0 - Z_1, Z_2 - Z_1)$ being the $\mathbb{U}$-Gram matrix (1.4) for the ordered matrix pair $(Z_0 - Z_1, Z_2 - Z_1)$. The positive sign exists exactly when the ordered triple (z_0, z_1, z_2) is counterclockwise oriented.

Proof. We apply formula (1.7). Since

$$F(t) = \begin{pmatrix} \beta^\ell x_\ell & \beta^\ell y_\ell \\ -\beta^\ell y_\ell & \beta^\ell x_\ell \end{pmatrix},$$

then

$$(\mathbb{J}\dot{F}(t))^T = \begin{pmatrix} -\dot\beta^\ell y_\ell & -\dot\beta^\ell x_\ell \\ \dot\beta^\ell x_\ell & -\dot\beta^\ell y_\ell \end{pmatrix}, \quad \ddot{F}(t) = \begin{pmatrix} \ddot\beta^s x_s & \ddot\beta^s y_s \\ -\ddot\beta^s y_s & \ddot\beta^s x_s \end{pmatrix}$$

and

$$\tfrac{1}{2}\text{trace}\{\ddot{F}(t).(\mathbb{J}\dot{F}(t))^T\} \quad = \dot{\beta}^\ell \ddot{\beta}^s \det\begin{pmatrix} x_\ell & y_\ell \\ x_s & y_s \end{pmatrix}$$

$$= \sum_{m=0}^{2} \det\begin{pmatrix} \dot{\beta}^m & \dot{\beta}^{m+1} \\ \ddot{\beta}^m & \ddot{\beta}^{m+1} \end{pmatrix} \det\begin{pmatrix} x_m & y_m \\ x_{m+1} & y_{m+1} \end{pmatrix},$$

where the values of the summation index m are taken by modulo 3. By using the barycentric identity $\sum_{\ell=0}^{3} \beta^\ell(t) = 1$ it is easy to check, that

$$\det\begin{pmatrix} \dot{\beta}^0 & \dot{\beta}^1 \\ \ddot{\beta}^0 & \ddot{\beta}^1 \end{pmatrix} = \det\begin{pmatrix} \dot{\beta}^1 & \dot{\beta}^2 \\ \ddot{\beta}^1 & \ddot{\beta}^2 \end{pmatrix} = \det\begin{pmatrix} \dot{\beta}^2 & \dot{\beta}^0 \\ \ddot{\beta}^2 & \ddot{\beta}^0 \end{pmatrix} =: \delta(t).$$

Then $\delta(t)$ is common multiplier for the factors of the last sum and

$$\frac{1}{2}\text{trace}\{\ddot{F}(t).(\mathbb{J}\dot{F}(t))^T\} = \delta(t)\sum_{m=0}^{2} \det\begin{pmatrix} x_m & y_m \\ x_{m+1} & y_{m+1} \end{pmatrix} = \delta(t)\det\begin{pmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}.$$

The last determinant is the well known formula for $\pm 2S_\Delta$, where S_Δ is the area of the basic triangle and the sign $+(-)$ corresponds to the counterclockwise (clockwise) orientation of its vertices.

From the barycentric identity for β^ℓ-s it follows $\sum_{\ell=0}^{2} \dot{\beta}_\ell = 0$ and

$$\dot{F}(t) = \dot{\beta}^0(Z_0 - Z_1) + \dot{\beta}^2(Z_2 - Z_1).$$

By using (1.5) we get $\det \dot{F}(t)$ as the described in Theorem **2.1** quadratic form in $(\dot{\beta}^0, \dot{\beta}^2)$. Moreover it is not difficult to see that $\det \dot{F}(t)$ is one and the same for the cyclic index-pairs $(0,2), (2,1), (1,0)$. □

We apply Theorem **2.1** to the curve (2.2) counting its matrix representation $F(t) = Z(t)$ in (2.3) and concrete $\beta^\ell(t) = b^\ell(u,t)$, $\ell = 0,1,2$. We calculate

$$\delta(t) = \det\begin{pmatrix} \dot{b}^2 & \dot{b}^0 \\ \ddot{b}^2 & \ddot{b}^0 \end{pmatrix} = (\dot{b}^0)^2\Big(\frac{\dot{b}^2}{\dot{b}^0}\Big)^{\bullet}.$$

We represent the derivatives $\dot{b}^\ell$, $\ell = 0,2$ in following determinant forms:

$$\dot{b}^2 = \Big(\frac{t^2}{B_2}\Big)^{\bullet} = \frac{\Delta_2}{(B_2)^2}, \quad \dot{b}^0 = \Big(\frac{(1-t)^2}{B_2}\Big)^{\bullet} = \frac{\Delta_0}{(B_2)^2},$$

where

$$\Delta_2 = \begin{vmatrix} 2t & \dot{B}_2 \\ t^2 & B_2 \end{vmatrix}, \quad \Delta_0 = \begin{vmatrix} 2(t-1) & \dot{B}_2 \\ (t-1)^2 & B_2 \end{vmatrix}.$$

Now

$$\delta(t) = \frac{(\Delta_0)^2}{(B_2)^4}\left(\frac{\Delta_2}{\Delta_0}\right)^{\bullet} = \frac{1}{(B_2)^4}(\dot{\Delta}_2\Delta_0 - \dot{\Delta}_0\Delta_2)$$

$$= \frac{1}{(B_2)^4}\left(\begin{vmatrix} 2 & \ddot{B}_2 \\ t^2 & B_2 \end{vmatrix}\begin{vmatrix} 2(t-1) & \dot{B}_2 \\ (t-1)^2 & B_2 \end{vmatrix} - \begin{vmatrix} 2 & \ddot{B}_2 \\ (t-1)^2 & B_2 \end{vmatrix}\begin{vmatrix} 2t & \dot{B}_2 \\ t^2 & B_2 \end{vmatrix}\right).$$

The simplification implies

$$\delta(t) = \frac{2}{(B_2)^3}\left[t(1-t)\ddot{B}_2 + (2t-1)\dot{B}_2 - 2B_2\right].$$

For the quadratic form $B_2(u,t)$ of the curve (2.2) we have

$$B_2 = 2t(1-t)(u-1)+1, \quad \dot{B}_2 = 2(1-u)(2t-1), \quad \ddot{B}_2 = 4(1-u).$$

After corresponding substitutions and simplification we obtain $\delta(t)$ in the form

$$\delta(t) = \frac{-4(B_2+u)}{(B_2)^3}.$$

Further for the concrete $\dot{F}(t) = \dot{b}^0(Z_0 - Z_1) + \dot{b}^2(Z_2 - Z_1)$ from (1.5) we get

$$\det \dot{F}(t) = (\dot{b}^0, \dot{b}^2) \overset{(0,2)}{\Gamma} (\dot{b}^0, \dot{b}^2)^T = (B_2)^{-4}(\Delta_0, \Delta_2) \overset{(0,2)}{\Gamma} (\Delta_0, \Delta_2)^T.$$

Thus we proved the following consequence of Theorem **2.1**.

Corollary 2.1. *The signed curvature for the plane rational Bézier curve* $\overset{(u)}{c}_{\Delta}$*, defined by (2.2) is*

$$\overset{(u)}{k}_{\Delta}(t) = \pm S_{\Delta}\frac{-8(B_2)^3(B_2+u)}{\left[(\Delta_0, \Delta_2) \overset{(0,2)}{\Gamma} (\Delta_0, \Delta_2)^T\right]^{\frac{3}{2}}}.$$

2.2. Möbius and Zhukovski-type conformal images of plane rational Bézier curves

We consider the following general case. Let $C : Z = Z(t) = \begin{pmatrix} x(t) & y(t) \\ -y(t) & x(t) \end{pmatrix}$ be C^2-$\mathbb{U}$-matrix curve. Let $F(Z) \in C^2F\mathbb{U}(D)$, defined by (1.8) be $\mathbb{U}$-holomorphic function, generating the plane conformal map Φ given by (1.9) with Jacobi matrix J_{Φ}. Let $\Phi(C)$ be the conformal image of C:

$$\Phi(C) : Z = \Phi(t) = F(Z(t)) = \begin{pmatrix} u\big(x(t), y(t)\big) & v\big(x(t), y(t)\big) \\ -v\big(x(t), y(t)\big) & u\big(x(t), y(t)\big) \end{pmatrix}.$$

Theorem **1.3** and Corollary **1.2** imply

$$F'(Z) = H^T\partial_{\vec{h}}F = \partial_{\vec{e}_1}F = -\mathbb{J}\partial_{\vec{e}_2}F,$$

$$F''(Z) = (HP)^T \partial^2_{\vec{h}\vec{p}} F = (H^2)^T \partial^2_{\vec{h}^2} F = \partial^2_{\vec{e}_1^2} F = -\partial^2_{\vec{e}_2^2} F,$$

where $H, P \in SO(2,\mathbb{R})$ with corresponding unit vectors $\vec{h} = (h_1, h_2)$, $\vec{p} = (p_1, p_2)$. We note that the main-diagonal ($\searrow$) components of the $\mathbb{U}$-matrix $\partial^2_{\vec{h}\vec{p}} F$ are equal to $\partial^2_{\vec{h}\vec{p}} u = (h_1, h_2)\mathrm{Hess}(u)(p_1, p_2)^T$, where $\mathrm{Hess}(u)$ is the Hess matrix of the function $u(x, y)$ and its co-diagonal ($\nearrow$) components are formed with $v(x, y)$ likewise. Thus we get explicit $\mathbb{U}$-matrices for the derivatives

$$\dot{\Phi} = F'(Z)\dot{Z} = (H^T \partial_{\vec{h}} F).\dot{Z},$$

$$\ddot{\Phi} = F''(Z).\dot{Z}^2 + F'(Z).\ddot{Z} = ((HP)^T \partial^2_{\vec{h}\vec{p}} F).\dot{Z}^2 + (H^T \partial_{\vec{h}} F).\ddot{Z}.$$

Counting $\det \dot{\Phi} = (\det J_\Phi)(\det \dot{Z}) = (\det F'(Z))(\det \dot{Z})$ and (1.7) we express the signed curvature for the conformal image $\Phi(C)$ in the form

$$k_{\Phi(C)}(t) = \frac{\mathrm{trace}\{\ddot{\Phi}.(\mathbb{J}\dot{\Phi})^T\}}{2[(\det F'(Z))(\det \dot{Z})]^{\frac{3}{2}}}. \tag{2.4}$$

We are going to apply the described above to the conformal image of the plane rational Bézier curve (2.2) with $\mathbb{U}$-matrix representation (2.3) by

I. Möbius transformation and **II.** Zhukovski-type function.

I. The $\mathbb{U}$-matrix representation of any Möbius transformation is of the form

$$W = F(Z) = (AZ + B)(CZ + D)^{-1}$$

with matrix coefficients $A, B, C, D \in \mathbb{U}$ such that $T = AD - BC \neq O$ and $\mathbb{U}$-matrix argument Z. We consider the case: C – invertible and by letting $F(-C^{-1}D) = \Omega$ – the $\mathbb{U}$-matrix at infinity and $F(\Omega) = AC^{-1}$ this transformation is bijective over the extended $\mathbb{U}$-matrix group $\mathbb{U}^*$. We use Corollary **1.1** to rewrite $F(Z)$ in the form

$$\begin{aligned}
F(Z) &= \tfrac{1}{\det(CZ+D)}(AZ + B)(CZ + D)^T \\
&= \tfrac{1}{\det(CZ+D)}(AZ + B)(Z^T C^T + D^T) \\
&= \tfrac{1}{\det(CZ+D)}[AD^T Z + BC^T Z^T + AC^T Z Z^T + BD^T] \\
&= \tfrac{1}{\det(CZ+D)}[AD^T Z + BC^T Z^T + AC^T (\det Z)^2 + BD^T].
\end{aligned}$$

To describe the corresponding conformal map Φ by (1.9) we express the traces $\mathrm{tr}\{F(Z)\}$ and $\mathrm{tr}\{\mathbb{J}F(Z)\}$, by counting linearity: $\mathrm{tr}\{P + Q\} = \mathrm{tr}\{P\} + \mathrm{tr}\{Q\}$, $\mathrm{tr}\{\lambda P\} = \lambda \mathrm{tr}\{P\}$ and transposition invariance: $\mathrm{tr}\{P\} = \mathrm{tr}\{P^T\}$ for the matrix trace-function $\mathrm{tr} : \mathbb{U} \longrightarrow \mathbb{R}$. We get the Möbius conformal transformation Φ in the form

$$\Phi : \left| \begin{array}{l} u = \mathrm{tr}\{F(Z)\} = \dfrac{\mathrm{tr}\{(AD^T + CB^T)Z\} + (\det Z)^2 \mathrm{tr}\{AC^T\} + \mathrm{tr}\{BD^T\}}{\det(CZ + D)} \\ \\ v = \mathrm{tr}\{\mathbb{J}F(Z)\} = \dfrac{\mathrm{tr}\{\mathbb{J}(AD^T - CB^T)Z\} + (\det Z)^2 \mathrm{tr}\{\mathbb{J}AC^T\} + \mathrm{tr}\{\mathbb{J}BD^T\}}{\det(CZ + D)} \end{array} \right. .$$

Now by replacing Z with the $\mathbb{U}$-matrix (2.3) $Z(t) = b^\ell(u,t) Z_\ell$ for the plane Bézier curve $\overset{(u)}{c}_\Delta$ with (2.2) one gets its Möbius conformal image

$$\Phi\big(\overset{(u)}{c}_\Delta\big) : Z = \Phi(t) = (u(t), v(t)), \text{ where}$$

$$u(t) = \frac{b^\ell(u,t)\mathrm{tr}\{(AD^T + CB^T)Z_\ell\} + (\det(b^\ell(u,t)Z_\ell))^2 \mathrm{tr}\{AC^T\} + \mathrm{tr}\{BD^T\}}{\det(b^\ell(u,t)CZ_\ell + D)}$$

$$v(t) = \frac{b^\ell(u,t)\mathrm{tr}\{\mathbb{J}(AD^T - CB^T)Z_\ell\} + (\det(b^\ell(u,t)Z_\ell))^2 \mathrm{tr}\{\mathbb{J}AC^T\} + \mathrm{tr}\{\mathbb{J}BD^T\}}{\det(b^\ell(u,t)CZ_\ell + D)}.$$

As in the proof segments above Corollary **2.1** we have

$$\left\{ \begin{array}{l} \dot{Z}(t) = \dot{b}^0(Z_0 - Z_1) + \dot{b}^2(Z_2 - Z_1) \\ \qquad = (B_2)^{-2}[\Delta_0(Z_0 - Z_1) + \Delta_2(Z_2 - Z_1)] \\ \\ \det \dot{Z}(t) = (\dot{b}^0, \dot{b}^2) \overset{(0,2)}{\Gamma} (\dot{b}^0, \dot{b}^2)^T = (B_2)^{-4}(\Delta_0, \Delta_2) \overset{(0,2)}{\Gamma} (\Delta_0, \Delta_2)^T \\ \\ \ddot{Z}(t) = \ddot{b}^0(Z_0 - Z_1) + \ddot{b}^2(Z_2 - Z_1)\,, \end{array} \right. \tag{2.5}$$

where $\ddot{b}^s = \Big[(B_2)^{-2}\Delta_s\Big]^\bullet = (B_2)^{-3} \begin{vmatrix} \dot{\Delta}_s & 2\dot{B}_2 \\ \Delta_s & B_2 \end{vmatrix}$, $s = 0,\ 2$. More precisely we get

$$\ddot{b}^2 = \frac{1}{(B_2)^2} \begin{vmatrix} 2 - 4t(\ln|B_2|)^\bullet & \ddot{B}_2 - 2\dot{B}_2(\ln|B_2|)^\bullet \\ t^2 & B_2 \end{vmatrix},$$

$$\ddot{b}^0 = \frac{1}{(B_2)^2} \begin{vmatrix} 2 - 4(t-1)(\ln|B_2|)^\bullet & \ddot{B}_2 - 2\dot{B}_2(\ln|B_2|)^\bullet \\ (t-1)^2 & B_2 \end{vmatrix}.$$

Further for the $\mathbb{U}$-derivatives of $F(Z)$ we have

$$F'(Z) = \frac{\det T}{(\det(CZ + D))^2}[(CZ + D))^2]^T,$$

$$F''(Z) = \frac{-2\det T}{(\det(CZ + D))^3} C[(CZ + D))^3]^T,$$

where $T = AD - BC$. The corresponding replacements of F', F'' and (2.5) in $\dot{\Phi}(t)$, $\ddot{\Phi}(t)$ and formula (2.4) lead one to the signed curvature of $\Phi(\overset{(u)}{c}_\Delta)$.

II. The $\mathbb{U}$-matrix representation of Zhukovski-type function ($f(z) = \lambda z + \mu \frac{1}{z}$, $\lambda, \mu \in \mathbb{C} \setminus \{0\}$) is of the form

$$F(Z) = LZ + \frac{1}{\det Z} M Z^T$$

for nonzero matrices $L, M \in \mathbb{U}$. The corresponding conformal transformation is

$$\Phi : \left| \begin{array}{l} u = \operatorname{tr}\{(L + (\det Z)^{-1} M^T) Z\} \\ v = \operatorname{tr}\{(\mathbb{J}L - (\det Z)^{-1} \mathbb{J} M^T) Z\} \end{array} \right.$$

and the conformal image of the $\mathbb{U}$-matrix (2.3) $Z(t) = b^\ell(u,t) Z_\ell$ for the plane Bézier curve $\overset{(u)}{c}_\Delta$ with (2.2) is

$$\Phi(\overset{(u)}{c}_\Delta) : Z = \Phi(t) = F(Z(t)) = (u(t), v(t)), \quad \text{where}$$

$$u(t) = b^\ell(u,t) \operatorname{tr}\{(L + (\det(b^\ell(u,t) Z_\ell))^{-1} M^T) Z_\ell\}$$

$$v(t) = b^\ell(u,t) \operatorname{tr}\{(\mathbb{J}L - (\det(b^\ell(u,t) Z_\ell))^{-1} \mathbb{J} M^T) Z_\ell\}\,.$$

The $\mathbb{U}$-derivatives of $F(Z)$ are the $\mathbb{U}$-matrices

$$F'(Z) = L - (\det Z)^{-2} M (Z^2)^T, \quad F''(Z) = 2(\det Z)^{-3} M (Z^3)^T$$

which substitution, jointly with (2.5) in $\dot{\Phi}(t)$, $\ddot{\Phi}(t)$ and formula (2.4) express the signed curvature of $\Phi(\overset{(u)}{c}_\Delta)$.

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MEASURES OF MATHEMATICAL KNOWLEDGE FOR TEACHING AND UNIVERSITY MATHEMATICS COURSES DESIGN

Slagjana Jakimovik

Abstract

During the past few decades a significant body of research has been accumulated worldwide in the area of mathematical knowledge for teaching in primary schools. Numerous theoretical and empirical research studies have paved the way towards defining what it is that teachers need to know and be able to do to produce positive effects on the development of primary school students' mathematics competencies.

Taking into consideration results from a previous study on students' mathematics competencies when entering university teacher education studies, implications on the kind of mathematics courses that need to be developed within university studies for primary school teachers have been hypothesized in this paper. A pilot study has been conducted to illuminate the way towards a larger study on the correlation between mathematical knowledge for teaching built in university mathematics courses and teacher performance. Initial findings from the pilot study are discussed in the paper and recommendations for further explorations are formulated.

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1. Introduction

Research on mathematical knowledge for teaching is based on the assumption that there is *a knowledge base for teaching*, "a codified or codifiable aggregation of knowledge, skill, understanding, and technology, of ethics and disposition, of collective responsibility" (p.4 in [31]), knowledge which can be represented and communicated. The conceptualization of the domain of mathematical knowledge that teachers need to possess and employ in their work with students has been the focus of intensive scientific research efforts of various groups of scholars, especially in the last three decades. Schulman's introduction of the notion of *pedagogical content knowledge* ([30], [31]) has created a major impact on the consequent development of research studies of teachers' mathematical education. The theory of

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mathematical knowledge for teaching (MKT), from its early conception by Ball and her colleagues ([3], [4], [6]) has been revised and transformed through a series of steps of rigorous check and justification based on development of measures, their piloting and data analysis ([14], [15] ,[21]), associating teachers' MKT with student achievements and with mathematical quality of instruction ([5], [17], [18], [20]). Argument-based test validation ([19], [27]) has influenced further investigation, refinement and understanding of the domain ([8], [14], [16]). The understandings of the nature of MKT which resulted from the above mentioned studies and the questions raised inspired researchers to contribute to the development of the domain of MKT by adapting, using and validating the measures outside the U.S. ([10], [11]). The study of pre-service teachers' learning to learn to teach by planning and evaluating instruction as they unpack lesson-level mathematical learning goals (an example of a sub-construct of MKT) contributes to the development of university education for mathematics teachers ([24]). In [9], Blömeke and Delaney give an extensive review of the state of research in the assessment of teachers' mathematics knowledge across countries, in which they summarize underlying theoretical models, develop a model of factors hypothesized to influence the development of teachers' knowledge, describe the study design, the key instruments and the core results on the structure of the knowledge and its relevance to teacher education, teacher performance and student achievement.

Development and implementation of university mathematics courses for primary school teachers requires a substantial understanding of the mathematical knowledge teachers need to employ in their work with students 6-11 years of age. How teachers use their own mathematical knowledge in conducting classroom discourse in a community of young learners, in addressing students' educational needs by engaging them in genuine mathematics learning, in interpreting students' productions, in evaluating students' mathematical knowledge and skills, in lessons preparation or in appraising curriculum materials, is something that needs to be established before any meaningful consideration of university course design can begin. Early efforts for development of topic-specific instructional approaches aimed at creating effective learning opportunities for future primary school teachers are discussed in the paper. The research based theoretical framework for MKT is discussed and analysed as a platform for the ongoing study.

2. A research-based theoretical framework

Testing teacher competence in subject matter and pedagogical skills is not a new idea; over a century ago tests for licensing teacher candidates were used in the U.S. ([30]). In the last decades of the twentieth century, tests were treated as prerequisites for entry into teacher education university programs. In comparison to the tests administered in the late nineteenth century, where approximately 5% of the tests were assigned to the 'theory and practice of teaching', a century later almost 100% of the test do so, covering topics in organization, preparation and presentation of instructional plans, management, understanding youth, educational policies and procedures ([30]). Shulman posed the question "Where did the subject matter go?"

(p. 5 in [30]) and expanded on George Bernard Shaw's infamous aphorism "He who knows, does; he who cannot, *but knows some teaching procedures*, teaches" (italics added). Additionally, Shulman questioned the heavy emphasis on pedagogical skills as attributed to 'research based teacher competencies' by policy makers. Investigators ignored subject matter and used it as a context variable only. Shulman named "the absence of focus on subject matter among various research paradigms for the study of teaching as the 'missing paradigm' problem" (p.6 in [30]) and pointed out the seriousness of consequences for both policy and research.

Looking further back in the past, Shulman argues that "content and pedagogy were part of one indistinguishable body of understanding" (p.6 in [30]). The sharp distinction between subject matter knowledge and pedagogy, a development as recent as the second half of the twentieth century, has been questioned lately by many researchers of mathematics education. In recent years there has been a shift from only posing questions about teachers' skills for classroom management or for using a wide spectrum of teaching methods towards asking questions about what kind of knowledge are teachers explanations based on, how teachers' interpret and address students' misconceptions, how teachers discuss profound mathematical ideas, or represent them in a scientifically honest ways appropriate to their students' cognitive development and to previously experienced opportunities to learn. How teachers build a subject matter knowledge base necessary to become effective teachers largely depends on successfully answering the above questions.

Foundations placed within teacher education university programs, coupled with personal abilities to engage in reflective practices as a means for enhancing future work present themselves as the most obvious prerequisites. Yet, it is not clear how students who have successfully passed university courses translate their knowledge so as to be used in teaching primary school students. Whether an experienced professional or a young novice, a teacher needs to make decisions in order to promote the established educational goals, to confront and choose from a multitude of sources and teaching materials, among which textbooks sometimes burdened with scientific inaccuracies or with poor mathematical explanations or even with complete lack of examples illuminating the targeted mathematical ideas. A strong knowledge base to support any decision making proves to be essential.

2.1. Pedagogical content knowledge

In [30], Schulman proposed three categories of content knowledge for teaching: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Together with general pedagogical knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of educational ends, purposes and values with their philosophical and historical grounds, they form the knowledge base for teaching ([31]).

Subject matter knowledge, in this case mathematics content knowledge, includes both the substantive structures and the syntactic structures as defined by Schwab (cited in [30]). The substantive structures, or knowledge *of* mathematics as referred to by Ball ([2]), involve knowledge of the concepts and the principles of mathematics as a discipline. The syntactic structures, or knowledge *about* mathematics ([2]), involve the set of ways in which validity of the statements is attested. Teachers are required to represent to their students not only what is true in mathematics but also why it is true, why it is worth learning about and how it relates to other mathematical concepts as well as to ideas in other disciplines both in theory and in practice.

Pedagogical content knowledge (PCK) is described by Schulman as "that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding.", "… the category most likely to distinguish the understanding of the content specialist from that of the pedagogue" (p.8 in [31]). PCK is subject matter knowledge for teaching, including but not restricting to knowledge of topics taught at certain educational level, knowledge of various ways of formulation and representation of ideas and procedures, knowledge of validation processes most appropriate to the individual learners' cognitive abilities, as well as knowledge of the most common difficulties or misconceptions related to the learning of certain topics and the most efficient ways of addressing them.

Curricular knowledge, according to Schulman ([30]), consists of knowledge of curricular resources and their alternatives available to teachers, as well as curriculum materials available to students in mathematics and in other subjects. Teachers need to be fully aware of curriculum goals at previous, current and latter stages of education, and the vertical alignment of learning-teaching trajectories[1].

In [30], Schulman also suggested three forms of teacher knowledge: propositional knowledge (research based theoretical principals, practice based maxims, and norms), case knowledge (knowledge of specific and well documented events illustrating theoretical, practical and normative knowledge), and strategic knowledge. In this way Schulman offered guidelines for further research of knowledge required for teaching, for the design of university content courses for teachers and professional support programs, and for the development of professional examinations measuring all categories of mathematical knowledge (subject matter knowledge, PCK and curricular knowledge) in addition to measuring other categories of teacher knowledge and actual instructional practices.

Empirical studies linking teachers' effectiveness in producing student gains as measured by standardized tests, with the number of university mathematics courses taken and their level of mathematics sophistication, produced contradictory results, as discussed by Ball, Lubienski and Mewborn in [6]. What those studies did not take into account is the nature of teachers' mathematical knowledge, and how they learn to use it in their everyday work with students. Successful meeting of the challenges faced by developers of university programs and of mathematics and methods courses largely depend on the results from research findings on the nature and structure of the mathematical knowledge for teaching. University courses which provide for knowledge growth in teaching are a key component for success.

[1] Within the theory of Realistic Mathematics Education specific strands of learning-teaching trajectories have been explored and carefully mapped ([35]).

2.2. Mathematical knowledge for teaching

Based on the assumption that "if teaching entails helping others learn, then understanding what is to be taught is a central requirement of teaching" (p. 437 in [7]), a group of scholars at the University of Michigan focused their research on teachers' mathematical knowledge necessary for facilitating students' development of intellectual resources for not only knowing about, but for actively engaging in mathematics as one of the fundamental domains of human thought and inquiry. Organization of the classroom as a mathematical community of learners whose knowledge of mathematics evolves as students make conjectures and evaluate their validity as opposed to the teacher merely interpreting students' claims as correct or incorrect, according to Lampert ([23]), is one of the desired goals of teachers' work. Ball and her colleagues ([3], [4], [8]) approached the research of mathematical knowledge for teaching by grounding it on examination of actual teaching practices rather than on examining the curriculum or teachers' perspective on teaching mathematics due to the incomplete information each of them provides; one by only laying assumptions on and the other one by recounting about the mathematical demands of actual teaching practice. Observations of classroom practices brought to surface the variety and the depth of mathematical ideas elementary school teachers needed to know and be sensitive to. The researchers anticipated to see in their analysis concepts like properties of the decimal number system, operations with fractions, informal reasoning, and the like. What they uncovered defies conventional expectations: recurrent prominence of "ideas about similarity, equivalence mapping among representations, and even isomorphism" as well as "salient issues involving mathematical language – symbolic notation and definitions of terms …" (p.97 in [3]). None of these issues can be revealed by examining the school curriculum; yet, teachers encounter their appearance in classroom discourse and need to be able to address them flexibly and productively.

Examples of core tasks teachers attend to in everyday mathematics classes include, but are not restricted to establishing and facilitating classroom discourse in mathematical inquiry, representing mathematical ideas to students, interpreting and working with students' mathematical productions, determining the validity of students' mathematical arguments, offering and appraising mathematical explanations and justifications, selecting and enacting cognitively demanding mathematical activities, choosing tasks to assess levels of students' learning, all of which are "quintessential mathematical – not pedagogical – questions" (p.7 in [4]). Ball and Bass ([4]) argue that the work of teaching mathematics is essentially mathematical work, which unlike doing advanced mathematics research requires "decompression" or "unpacking" of ideas without losing sight of the mathematical horizons towards which mathematics education of their students is headed for. The belief that mathematics teaching is a "mathematically-intensive work, involving significant and challenging mathematical reasoning and problem solving" (p.13 in [4]) raises the standards for the mathematical education of teachers.

The initial hypothesis of the structure of MKT included several categories of knowledge: common content knowledge (CCK), specialized content knowledge

(SCK), knowledge of content and students (KSC), knowledge of content and teaching (KCT), and later the domain was conceptualized as including also knowledge at the mathematics horizons (KMH), and knowledge of curriculum ([16]). CCK, SCK and KMH are purely mathematical constructs; KCS and KCT are central elements of Schulman's notion of PCK.

Substantial efforts to design and test measures of teachers' mathematical knowledge for teaching elementary mathematics revealed its multidimensionality ([21]). A large pool of survey-based teaching problems hypothetically representing various components of MKT was generated. Three different forms were piloted and teachers' responses were factor analyzed in an attempt to answer the question of whether there was a single construct which can be called MKT or whether the items represented several distinct constructs requiring corresponding mathematical competencies. At the same time the writing of the items served as a way to further investigate the nature of MKT.

Further revision of the sub-construct of specialized content knowledge (SCK) was necessitated by the exploratory factor analysis of the responses which revealed that CCK and SCK did not differentiate themselves and that the subscales corresponding to each of them did not meet criteria for unidimensionality ([27]). The new definition of SCK started with including elements like teachers' ability to provide students with "decompressed" mathematical explanations and representations, and progressed towards identifying four major categories: responding to common student errors, evaluating students' understanding of content, knowledge of student developmental sequences, and common student computational strategies ([16]).

Another sub-construct which needed a redefinition was knowledge of content and students (KCS), which also did not meet the criteria for unidimensionality. Cognitive tracing interviews confirmed that respondents (teachers, non-teachers, and mathematicians) deployed general mathematical strategies to answer the items meant to represent KCS ([19]). Measurement of KCS was being reconsidered in terms of abandoning multiple-choice items in favour of open-ended questions ([27]). A question arises if the key part of the conceptualization of KCS as being distinct from common content knowledge holds in terms defined by Hill, Ball and Schilling – teachers having "strong knowledge of the content itself but weak knowledge of how students learn the content or *vice versa*" (p.378 in [16], emphasis added). Is it possible for teachers to prove to have strong knowledge of students' learning of mathematics without having strong knowledge of mathematics content themselves? Due to strong correlation between CK and KCS items, constructing items which distinguish teachers' mathematics content knowledge from teachers' knowledge about students' learning of mathematics might prove to be an extraordinarily difficult undertaking, as was observed by Hill, Ball and Schilling ([16]).

Application of the MKT measures for evaluation of the effectiveness of California's Mathematics Professional Development Institutes and the analysis of the results suggested that program focus on mathematical analysis, reasoning, and communication was a significant predictor of teachers' learning; "the more teachers' engage in mathematics in ways that afford them opportunities to explore and link alternative representations, to provide and interpret explanations, and to delve into meanings and connections among ideas, the more flexible and developed their knowledge will be" (p.346 in [15]).

In order to provide scientific evidence for the structure of MKT, Ball and her colleagues ([5]) developed and continued to refine large scale survey-based MKT measures. Using a measure consisting of CCK and SCK multiple-choice items, Hill, Rowan and Ball ([20]) found that teachers' mathematical knowledge was the strongest teacher-level predictor of student achievement gains in both first and third grades. The positive effect of teachers' content knowledge was comparable in size to student background characteristics like socio-economic status, ethnicity, and gender. The effect of teachers with higher MKT scores on students' gain over a course of a year of instruction persisted when teachers' content knowledge for teaching reading was entered in the model, which means that mathematics-specific knowledge, not general pedagogical skills, is the principal contributor to the effect ([17]).

Based on series of five case studies and related quantitative data, Hill et al. ([18]) showed that there is a significant positive association between levels of MKT and the mathematical quality of instruction (MQI), defined as a composite of several components (presence or absence of mathematical errors, responding to students appropriately or inappropriately, connecting classroom practice to mathematics, richness of mathematics, and mathematical language). Hill and her colleagues argued that possible mediating factors like teachers' beliefs about the nature of mathematics learning or about the use of curriculum materials were also shaped by teachers' mathematics knowledge.

Adaptation of the MKT measures for use in Ireland ([11]) and a subsequent validation study ([10]) resulted in the identification of the need for further refinement of the domain of KCS.

The research-based theory of MKT addresses the problems of teachers' knowledge and how this knowledge is held by them; the problem of helping them learn to use it requires teacher education university programs to offer qualitatively different mathematics and methods courses for primary school teachers from traditional ones. The above discussed findings support focusing efforts on the development of a mathematics curriculum for teachers which does not simply include topics from fifth grade or eight grade textbooks, but offers treatment of mathematics authentic to the work of teaching ([21]).

Although much has been done regarding the conceptualization of the domain of MKT, "existing research on aspects of teacher education, including standard teacher preparation programs … is not of sufficient rigor or quality to permit the (U.S. National Mathematics Advisory) Panel to draw conclusions about the features of professional training that have effects on teachers' knowledge, their instructional practice, or their students' achievements" (p.xxi in [25]).

3. Primary school teachers' mathematics education

Prospective primary school teachers in the Republic of Macedonia conclude their university education by receiving either a bachelor's degree in *primary teacher education* at the faculties of pedagogy or a bachelor's degree in *pedagogy* at the institutes of pedagogy. There is at least one substantial difference between the two kinds of university programs. The primary school teaching programs require students to take at least one course in mathematics (specialized for primary school teachers) as opposed to the pedagogy programs, with neither mathematics course requirements nor offerings. At the same time, the percent of secondary school graduates who take the National Matriculation Exam in Mathematics is in steep decline (with less than 12 % graduates taking it at the end of the 2012/13 school year). Yet, both of the above mentioned programs accept secondary school graduates from general and vocational schools, for some of which mathematics is not a part of the curriculum past the first year of secondary school. Formal pre-service teacher education is still based on the remnants of the traditional view that students know "enough" mathematics when they enter the program, they just "need to learn how to teach it". In stark contrast to the current state of teacher education in Macedonia stands the recommendation of NMAP ([25]) to strengthen the mathematics education of elementary and middle school teachers by providing them with ample opportunities to learn mathematics for teaching from an advanced perspective, beyond the level they are assigned to teach and in a way that connects it with other important mathematics.

Results from a mathematics entry test administered to primary school students at the beginning of the semester in which they took the first mathematics course presented a grim picture: the percent of students who couldn't answer correctly to any of the test problems rose dramatically from less than 10% (out of 172 students) in 2006/07 academic year to almost 45% (out of 58 students) in the 2009/10 academic year ([22]). The test consisted of 15 open-ended items covering primary and lower secondary school topics in numbers, operations and basic algebra.

The low mathematics achievements of the students entering the primary teacher education program have made the goal of promoting profound understanding of mathematics needed for teaching an important part of the design and the implementation of the mathematics courses. Lampert ([23]) provided an "existence proof" that the type of learning of mathematics consistent with knowing and doing mathematics as a discipline is possible in a typical primary school classroom setting under regular conditions. Attempting the same in a teacher education classroom setting allows prospective teachers to experience "the intellectually generative sort of mathematical activities" (p.33 in [23]) expected of them to provide for their future students.

Another long established practice of teacher education programs is the complete lack of connection between mathematics courses and methods courses. The only requirement is that students pass the mathematics courses exams to be allowed to take the methods courses. University mathematicians teach the mathematics courses and the methods courses are taught by university pedagogues and there is no emphasis on the integral relationship between subject matter knowledge and teaching. Since students don't participate in school classroom practice focused on mathematics instruction until they enroll for the methods courses, at the time they take the mathematics courses they cannot form any preconception of the depth of the mathematics knowledge entailed in teaching primary school mathematics. In this sense, the fundamental distortion of knowledge resulting from the separation of substance from method as called by Dewey in [12] (referenced in [3]) still persists within primary school teacher university education. Scholastic knowledge even when unconsciously assumed as irrelevant to method disregards the fact that "how an idea is represented is part of the idea, not merely its conveyance" (p.85 in [3]). In [3], Ball and Bass discuss Dewey's belief that recognition and creation of "genuine intellectual activity" in students require teachers to employ methods originating from disciplinary knowledge.

4. An excerpt from a university classroom learning experience

Mathematics 1 is a course which covers topics in numbers and basic algebra. Geometry and measurement represent a major part of the Mathematics 2 course. Theoretically the main goals of the course closely relate to Schoenfeld's assumptions on the nature of mathematics and the nature of humans as learners ([28]). The purpose of mathematics instruction is to make provisions for students to learn to think mathematically, which means not only learning about facts and becoming proficient with mathematical computations, but also understanding the ideas and the connections between them, being able to apply them flexibly and meaningfully in problem solving situations, and adopting the view that mathematics is highly structured and complex even at the elementary levels. On practical grounds the course is designed around the idea of a classroom participation structure as discussed by Lampert ([23]) – an agreement among the classroom participants on the common grounds for interaction and collaboration including the expectations, the rights and the obligations.

The goals of the course and the grounds for collaboration in the classroom are clearly established at the beginning of the semester. Students are responsible for their own learning and for contributing productively to the learning community they are constituent members of. During each class meeting homework problems are assigned by the teacher and are not collected. Computational tasks, problem solving tasks and assignments requiring students to generate examples of concepts and procedures learned in the classroom and represent them in various equivalent forms are equally emphasized. Explicit instruction[2] is provided with topics deemed to present substantial difficulties for students. At the beginning of each class meeting solutions to homework problems are presented and discussed by the students who wish to do so. Points are awarded to each student who participates in the homework discussion as well as in the classroom discourse at any point of time, and the total number of points earned this way carries 10 % of the final grade. The awarding of

[2] The term *explicit instruction* is used in the sense adopted by NMAP ([25]), meaning the university teacher provides students with clear models for solving a problem type, with multiple opportunities to practice newly learned strategies and with extensive feedback.

points does not depend on the correctness of the solution offered; the students are expected to explain and justify their approaches. The final exam, which carries 30 % of the grade, consists of questions asking students to write about specific mathematical concepts or procedures, and illustrate them by employing various representations.

4.1. Generating representations vs. carrying out calculation

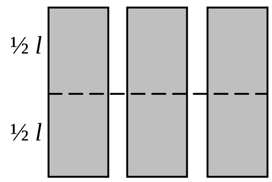
The paper is focused on the classroom practice regarding the topic of division by rational numbers represented as fractions, and on students' understandings of the topic as manifested in their work on the final exams. Two dimensions of MKT are explored: the ability to calculate division by a fraction and the ability to generate a representation justifying the calculation procedure, the first representing common content knowledge and the second representing specialized content knowledge.

During a class meeting in the semester in question, the students (Group 1) were asked to create a word problem illustrating division by rational numbers as fractions and to justify the procedure. Up till that point students were engaged in working on analogous assignments involving addition, subtraction and multiplication of fractions and in using various representations of rational numbers. After a few unsuccessful attempts which lead some of the students to come up with examples of division by a whole number or examples of multiplication by a fraction, there was a general agreement in the classroom that it is difficult to come up with real life situations involving division by a fraction. Similar to Ball's observations in her interviews with prospective teachers ([2]), students in Group 1 only considered the partitive model of division. Some students even commented that one cannot divide a number of chocolates to a fraction of a child. Next, students were engaged in the process of creation of the following word problem and the justification of the division by fractions procedure. The idea for the problem emerged as a result of the students' answers to the invitation to name a few things that come in sizes of fractions, and obviously people do not. One of the students pointed to the fact that water bottles (drinks packages) came in sizes of ¼ liter, ½ liter, 1½ liters, 1 liter, 2 liters, and other sizes. Students built the problem on that idea.

On a warm spring day a primary school teacher plans to take a small group of students to a field trip. Each child is to take a small bottle of water (a ½ liter in volume). The teacher anticipates that some of the children will drink their water before the trip ends and will need a refill, so she brings 3 large bottles of 1 liter volume.

a. How many empty small bottles can the teacher refill?

After depicting the bottles by rectangles and representing the problem by a number expression, students linked it to the procedure of multiplying the dividend by the divisor's reciprocal fraction, commenting that dividing by ½ means multiplying by 2 since there are two halves of a liter in 1 liter.

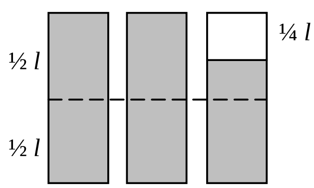


$$3 : \frac{1}{2} = 3 \cdot 2 = 6$$

Next, students worked on the following modification of the problem.

b. *Before refilling the students' bottles, the teacher had already drunk ¼ liter herself. How many empty small bottles can she refill now?*

After establishing that there were 2¾ liters of water left, the students made a sketch and modeled the problem by an expression, then solved it in a way to reflect the pictorial representation, which meant applying the distributive property of division over addition.



$$2\frac{3}{4} : \frac{1}{2} = \left(2 + \frac{3}{4}\right) : \frac{1}{2} = \left(2 : \frac{1}{2}\right) + \left(\frac{3}{4} : \frac{1}{2}\right) = 4 + 1\frac{1}{2} = 5\frac{1}{2}$$

Students remarked that it was crucial to phrase the question properly so as to make real sense: How many halves there are in two ones and three quarters? Then, the answer made sense as well: five and a half halves.

One of the examples created in the classroom involved a ribbon 2½ meters long cut in a third of meter long pieces. The example provided students with the opportunity to use the number line to represent division by a rational number. Unfortunately, the number line is persistently being avoided from the mathematics instruction in the early grades of primary education in Macedonia as there is no referral to it within the national mathematics curriculum grades I-IV and the state approved mathematics textbooks. In contrast, one of the main findings of the U.S. National Mathematics Advisory Panel ([25]) regarding learning processes emphasizes the ability to represent fractions on a number line as a key mechanism linking conceptual and procedural knowledge, the second one likely to have a major impact on students' problem-solving performance.

Another example was constructed by a student up to the formulation of the question. It involved 5¾ containers of party food leftovers. Most of the students still had problems formulating the question so as to illustrate a division by fractions instead of division by a whole number. Commonly used patterns of thinking seemed to lead the students toward division by a whole number, the reasons of which can be traced back to the way primary school mathematics textbooks treat the topic of division. Word problems are always focused on the question of how many elements in each subset if there are a given number of subsets, and never the question how many subsets if each is to have a given number of elements.

The next class meeting students presented their homework assignments involving creating word problems with appropriate representations illustrating fraction arithmetic. A very few students attempted generating a representation of division by fractions.

In the previous semester, with a different group of students (Group 2), up till multiplication and division by fractions, the same sequence of lessons was followed as with Group 1: various representations of rational numbers as fractions (points on a number line, area model, a number of equivalent subsets of a set partitioned in a number of subsets, division of numbers, ratios), comparison of fractions, addition and subtraction of fractions. In Group 2 multiplication and division by a rational number as a fraction was worked on only in a manner which promotes computational skills and the ability to apply those skills to solve word problems. Unlike with addition and subtraction, students in Group 2 were neither asked explicitly to generate examples of word problems to represent multiplication and division by fractions, nor to justify the procedure using pictorial representations. The approach did not challenge students' beliefs regarding whether the procedure for division by fractions constitutes an arbitrary convention to be memorized or a logical construct which can be justified. The earlier course of actions was motivated mainly by the national curriculum goals for I-V grade mathematics, in which fraction arithmetic is restricted only to addition and subtraction of fractions with equal denominators, and consequently, the lack of need for teachers to be able to explain to students why division by a fraction is done via multiplication by that fraction's reciprocal. There are reasons to believe that an opportunity to promote students' understanding of fundamental mathematical ideas and their representations was missed.

4.2. An analysis of the results

Evaluation of the course effectiveness means evaluation of the student achievement gains. Since no entry test was administered at the beginning of the semester, a comparison of the responses to two items was used to evaluate the students' growth in understanding of division by rational numbers as fractions between the students in Group 1 and in Group 2. The results of the students in Group 1 are drawn from the midterm exam papers for the first item, and from the final exam papers on the second item (see the Appendix). The results of the students in Group 2 are obtained by a short test administered at the final exams. Follow up interviews were conducted with more than a half of the students in each group who did not leave any written response to the second item or gave a completely unrelated response. Additional data about students' performances in the course was gathered in the form of the total number of points earned by the student's contribution to the classroom learning community and the student's overall course grade ranging, from 5 (F, fail) through 10 (A, excellent).

The first item measures students' computational skills and has no discriminative power in exposing degrees of students' understanding. The second item measures their conceptual understanding in a way which fully illuminates their abilities to generate an adequate representation and a justification of division by fractions. The results on the items are given in Table 1.

Table 1. Prospective primary school teachers' responses on *Division by fractions* (%)

	Group 1 (n = 28)	Group 2 (n = 28)	Totals
Item 1. *Computational Skills*			
Correct responses	86	82	84
Incorrect responses	14	18	16
Item 2. *Conceptual Understanding*			
Correct responses			
A complete answer (example and justification)	11	0	5
Appropriate example without justification	11	4	7
Incorrect responses			
An example of division by a whole number	18	14	16
An example of multiplication of fractions	4	0	2
Erroneous or nonrelated response	50	25	38
No response	7	57	32

More than four fifths of the students in each group computed the quotient of the given fractions correctly. Yet, only slightly more than a fifth of the students in Group 1 offered an appropriate example to illustrate division by fractions, and only half of these examples were supported by a corresponding pictorial representation. Students in Group 2 had even lesser success in providing a correct example; only one student generated an appropriate word problem with no pictorial representation.

No significant difference between the two groups of students with respect to their grades (grouped in grades of fail, lowest passing, middle passing, and highest passing) was found using the chi-square test, c^2 (3, n = 58) = 3.02, p = 0.39, Cramer's V = 0.23. The Man-Whitney U test did not discover a significant difference in classroom participation points between Group 1 (Md = 2.5, n = 28) and Group 2 (Md = 0.5, n = 28). The chi-square test (with Yates's correction for continuity) showed no significant difference between the two groups of students with respect to their computational skills, c^2 (1, n = 58) = 0.12, p = 0.73, phi = 0.09.

If students who produced a correct response (complete or incomplete) are represented as one category, students who offered a representation of division by a whole number or of multiplication by a fraction as another category, students who produced erroneous or non-related answer as a third category, and students with no response as a fourth category, the chi-square test of independent samples shows that the difference between the two instructional groups of students with respect to their success in generating a representation of division by rational numbers as fractions is significant, c^2 (3, n = 58) = 17.19, p = 0.001, Cramer's V = 0.55.

Direct logistic regression was conducted to measure the effects of the instructional approach, students' final grades, their classroom participation, and their computational skills on their success in constructing an illustrative example justifying the procedure for division by fractions. The model was statistically significant, c^2 (4, N=58) = 13.85, $p < 0.01$. The model as a whole explains between 21.9 % and 41.4 % of variance, and correctly classifies 94.6 % of cases. As visible from Table 2, only the students' overall course grades were significant predictors of students' understanding of division by fractions manifested through the ability to generate an appropriate representation, which might lead towards hypothesizing that the overall grade correctly identifies students with conceptual understanding of a specific topic, division by fractions, although it is a compound measure of student's procedural and conceptual knowledge of various topics.

Table 2. Predicting correct responses on *Generating representation of division by fractions*

	B	S.E.	Wald	df	Sig.	Exp(B)	95.0% C.I. for Exp(B)	
							Lower	Upper
Instructional group	1.87	1.26	2.21	1	0.14	0.15	.013	1.82
Final grade	0.82	0.41	4.05	1	0.04	2.27	1.02	5.04
Classroom participation	-0.07	0.13	0.33	1	0.57	0.93	0.73	1.19
Computational skills	17.78	1.22×10^4	0.00	1	1.00	5.24×10^7	0.00	.
Constant	-24.77	1.22×10^4	0.00	1	1.00	0.00		

Variables entered on step 1: Instructional group, Final grade, Classroom participation, Computational skills.

For numerous reasons general conclusions may not be drawn from these findings, yet they can serve as guidelines for development of a sound approach to measuring the effectiveness of the course.

4.3. Discussion of the results

These findings suggest that the mathematics instruction offered on the topic did not result in a significant increase of students' conceptual understanding. Follow-up interviews revealed a possible obstacle to students' learning. The exercise in generating this kind of an example by oneself was not perceived as a task worth spending one's time and efforts; many students simply counted on passing the course by responding to "easier" questions. This finding closely resonates with Schoenfeld's conjecture ([28]) that as a result of their experience in primary and secondary school mathematics courses students may develop beliefs regarding the nature of mathematics and mathematics learning which make them unable to engage in thinking mathematically in a productive manner. According to Schoenfeld ([28]) the disasters of "well taught"[3] mathematics courses were students' beliefs that mathematics problems are actually tasks that can be solved in a few minutes if the student understood the material, that is, if the problem required prolonged or deeper thinking the student is wasting her/his time on something that requires application of skills they had obviously not acquired from lectures. In other words, previous experiences with mathematics courses have reinforced students' beliefs that the ability to make sense of mathematics is reserved for the few gifted ones, that mathematics knowledge is "passed down from above" to students as passive learners whose only goal is to accurately perform the procedures as prescribed by the teachers or by the textbooks.

Classroom participation points did not proved to be a significant indicator of students' conceptual understanding of division by fractions; yet, this finding does not contradict common sense as they are considered as a partial contributor to students' overall achievements in the course. Some of the students engaged in classroom discourse with the explicit purpose of collecting points on cognitively less demanding tasks, resisting applying themselves in doing intellectually challenging mathematics.

The results obtained on the second item are similar to the results obtained by Ball ([2]) from the interviews with 25 prospective elementary school teachers and 10 prospective secondary school teachers within a larger study. Prospective teachers were asked to generate a mathematically appropriate representation of 1¾ divided by ½. Only four of the secondary school teacher candidates and none of the elementary candidates was able to represent the operation. Forty percent of the elementary school teachers gave an inappropriate representation among which the most prevalent was a representation of division by 2 instead of ½. Prospective elementary school teachers who were unable to generate any representation believed that division by fractions could not be represented in a real-life context which resonates closely to the classroom episode described in section 4.1. Ball, Lubienski and Mewborn ([6]) also referred to the results obtained by Simon ([32]) as he asked 33 prospective elementary teachers to a write a story problem for which ¾ divided by ¼ would represent the operation used to solve the problem; 24 students produced inappropriate representations, with fraction multiplication as the most common incorrect answer. Similar results regarding prospective teachers' limited understanding of multiplication and division by fractions were found by Tirosh et al. ([33]), who reported that prospective teachers held a narrow view of rational numbers and sparsely connected knowledge of entirely rigid procedural nature.

Although the instructional approach introduced in Group 1 did not prove to be a good predictor of students' ability to generate an appropriate example, it was not very far off. One of the indications that the course treatment of the general topic of

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[3] Under "well taught" mathematics courses Schoenfeld ([28]) meant courses satisfying widely accepted standards like reaching of curriculum goals, quality of instruction, good classroom management, and students' success on standard performance measures.

rational numbers represented as fractions had positive effects on students' understanding may be observed through the representations that were used by the students. Half of students who responded correctly used the number line representation of fractions – a departure from the usual primary school textbook treatment of the topic, which is in favor of circular foods in word problems with fractions. Relying on the possibility of teachers learning about the substance and nature of mathematics from school textbooks may be unfounded; textbook authors and commercial publishers may well be bound by policy limitations and considerations other than scientific rigor and educational research-based evidence.

5. Conclusions

Contemporary education is based on assumptions about the benefits from student-centered forms of learning and inquiry-based classroom. The corresponding forms of instruction raise the bar even higher with respect to teacher comprehension. As members of the scholarly community, teachers have a special responsibility in communicating profound mathematical ideas to their students and in emphasizing how these ideas emerge and are validated in the domain. The teachers' deep understanding of mathematics as well as their beliefs regarding the nature of mathematics and the purposefulness of studying mathematics has important consequences on their teaching practices and their effectiveness. The wisdom of practice cannot compensate the shortcomings of pre-service education programs; teachers need a strong knowledge base on which they can build their judgments and actions.

Mathematicians designing and/or teaching mathematics courses to prospective teachers have to address possible misconceptions and gaps in teachers' knowledge of particular topics, encourage them to deepen their understanding of mathematics as a human activity and facilitate raising prospective teachers' self-confidence in doing mathematics ([33]). Assisting students in the rebuilding of their knowledge of mathematics as a coherent and highly structured body is a challenge on both practical and theoretical grounds. University course design and practical implementation in the university classrooms serve the purpose of enhancing learning opportunities for students as well as offering them experiences on basis of which they can model their future instructional approaches and practices.

An effort in attempting a certain instructional approach on a specific mathematics topic in a university classroom setting and in measuring the effects on student understanding of the topic has been presented in the paper with the aim of using this attempt as a step towards designing a more general approach in teaching a university mathematics course to prospective primary school teachers and in designing a measure of the course effectiveness. The findings regarding the topic of division by fractions point towards the need to further differentiate steps in assigning mathematical tasks which offer students first to revise their understanding of division by whole numbers, then to analyze and classify word problems involving fraction arithmetic, discuss various representations of rational numbers and apprise their appropriateness in illustrating the assigned construct, and then to engage in generating a complete set of representations with the aim to explain a concept or to justify a procedure. For example, the problem Simon posed to prospective students in his study ([32]) to write three different realistic story problems which could be solved by dividing 51 by 4 and for which the answers would be 12 ¾, 13, and 12 respectively, is a valuable instructional step which would have guided students to rethink the meaning of division before asking them to generate a realistic story problem of division by fractions. In [1], An, Culm and Wu bring forward the good practice of Chinese teachers focused on establishing explicit connections between various models and abstract thinking, and on developing reflection as a critical learning strategy. Another important clue which has been disregarded in the course is that it takes time for learners to consolidate the connections made and that after certain instructional time distance incorporating tasks which require students to draw from these connections would contribute towards their consolidation.

Adjustments to improve the above described approach are to be made and the approach may be attempted to other course topics. The process of improvement may benefit from the model proposed by Hiebert, Morris and Glass ([13]) to treat lessons as experiments by setting learning goals in terms of students' thinking, designing lessons to support achievement of the goals, evaluate the lessons' effectiveness using data on changes in students' thinking and revise the lessons based on the analysis of the evaluations and on carefully formulated hypothesis for further improvement.

At the same time, important note is taken from research studies ([29], [36]) which, after reporting success in assisting students in deepening their conceptual knowledge of mathematics and in adopting the disposition to create classrooms of young learners actively engaged in mathematical inquiry, found that the effects of the pre-service intervention did not translate equivalently in novice teachers' work with children. Although disciplinary study is absolutely necessary to equip pre-service teachers with a set of intellectual tools and a disposition to engage in mathematical inquiry, it proved insufficient in overcoming deeply held beliefs and attitudes which were incongruent with the epistemological orientation of the education program. University teacher education developers have to take into account the intellectual qualities and habits of mind teacher candidates bring to their studies, and also to accept the responsibility of providing professional support to novice teachers during their induction years, when innovative practices get challenged by established traditional school settings ([36]).

Using video-taped classroom segments with primary teachers' reactions to salient mathematics emerging in classroom discourse may provide a useful vehicle in bridging the existing gap between mathematics and methods courses and in motivating students to apply themselves in becoming mathematically proficient.

Further steps involve constructing comprehensive measures to be applied as pre-tests and post-tests to evaluate the mathematics and methods courses achievement gains of students within different university programs leading towards a degree which allows primary teaching employment, as proposed by Ball, Thames and Phelps in [8], validating the measures and analyzing the results with the purpose of offering theoretical recommendations and practical guidelines in reforming mathematics education of future primary school teachers.

Translating the MKT measures developed in the USA to evaluate in-service teachers' knowledge or prospective teachers' achievement gains within different university programs in Macedonia is not plausible for numerous reasons. Mathematics education, like education in general, is predetermined by many factors, among which the country's specific cultural, historical and social settings, nationally defined curricular goals, traditionally established instructional practices, beliefs, expectations, and assessment procedures ([1], [26]). Even an extremely careful adherence to explicit guidelines for adapting teaching knowledge items may result in an inaccurate measurement, as cautioned by Delaney et al. in [11] and by Delaney in [10]. Such attempts in adapting the MKT measures requires validation work to be done ([10]), as would be done in creating entirely new set of measures grounded in the actual teaching practice in classrooms in Macedonia. However, close investigation of the MKT measures provides useful resources for the development of authentic research approaches in mathematics education of primary school teachers. Another significant source of information and guidelines is provided by the comparative study TEDS-M ([34]) focused on the outcomes of teacher education programs in 17 countries around the world, with respect to the acquisition of knowledge and beliefs. Due to the similarities found in the rankings of countries in TEDS-M and in TIMSS, a cyclic relationship has been hypothesized, which means that student achievement may be improved by increasing teachers' professional knowledge ([9]). The kind of research outlined above may be attempted only by establishing collaboration of a multidisciplinary group of scholars: mathematicians, mathematics educators, cognitive and educational psychologists, teaching experts, experts in statistical analysis of data, and above all, practicing primary school teachers.

Some of the principal messages of the National Mathematics Advisory Panel in its final report [25] are that society needs to "recognize that mathematically knowledgeable classroom teachers have a central role in mathematics education" and that "the nation needs to build capacity for more rigorous research in education so that it can inform policy and practice more effectively" (p. xiv). These messages hold powerful truths which transcend national boundaries.

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Appendix

1. Perform the following operations and express the resulting fraction in simplest form.

   a. $\frac{3}{8} \cdot \frac{6}{9} =$

   b. $\frac{3}{8} : \frac{7}{12} =$

2. Create a simple word problem to illustrate division by a rational number represented as a fraction. Sketch a pictorial representation to link the word problem to the appropriate number expression and justify the procedure of division by fractions.

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AN OPEN PROBLEM OF LJUBOMIR ILIEV RELATED TO THE MITTAG-LEFFLER FUNCTION AND FRACTIONAL CALCULUS OPERATORS

Virginia Kiryakova

Dedicated to the 100th Anniversary of Acad. Ljubomir Iliev

Abstract

This is a short survey on an open problem posed by Academician Ljubomir Iliev in his studies on constructive theory of Laguerre functions. It is stated in notions as the Jensen polynomials and zeros of polynomials and entire functions. However, the linear differential operator $\mathcal{D}_\alpha$ introduced by Iliev and involved in this problem is related to the most popular special function of the fractional calculus (FC) – the Mittag-Leffler (M-L) function, and the operator itself can be considered as a special case of the operators of FC and of the Gelfond-Leontiev (G-L) operators of generalized differentiation. From this point of view, Iliev's open problem can be formulated also in a more general setting, related to the multi-index M-L functions ([15], [16]) and to the operators of generalized FC ([14]).

Together with the problem as stated by Iliev in [12], [13], and its alternative interpretations, we provide the readers with some additional literature closely related to techniques and results possibly helpful in understanding and solving the open problem – the basic among them – by Craven and Csordas [1], [2], Dzrabashjan [6], Ostrovskii and Peresyolkova [20], Popov [21], etc.

We hope that the provided information would stimulate researchers to try solving Iliev's problem, using the tools of the contemporary theory of the Mittag-Leffler function and of the FC operators.

MSC 2010: 30C15, 30D20, 33E12

Key Words and Phrases: zeros of polynomials and entire functions; Jensen polynomials; Mittag-Leffler function; multiplier sequences, Gelfond-Leontiev operators; operators of fractional calculus

1. Preliminaries

1.1. Mittag-Leffler function

For a long time, the *Mittag-Leffler* (*M-L*) *function*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}\,, \quad \alpha > 0,\ \beta > 0, \tag{1.1}$$

has been totally ignored in the common handbooks on special functions and existing tables of Laplace transforms, although E_α was introduced yet in 1902-1905 and available in the third volume of the classical Bateman-Erdélyi Project [7], 1954; and $E_{\alpha,\beta}$ appeared in 1953. Recently, the interest in this remarkable entire function (the simplest one of prescribed order $1/\alpha$) has been enormously increased in view of its *important role in Fractional Calculus* (FC) and related differential and integral equations of fractional (that is arbitrary non-integer) order and applications in treating fractional order model and systems that describe the fractality of the real physical and social world better than the integer order approximations. Thus, the Mittag-Leffler (M-L) function exited from its isolated life as *Cinderella of the Special Functions*, to become the *Queen-function of Fractional Calculus* ([9]), and has been included as a separate item 33E12 in MSC 2000. Nowadays, almost each paper on fractional order differential equations and models involves this special function or its generalizations (a long list to be found in [15], [16]) and many books on FC have chapters or are entirely devoted to it, see e.g. [14], [22], [18].

It is then an interesting fact to mention that two of the pioneers of the contemporary Bulgarian mathematics, had applied the M-L function (although not mentioning explicitly its name, but using the already adopted notation E_α) in their studies in the area of *Complex Analysis*. In 1930, *Nikola Obrechkoff* [19] defined a general method of summation (more precisely, a Borel type method for analytical continuation of complex functions defined by convergent power series), specified it also for the so-called Mittag-Leffler summation and studied the asymptotics of the M-L function. Later, since 1969, *Ljubomir Iliev* [12], [13] used the M-L function in the constructive theory of Laguerre functions. On the set of the power series, he defined a linear differential operator $\mathcal{D}_\alpha$ generated by means of the M-L function $E_{\alpha,\alpha}(z)$. It happens that this operator is a special kind of the Gelfond-Leontiev operators of generalized differentiation [8], when the multipliers' sequence includes the coefficients of the M-L function as a generating entire function. And it is also an operator for differentiation of fractional (that is arbitrary, not obligatory integer) order $\alpha > 0$, for which (1.1) is an eigen-function.

1.2. Preliminary notions from the theory of Laguerre entire functions

Definition 1.1. (e.g. [13], Ch. 1). An infinite sequence $\{\alpha_n\}$ is said to be an *α-sequence*, denoted by $\{\alpha_n\} \in \alpha$, if for every polynomial

$$b(z) = b_0 + b_1 z + \cdots + b_n z^n$$

possessing real zeros only, the polynomial

$$b(z) \star \{\alpha_n\} = b_0\alpha_0 + b_1\alpha_1 z + \cdots + b_n\alpha_n z^n$$

has real zeros only. Related definition concerns the so-called *β-sequences*, $\{\beta_n\} \in \beta$, if for every polynomial $b(z)$ with only real nonpositive zeros, the polynomial $b(z) \star \{\beta_n\}$ has real zeros only. Obviously, $\alpha \subset \beta$.

For the first time Laguerre (1898) considered special cases of α-sequences, similar definitions but in different terms were given in a paper by Pólya and Schur (1914), revealing some important applications and properties of these sequences. In Chapter 1 of [13], also in [11], [12], Iliev describes and studies several other properties, involving the classes L_1, resp. L_2, of entire functions which are either polynomials possessing real nonpositive zeros only (resp. real zeros only), or are limits of such polynomials in every finite domain; including algebraic characteristics of L_1 and L_2, and algebraic and transcendental *criteria for the α- and β-sequences* in their terms (some of them coming also from Obreshkoff's works).

The notion for the *Jensen polynomials* is also introduced: Let the series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots := \gamma_0 + \frac{\gamma_1}{1!}z + \frac{\gamma_2}{2!}z^2 + \cdots + \frac{\gamma_n}{n!}z^n + \cdots \qquad (1.2)$$

be convergent in a neighbourhood of $z = 0$. If $D = d/dz$ denotes the *differentiation operator*, then the polynomials

$$J_n(f, z) = f(D)\, z^n = \sum_{k=0}^{n} \binom{n}{k} \gamma_k z^{n-k} := J_n(z), \quad n = 1, 2, \ldots \qquad (1.3)$$

are said to be the *Jensen polynomials for the function* $f(z)$ (with respect to the operator D).

2. L. Iliev's open problem

Below $\alpha > 0$ is assumed an arbitrary parameter. Let us reproduce the notations and terminology as stated in Iliev [13], Ch. 2: Constructive Theory of Laguerre Entire Functions, §2.6, namely:

2.6. Define the linear differential operator $\mathcal{D}_\alpha$ which acts on the set f of the power series by the equalities:

a) $\mathcal{D}_\alpha\, z^n = \dfrac{\Gamma(\alpha(n+1))}{\Gamma(\alpha n)}\, z^{n-1}, \quad \alpha > 0, \; n = 1, 2, \ldots,$

$$\mathcal{D}_\alpha^k\, z^n = \mathcal{D}_\alpha\, \mathcal{D}_\alpha^{k-1}\, z^n = \frac{\Gamma(\alpha(n+1))}{\Gamma(\alpha(n-k+1))}\, z^{n-k}, \quad \mathcal{D}_\alpha = \mathcal{D}_\alpha^1, \; k=1,2,\dots, \tag{2.1}$$

$$\mathcal{D}_\alpha\, C = 0, \quad \frac{1}{\Gamma(0)} = 0, \quad C = \text{const},$$

b) $\mathcal{D}_\alpha\, C z^n = C \mathcal{D}_\alpha\, z^n, \quad C = \text{const},$

c) $\mathcal{D}_\alpha\,(z^n + z^m) = \mathcal{D}_\alpha\, z^n + \mathcal{D}_\alpha\, z^m,$

where $\Gamma(z)$ is the Euler function.

1. Let (*Our Note*: Actually, Iliev used a denotation $E_\alpha(z)$ for below, but compared to universally adopted definition (1.1), it should be $E_{\alpha,\alpha}(z)$):

$$E_{\alpha,\alpha}(z) = \frac{1}{\Gamma(\alpha)} + \frac{z}{\Gamma(2\alpha)} + \frac{z^2}{\Gamma(3\alpha)} + \dots + \frac{z^k}{\Gamma(\alpha(k+1))} + \cdots .$$

Since $\Gamma(\alpha(n+1))/\Gamma(\alpha n) \to \infty$ as $n \to \infty$, the function $E_{\alpha,\alpha}(z)$ is entire. Moreover, $E_{1,1}(z) = \exp z$ and

$$\mathcal{D}_\alpha\, E_{\alpha,\alpha}(z) = E_{\alpha,\alpha}(z). \tag{2.2}$$

Set (*Our Note*: this is the fractional order analogue of the binomial coefficients)

$$\binom{n}{k}_\alpha := \frac{\Gamma(\alpha(n+1))}{\Gamma(\alpha(k+1))\,\Gamma(\alpha(n-k+1))}, \quad n=1,2,\dots, \quad k=0,1,2,\dots . \tag{2.3}$$

For

$$f(z) = a_0 + a_1 z + a_2 z^2 + \dots = \frac{\gamma_0}{\Gamma(\alpha)} + \frac{\gamma_1}{\Gamma(2\alpha)} z + \frac{\gamma_2}{\Gamma(3\alpha)} z^2 + \cdots, \tag{2.4}$$

the polynomial

$$J_n^\alpha(f,z) := f(\mathcal{D}_\alpha)\, z^n = \binom{n}{0}_\alpha \gamma_0 z^n + \binom{n}{1}_\alpha \gamma_1 z^{n-1} + \binom{n}{2}_\alpha \gamma_2 z^{n-2} + \dots + \binom{n}{n}_\alpha \gamma_n, \tag{2.5}$$

$n = 1,2,\dots$, will be called the *Jensen polynomial of degree* n *of* $f(z)$ *with respect to the operator* $\mathcal{D}_\alpha$.

Our Note 1. Observe that for $\alpha = 1$, (2.4) and (2.5) coincide with (1.2), (1.3), since $\Gamma(k+1) = k!$, $k = 0,1,2,\dots$.

Employing the methods from Ch. 1 (Laguerre Entire Functions) in [13], it is established that if $f(z)$ is an entire function, then in any finite domain:

$$\lim_{n\to\alpha} \left(\frac{z}{\alpha_n}\right) J_n^\alpha\left(f, \frac{\alpha_n}{z}\right) = f(z), \quad \text{where } \alpha_n := (\alpha n)^\alpha. \tag{2.6}$$

From (2.5), for the Jensen polynomials with respect to the operator $\mathcal{D}_\alpha$ of the function $f(z) = E_{\alpha,\alpha}(z)$, we obtain for $n = 1,2,\dots$:

$$J_n^\alpha(E_{\alpha,\alpha},z)=\binom{n}{0}_\alpha z^n+\binom{n}{1}_\alpha z^{n-1}+\binom{n}{2}_\alpha z^{n-2}+\cdots+\binom{n}{n}_\alpha=\sum_{k=0}^{n}\binom{n}{k}_\alpha z^{n-k}. \tag{2.7}$$

If α is a positive *integer*, then the sequence $\{\Gamma(k+1)/\Gamma(\alpha(k+1))\}\in\alpha$, so that in this case the zeros of $J_n^\alpha(E_{\alpha,\alpha},z)$ are real. *The question* which is the domain A_α, where all the zeros of the polynomials $J_n^\alpha(E_{\alpha,\alpha},z)$, $n=1,2,\dots$, lie, for $\alpha>0, [\alpha]\neq\alpha$ (that is, for α not integer), *remains open.*

Our Note 2. Observe a conflict of notations above: the symbols α have different meanings. Once it is for the notion α-sequence adopted by Def. 1.1 (in Section 4 we compare with the notion "multiplier sequence"), then in $\mathcal{D}_\alpha$, $E_{\alpha,\alpha}$, J_n^α, A_α it is an arbitrary parameter $\alpha>0$.

Further, Iliev gives some notations as:

2. Setting

$$(x+y)_\alpha^n=\sum_{k=0}^{n}\binom{n}{k}_\alpha x^k y^{n-k},\quad \alpha>0,\ n=1,2,\dots,$$

he introduces the so-called α-*sum* and provides related identities. He also notes that

$$D_\alpha\, z^n=\lim_{h\to 0}\frac{(z+h)_\alpha^n-z^n}{h}.$$

Finally, in **2.7.** Iliev summarizes the open problems posed by his studies in Ch. 2, [13], and among them the above mentioned one is stated explicitly as (see p. 43):

Problem 2.5. *For* (*arbitrary, i.e. not integer*) $\alpha>0$, *which is the domain* A_α, *where do all zeros of the Jensen polynomials* $J_n^\alpha(E_{\alpha,\alpha},z)$, $n=1,2,\dots$, *defined by* (2.7) *and* (2.3), *lie?*

3. Fractional calculus and Gelfond-Leontiev operators

3.1. Gelfond-Leontiev operators of generalized differentiation and integration

In 1951 Gelfond and Leontiev [8] introduced an operation, more general than the classical differentiation $D=d/dz$. To clarify the exposition we state the definition in its simplest form, and details an be seen in Kiryakova [14].

Definition 3.1. Let the function

$$\varphi(\lambda)=\sum_{k=0}^{\infty}\varphi_k\,\lambda^k \tag{3.1}$$

be an entire function of order $\rho>0$ and type $\sigma\neq 0$ such that the condition

$$\lim_{k\to\infty} k^{1/\rho} \sqrt[k]{|\varphi_k|} = (\sigma e \rho)^{1/\rho} \tag{3.2}$$

holds. For a function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

analytic in a disk $|z| < R$, the operation

$$D_\varphi f(z) = \sum_{k=1}^{\infty} a_k \frac{\varphi_{k-1}}{\varphi_k} z^{k-1} \tag{3.3}$$

is said to be a *Gelfond-Leontiev* (*G-L*) *operator of generalized differentiation* with respect to (or generated by) the (entire) function φ.

From the theory of entire functions, it is known that (3.2) always holds but with $\limsup\limits_{k\to\infty}$. However, the condition as in (3.2) yields that there exists $\lim\limits_{k\to\infty} \sqrt[k-1]{|\frac{\varphi_{k-1}}{\varphi_k}|} = 1$ and therefore, by the Cauchy-Hadamard formula, the image-series (3.3) has the same radius of convergence as the original $f(z)$.

Evidently, if we take $\varphi(\lambda) = \exp \lambda$, i.e. if $\varphi_k = 1/\Gamma(k+1)$, $k = 0, 1, 2, \dots$, then the G-L differentiation operator (3.3) reduces to the classical differentiation $D_{\exp} = D = d/dz$.

Along with the generalized differentiation, the *G-L operators of generalized integration* are considered, of the form:

$$I_\varphi f(z) = \sum_{k=0}^{\infty} a_k \frac{\varphi_{k+1}}{\varphi_k} z^{k+1}, \quad \text{resp.} \quad I_{\exp} f(z) = \sum_{k=0}^{\infty} a_k \frac{z^{k+1}}{k+1}, \tag{3.4}$$

and for the generating function $\varphi(\lambda) = \exp \lambda$ one gets the classical integration $I_{\exp} = I$ (of order 1).

In a more general setting, the G-L operators (3.3), (3.4) can be viewed as *Hadamard products* by a function

$$g(z) = \sum_{k=0}^{\infty} b_k z^k,$$

and if $b_k \to \infty$ for $k \to \infty$ as is in the cases $b_k = k$, and $b_k = \varphi_{k-1}/\varphi_k$, the operation $D\{g; f\} = f(z) \star g(z) = \sum\limits_{k=0}^{\infty} a_k b_k z^k$ can be considered as a generalized differentiation, while for $b_k \neq 0$, $k = 1, 2, \dots$, its inverse operation $I\{g; f\} = \sum\limits_{k=0}^{\infty} \dfrac{a_k}{b_k} z^k$ should be a generalized integration, or Hadamard product with the "reciprocal" function $g_*(z) = \sum\limits_{k=0}^{\infty} z^k / b_k$.

Evidently, *Iliev's operator $\mathcal{D}_\alpha$ is a kind of G-L operator of generalized differentiation, related to a M-L function* (1.1) as its generating entire function...

But to make the picture more clear, let us first remind some notions for the operators of fractional calculus.

3.2. Operators of fractional calculus

Fractional Calculus (FC) is an extension of the classical Calculus, when differentiation and integration can be done not only integer number of times, but these operations can be of fractional (i.e. arbitrary) orders. Detailed theory is available for example, in [23], [14], [22] and in many dozens of newer books. The most popular definition for integration of arbitrary order $\alpha>0$ is the *Riemann-Liuouville (R-L) fractional integral* I^α, considered as extension of the n-fold integration I^n via substitution the factorial function $n!$ by the Gamma-function:

$$I^n f(z)=\frac{1}{(n-1)!}\int_0^z (z-\xi)^{n-1}f(\xi)d\xi \quad\longrightarrow\quad I^\alpha f(z)=\frac{1}{\Gamma(\alpha)}\int_0^z (z-\xi)^{\alpha-1}f(\xi)d\xi$$

$$\longrightarrow \text{ after suitable substitution } \longrightarrow I^\alpha f(z)=\frac{z^\alpha}{\Gamma(\alpha)}\int_0^1 (1-\sigma)^{\alpha-1}f(z\sigma)d\sigma. \tag{3.5}$$

The problem to define then the fractional order differentiation is solved by composing integer order differentiation and R-L fractional order integral (3.5), namely: by means of the *R-L fractional derivative*:

$$D^\alpha f(z):=D^n I^{n-\alpha}f(z)=\left(\frac{d}{dz}\right)^n\left\{\frac{1}{\Gamma(n-\alpha)}\int_0^z (z-\xi)^{n-\alpha-1}f(\xi)d\xi\right\}, \tag{3.6}$$

$n-1<\alpha\leq n,\ n\in\mathbb{N}$, or its alternative, called as *Caputo fractional derivative*,

$$^*D^\alpha f(z)=I^{n-\alpha}D^n f(z)=\frac{1}{\Gamma(n-\alpha)}\int_0^z (z-\xi)^{n-\alpha-1}f^{(n)}(\xi)d\xi, \tag{3.7}$$

the latter definition avoiding: 1) the fact strange for classical analysts, that in general $D^\alpha\{\text{const}\}\neq 0$, and 2) the appearance of fractional order derivatives in the initial conditions for Cauchy problems for fractional differential equations with D^α, which are physically difficult to interpret. A greater freedom and much more applications of the fractional derivatives and integrals can be achieved by involving additional parameters in definitions (3.5),(3.6),(3.7) – thus using the name *Erdélyi-Kober operators*, or by replacing the kernel elementary function $K(\sigma)=\sigma^{\alpha-1}/\Gamma(\alpha)$ in (3.5) by means of a suitable special function $K(\sigma)$, say a Meijer's G-function or Fox's H-function – as in the *Generalized FC*, Kiryakova [14]. Specially, the *Erdélyi-Kober* (*E-K*) *operator of fractional integration* of order $\alpha>0$ with additional parameters $\gamma\in\mathbb{R}$ and $\beta>0$, is defined by

$$I_\beta^{\gamma,\alpha}f(z)=\frac{\beta}{\Gamma(\alpha)}z^{-\beta(\gamma+\alpha)}\int_0^z (z^\beta-\xi^\beta)^{\alpha-1}t^{\beta(\gamma+1)-1}f(\xi)d\xi \tag{3.8}$$

$$= \frac{1}{\Gamma(\alpha)} \int\limits_0^1 (1-\sigma)^{\alpha-1}\sigma^{\gamma} f(z\sigma^{1/\beta})d\sigma; \;\; \gamma=0, \beta=1 \text{ giving } I_1^{0,\alpha} f(z) = z^{-\alpha} I^{\alpha} f(z).$$

Then, the corresponding *E-K fractional derivatives* are defined by analogy with (3.6) and (3.7), using suitable polynomials of the Euler differentiation $z\dfrac{d}{dz}$ and integer $n-1<\alpha\le n$, resp. – the R-L type variant:

$$D_{\beta}^{\gamma,\alpha} f(z) := D_n \, I_{\beta}^{\gamma+\alpha,n-\alpha} f(z) = \prod_{j=1}^{n}\left(\frac{1}{\beta} z\frac{d}{dz}+\gamma+j\right) I_{\beta}^{\gamma+\alpha,n-\alpha} f(z), \quad (3.9)$$

and – the Caputo type:

$$^{*}D_{\beta}^{\gamma,\alpha} f(z) := I_{\beta}^{\gamma+\alpha,n-\alpha} D_n f(z) = I_{\beta}^{\gamma+\alpha,n-\alpha} \prod_{j=1}^{n}\left(\frac{1}{\beta} z\frac{d}{dz}+\gamma+j\right) f(z). \quad (3.10)$$

The details of introducing these E-K fractional derivatives and their properties can be found in Kiryakova [14] and Kiryakova and Luchko [17].

According to their definitions, and to the rules of the FC theory, the R-L and the E-K operators of integration are resp. right inverse to the R-L (Caputo) and E-K operators of differentiation, namely:

$$D^{\alpha} I^{\alpha} f(z) = {}^{*}D^{\alpha} I^{\alpha} f(z) = f(z); \; D_{\eta}^{\gamma,\alpha} I_{\beta}^{\gamma,\alpha} f(z) = {}^{*}D_{\beta}^{\gamma,\alpha} I_{\beta}^{\gamma,\alpha} f(z) = f(z), \quad (3.11)$$

see Kiryakova et al. [14], [17].

A term-by-term application of the R-L and of the E-K operators of fractional order integration and differentiation to power series $f(z) = a_0 + a_1 z + a_2 z^2 + ...$, convergent in disks $|z| < R$ (as considered by Iliev), is an easy task, not requiring special care about functional spaces (as in the common case theory of FC) and domains in $\mathbb{C}$, with except the reminder that we determine a unique branch of $\arg z$ $(0 \le \arg z \le 2\pi)$ so to avoid multiplicities in terms as z^{μ}. Then it is easy to see that, for example, the R-L integral has the representation

$$I^{\alpha} f(z) = z^{\alpha-1} \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} z^{k+1},$$

that is, very close as form to a G-L generalized integration.

Let us consider now the *G-L operators of generalized integration and differentiation with respect to (generated by) the Mittag-Leffler function* $\varphi(z) = E_{\alpha,\beta}(z)$, that is with coefficients $\varphi_k = 1/\Gamma(\alpha k+\beta)$. In Dimovski and Kiryakova [5], Kiryakova [14] and later works, we called them *Dzrbashjan-Gelfond-Leontiev operators*, in honor to Armenian mathematician Dzrbashjan who was one of the first to study the 2-indices M-L function with great details and in complex plane. For a function $f(z) = a_0 + a_1 z + a_2 z^2 + ...$, analytic in $|z| < R$, the definitions are, respectively:

$$D_{\alpha,\beta} f(z) = \sum_{k=1}^{\infty} a_k \frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k-1)+\beta)} z^{k-1},$$
$$I_{\alpha,\beta} f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(\alpha k+\beta)}{\Gamma(\alpha(k+1)+\beta)} z^{k+1}. \tag{3.12}$$

It is proved (see e.g. Kiryakova [14], etc.) that these D-G-L operators of the forms (3.3), (3.4) are representable also *as Erdélyi-Kober operators of fractional integration and differentiation of order* $\alpha > 0$ of the forms (3.8), (3.10), as follows:

$$I_{\alpha,\beta} f(z) = z\, I_{1/\alpha}^{\beta-1,\alpha} f(z), \quad D_{\alpha,\beta} f(z) = {}^{*}D_{1/\alpha}^{\beta-1,\alpha} z^{-1} f(z), \tag{3.13}$$

and thus, they can be also analytically continued outside of disks, to starlike domains with respect to the origin $z = 0$.

In the special case $\beta = 1$ the operator $I_{\alpha,1}$ has been studied by Dimovski [4] as an example of a linear integration operator preserving the space of analytic functions in starlike domains, for which he found a convolution operator, the commutant and a convolutional representation of the commutant of a fixed integer power of $I_{\alpha,1}$. Later, in Dimovski and Kiryakova [5] we established similar results for the more general D-G-L operator $I_{\alpha,\beta}$, and in Kiryakova [14], Ch.2, a detailed study of the operators $D_{\alpha,\beta}$, $I_{\alpha,\beta}$, was exposed, combined with a Laplace type integral transformation (called Borel-Dzrbashjan transform, $\mathcal{B}_{\alpha,\beta}$) playing the same role in the operational calculus for these operators as the Laplace transform for the classical differentiation and integration.

For further generalizations, see e.g. Kiryakova [14, Ch.5], [15]. For example, instead of the M-L function (1.1), we introduced and studied its multi-index generalization (Kiryakova [15],[16]) with set of real indices $\alpha_1, \dots, \alpha_m > 0$ and $\beta_1, \dots, \beta_m,\ m = 1, 2, 3, \dots$:

$$E_{(\alpha_i),(\beta_i)}(z) := E_{(\alpha_i),(\beta_i)}^{(m)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k+\beta_1)\dots\Gamma(\alpha_m k+\beta_m)}\,, \tag{3.14}$$

called *multi-index M-L function*. We proved that (3.14) is an entire function of order $\rho = (\alpha_1 + \dots + \alpha_m)^{-1}$ and type $\left(\frac{1}{\alpha_1\rho}\right)^{\alpha_1\rho} \dots \left(\frac{1}{\alpha_m\rho}\right)^{\alpha_m\rho} > 1$ if $m > 1$, and many of its analytical properties as example of H-function and of the Wright ${}_1\Psi_m$-function. Many special functions appearing previously as solutions of fractional order differential equations and fractional order models are sjhown to be very particular cases of this new class of special functions of fractional calculus, see [16].

Then, similarly to D-G-L operators (3.12), we introduced and studied *G-L operators* (3.4), (3.3) *generated by the multi-index M-L function* $E_{(\alpha_i),(\beta_i)}(z)$, that is,

$$
\begin{aligned}
D_{(\alpha_i),(\beta_i)} f(z) &= \sum_{k=1}^{\infty} a_k \, \frac{\Gamma(\alpha_1 k + \beta_1) ... \Gamma(\alpha_m k + \beta_m)}{\Gamma(\alpha_1 (k-1) + \beta_1) ... \Gamma(\alpha_m (k-1) + \beta_m)} \; z^{k-1}, \\
I_{(\alpha_i),(\beta_i)} f(z) &= \sum_{k=0}^{\infty} a_k \, \frac{\Gamma(\alpha_1 k + \beta_1) ... \Gamma(\alpha_m k + \beta_m)}{\Gamma(\alpha_1 (k+1) + \beta_1) ... \Gamma(\alpha_m (k+1) + \beta_m)} \; z^{k+1},
\end{aligned} \tag{3.15}
$$

such that $D_{(\alpha_i),(\beta_i)} \, I_{(\alpha_i),(\beta_i)} \, f(z) = f(z)$, and appearing as special cases (similarly to relation (3.13)) of the so-called *multiple E-K operators* $D^{(\beta_i-1),(\alpha_i)}_{(1/\alpha_i),m}$, $I^{(\beta_i-1),(\alpha_i)}_{(1/\alpha_i),m}$ in the *generalized fractional calculus*, Kiryakova [14].

Note that the multi-index *M-L functions are eigen functions of the D-G-L operators* (3.12), since the following (generalized) differential relation holds as an extension of (2.2):

$$
D_{(\alpha_i),(\beta_i)} \, E_{(\alpha_i),(\beta_i)}(\lambda z) = \lambda \, E_{(\alpha_i),(\beta_i)}(\lambda z), \quad \lambda \neq 0,
$$

(as Lemma 4.3 in Kiryakova [15]).

3.3. New interpretations of Iliev's operator and problem, and possible generalizations

The differential operator $\mathcal{D}_\alpha$ of Iliev, defined by a), b), c) in Section 2, and its linear right inverse integration operator (if we denote it by $\mathcal{I}_\alpha$: $\mathcal{D}_\alpha \mathcal{I}_\alpha \, f(z) = f(z)$), taken of a power series $f(z) = a_0 + a_1 z + a_2 z^2 + ...$, convergent in $|z| < R$, have evidently the forms:

$$
\mathcal{D}_\alpha \, f(z) = \sum_{k=1}^{\infty} a_k \, \frac{\Gamma(\alpha(k+1))}{\Gamma(\alpha k)} \; z^{k-1}, \quad \mathcal{I}_\alpha \, f(z) = \sum_{k=0}^{\infty} a_k \, \frac{\Gamma(\alpha(k+1))}{\Gamma(\alpha(k+2))} \; z^{k+1}, \tag{3.16}
$$

and therefore, they appear to be *Gelfond-Leontiev operators of generalized differentiation and integration with respect to the Mittag-Leffler function* $E_{\alpha,\alpha}$ (that is with $\beta = \alpha$ in (1.1)).

On the other side, the same operators, according to (3.13), can be considered as *Erdélyi-Kober operators of fractional differentiation and integration* (3.9),(3.8) of order $\alpha > 0$, as follows:

$$
\mathcal{D}_\alpha \, f(z) = {}^{*}D^{\alpha-1,\alpha}_{1/\alpha} \, z^{-1} \, f(z) = D^{\alpha-1,\alpha}_{1/\alpha} \, z^{-1} \, f(z), \quad \mathcal{I}_\alpha \, f(z) = z \, I^{\alpha-1,\alpha}_{1/\alpha} \, f(z). \tag{3.17}
$$

Thus, *Problem 2.5 can be stated as*: to find the domain where all zeros of the Jensen polynomials $J^\alpha_n(E_{\alpha,\alpha}, z)$ lie, if we take in mind that in their definition (2.5) the operator $\mathcal{D}_\alpha$ is meant as G-L operator or E-K operator of fractional differentiation. And possibly, it can be attacked by tools from the theory of Mittag-Leffler functions and of the fractional calculus.

The information in next section also suggests that this can be a right way.

Meanwhile, one can consider Jensen polynomials as in (2.5) but instead of the operator $\mathcal{D}_\alpha$ of (3.16), (3.17) to take multi-index analogues as $\mathcal{D}_{(\alpha,\dots,\alpha)}$ or more generally, $\mathcal{D}_{(\alpha_1,\alpha_2,\dots,\alpha_m)}$, related to the multi-index M-L functions (3.14): $E_{(\alpha,\dots,\alpha),(\alpha,\dots,\alpha)}(z)$ or $E_{(\alpha_1,\dots,\alpha_m),(\alpha_1,\dots,\alpha_m)}(z)$. Evidently (by the reasons given next in (4.2)), for positive integer α and $\alpha_1,\dots,\alpha_m$ all the zeros of the Jensen polynomials will be real. It is a hypothesis that the techniques to be used to resolve the open problem for non-integer multi-indices will be quite similar to those for single non-integer index α. See for example, the paper by Craven and Csordas [2] devoted to multiplier's sequences related to special functions including the mentioned ones.

4. Additional literature that can be related to the solution of Iliev's problem

First, let us comment Iliev's remark (text before Our Note 2.) before to state his open problem: "If α is a positive integer, then the sequence $\{\Gamma(k+1)/\Gamma(\alpha(k+1))\}$ is an α-sequence, so in this case (all) the zeros of the (Jensen) polynomial $J_n^\alpha(E_{\alpha,\alpha},z)$ are real."

To avoid or duplicate the notion *α-sequence*, let us note that many authors use the notion "(*Laguerre*) *multiplier sequence*", see e.g. the paper by Craven and Csordas [1], available at the website of "Serdica Math. J.", `http://www.math.bas.bg/~serdica/1996/1996-515-524.pdf`, in **Def. 2.2** (we are changing the notations to have uniformity of exposition):

"A sequence $A=\{\alpha_k\}_{k=0}^\infty$ of real numbers is called a *multiplier sequence* if, whenever the real polynomial $b(z)=\sum\limits_{k=0}^n b_k z^k$ has only real zeros, the polynomial $A[b(z)]=\sum\limits_{k=0}^n \alpha_k b_k z^k$ also has only real zeros." Then, in their papers Craven and Csordas, similarly to Iliev and to many other authors on this topic (e.g. [1], [2], [3], etc.), present several criteria for a sequence to be a multiplier sequence, or find new ones.

It sounds worth for the occasion of this memorial conference, to cite also Craven and Csordas's words in beg. of [1]: "In this century, *the Bulgarian mathematicians have played a prominent role* in several areas of mathematics and, in particular, *in the theory of distribution of zeros of polynomials and entire functions*. The theory of multiplier sequences commenced with the work of Laguerre and was solidified in the seminal work of Pólya and Schur. *Subsequently, this theory gained prominence at the hands of renowned mathematicians as Iliev,...* and *Obreshkov*, just to mention a few names."

On the other hand, the use of the notion multiplier sequence sounds much *closer* (!) *to the topic of the Gelfond-Leontiev operators* of § 3.1, where the

(G-L) operators of generalized differentiation are created by means of multiplier sequences involving coefficients of entire functions as $\exp z$, M-L function, etc.

The above mentioned Iliev's observation is based on the fact that in case of integer $\alpha > 0$, the sequence

$$A = \{\alpha_k\} = \{k!/\Gamma(\alpha(k+1))\} = \{\Gamma(k+1)/\Gamma(\alpha(k+1))\} \in \alpha, \tag{4.1}$$

that is, it is a multiplier sequence. Then, the Jensen polynomial $J_n^\alpha(E_{\alpha,\alpha}, z)$ can be represented as a Hadamard product of multiplier sequences and the polynomial $b(z) = (1+z)^n$ having real zeros only (n-tuple zero $z = -1$) in the following way, starting from (2.7):

$$J_n^\alpha(E_{\alpha,\alpha}, z) = \sum_{k=0}^{n} \frac{\Gamma(\alpha(n+1)}{\Gamma(\alpha(k+1))\Gamma(\alpha(n-k+1))} z^{n-k}$$

$$= \frac{\Gamma(\alpha(n+1))}{\Gamma(n+1)} \sum_{k=0}^{n} \alpha_k \cdot \alpha_{n-k} \cdot \binom{n}{k} z^{n-k} = \text{const}\, A \star A \star (1+z)^n. \tag{4.2}$$

Practically, the open problem posed by Iliev, *Problem 2.5 can be separated in two parts*:

2.5.1. If there are some ranges of values for $\alpha > 0$ not integer, so $A = \{\alpha_k\} = \{k!/\Gamma(\alpha(k+1))\}$ will be still a multiplier sequence (that is an α-sequence)?

2.5.2. If (when) this sequence $A = \{\alpha_k\}$ is not α-sequence, then is it possible to describe the domain where the zeros of the Jensen polynomials $J_n^\alpha(E_{\alpha,\alpha}, z)$ do lie?

Specially, for the first part 2.5.1, all traces lead to some very *subtle results from the theory of Mittag-Leffler functions* $E_{\alpha,\alpha}(z)$, $E_{\alpha,\beta}(z)$, *related to distributions of their zeros, asymptotical behaviour*, etc. (and in this way, also connected somehow to the instruments of fractional calculus).

From this point of view, *we would like to attract the readers attention to some additional literature* which can be useful to go further in understanding and resolving the problems 2.5.1 and 2.5.2.

A. The *remarkable recent paper by Craven and Csordas* [2], where the authors expose a great amount of information and results, relating rather wide classes of special functions to the problems for Laguerre multiplier sequences. There the discussions are in the rather general setting of the *Fox-Wright functions* ${}_p\Psi_q$, called special functions of fractional calculus in the sense of Kiryakova [16], Podlubny [22], etc. Let us emphasize on the passage from [2], p.110: "*These problems appear to be new and are ostensibly difficult.* While at the present time we are unable to solve them, we wish to call attention to them and to related questions ...", and specially to §2 there: Multiplier sequences and the Mittag-Leffler function. In that paper, Example 2.6 (p.113), the authors stress on the same fact as Iliev, but written in other words, referring to Pólya: that the M-L function $E_{\alpha,\alpha}(z)$ belongs to the class of the so-called Laguerre-Pólya functions, when $\alpha = 2, 3, 4, ...$ And for the case of *not integer* α, they refer to Example 2.8 (p.114) and to the paper by Ostrovskii and Peresyolkova [20], p.284. Also the paper of Popov [21] is mentioned (see Remark 2.10, p.114).

The discussion in [2] can be useful also for attacking the extension of the Iliev's problem to the case with multi-index Mittag-Leffler functions (3.14) from §3.3, since these functions are the most representative case of the Fox-Wright functions ${}_p\Psi_q$ $(p = 1, q = m)$, according to Kiryakova [16], and even the hyper-Bessel functions [14, Ch.3] are considered as more particular cases. The paper finishes with a rather general Conjecture 3.17 on the Fox-Wright function, and its References include items by Dzrbashjan (as [6]), by the Bulgarian author Obrechkoff, and by many well-known authors working in the field of fractional calculus.

B. The *important paper by Ostrovskii and Peresyolkova* [20]. It is essentially based (again) on the book by Dzrbasjan [6], discusses some earlier results by Wiman (1905), whose proof sounds incomplete, and provides results for the zeros of the M-L function $E_{\alpha,1}(z)$ when, e.g. $0 < \alpha < 2$. Observe that this paper [20] is essentially referred to in Popov [21].

C. *Several recent papers on distributions of the zeros of the Mittag-Leffler functions*, most of *whose authors work basically in fractional calculus*, as the very recent paper (2013) by Hanneken, and earlier ones by Mainardi, Gorenflo, Luchko, Hilfer, Diethelm, etc., and from the References list of [2].

The links between the cited papers from Sect. 4 come to confirm the suggested relations between Iliev's problem (Sect. 2), the topics of Gelfond-Leontiev and fractional calculus operators (Sect. 3) and the special functions of fractional calculus.

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ISOMETRIES BETWEEN FUNCTION SPACES

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Abstract

Let A and B be complex-linear subspaces of $C(X)$ and $C(Y)$, respectively, where X and Y are compact Hausdorff spaces. Suppose that for each distinct points $x, y \in X$ there exists $f \in A$ such that $f(x) \neq f(y)$. Under an additional assumption on the Choquet boundary $\mathrm{Ch}(A)$ of A, we prove that if S is a real-linear isometry of A onto B, then there exist an open and closed subset K of $\mathrm{Ch}(B)$, a homeomorphism $\phi\colon \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ and a unimodular continuous function u on $\mathrm{Ch}(B)$ such that

$$S(f)(y) = \begin{cases} u(y)f(\phi(y)) & y \in K \\ \overline{u(y)f(\phi(y))} & y \in \mathrm{Ch}(B) \setminus K \end{cases}$$

for all $f \in A$.

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1. Introduction

Let $C(X)$ be a complex Banach space of all complex-valued continuous functions on a compact Hausdorff space X with respect to sup-norm $\|\cdot\|$. A normed linear subspace $A \subset C(X)$ is called a *function space* on X provided that A contains the constant functions and separates the points of X in the following sense; for each distinct points $x, y \in X$ there exists $f \in A$ such that $f(x) \neq f(y)$. An *isometry* S between function spaces A and B is a map, not necessarily linear, $S\colon A \to B$ that satisfies $\|S(f) - S(g)\| = \|f - g\|$ for all $f, g \in A$. The study of isometries between function spaces has a rather long history. In 1932, Banach obtained what is so called the Banach-Stone theorem for compact metric spaces X and Y. Stone has proven that the result holds for compact Hausdorff spaces in 1937.

Theorem 1.1 (Banach-Stone theorem). *If $S\colon C(X) \to C(Y)$ is a surjective, complex-linear isometry, then there exist a homeomorphism $\phi\colon Y \to X$*

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and a unimodular, continuous function $u\colon Y \to \{z \in \mathbb{C} : |z| = 1\}$ *such that* $S(f)(y) = u(y)f(\phi(y))$ *for all* $f \in C(X)$ *and* $y \in Y$.

The Banach-Stone theorem has been generalized in several directions (see [1, 2, 5, 10, 11]). One of important results on isometries was proven by Mazur and Ulam [8].

Theorem 1.2 (Mazur-Ulam theorem)**.** *Let* E *and* F *be (real or complex) normed linear spaces. If* $S\colon E \to F$ *is a surjective isometry, then* $S - S(0)$ *is real-linear.*

It is obvious that the shift $S - S(0)\colon E \to F$ of S from the Mazur-Ulam theorem is a surjective isometry. Thus the description of real-linear isometries is an essential role in the study of surjective isometries.

2. Main results

Definition 2.1. Let A be a function space on a compact Hausdorff space X and A^* the dual space of A, that is the set of all bounded, complex-linear functionals on A with the operator norm. The unit ball $A_1^*$of A^* with the weak $*$-topology is a compact and convex set, and thus A_1^* has extreme points. Let $Ext(A_1^*)$ be the set of all extreme points of A_1^*. Then the set $\{x \in X : \delta_x \in Ext(A_1^*)\}$ is called the *Choquet boundary* of A, and is denoted by $\mathrm{Ch}(A)$, where δ_x is a point evaluation, i.e. $\delta_x(f) = f(x)$ for all $f \in A$. Then

$$Ext(A_1^*) = \{\alpha\delta_x : \alpha \in \mathbb{T} \text{ and } x \in \mathrm{Ch}(A)\}.$$

Definition 2.2. A function space A is said to be *triple separating*, if for each triple of distinct points $x, y, z \in \mathrm{Ch}(A)$ there exists $f \in A$ such that $f(x) = 1$ and $|f(y)|, |f(z)| < 1/\sqrt{2}$.

Example 2.1. Let A be a function space on a compact Hausdorff space X. If A is closed under multiplication, then A is triple separating. In fact, if $x, y, z \in \mathrm{Ch}(A)$ are distinct points, then there exist $f_1, f_2 \in A$ such that $f_1(x) = 1 = f_2(x)$ and $f_1(y) = 0 = f_2(z)$; this is possible since A separates the points of X. Then $f = f_1 f_2 \in A$ satisfies $f(x) = 1$ and $f(y) = 0 = f(z)$. Hence A is triple separating.

Example 2.2. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $A = \{az + b \in C(\mathbb{T}) : a, b \in \mathbb{C}\}$. Then A is a function space on $\mathbb{T}$ with $\mathrm{Ch}(A) = \mathbb{T}$. We see that there is no function $f \in A$ such that $f(1) = 1$ and $|f(i)|, |f(-i)| < 1/\sqrt{2}$. Consequently A is not triple separating.

Theorem 2.1. *Let A and B be triple separating function spaces. If S is a real-linear isometry of A onto B, then there exist an open and closed subset K of* $\mathrm{Ch}(B)$*, a homeomorphism* $\phi\colon \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ *and a unimodular continuous function* $u\colon \mathrm{Ch}(B) \to \mathbb{T}$ *such that*

$$S(f)(y) = \begin{cases} u(y)f(\phi(y)) & y \in E \\ \overline{u(y)f(\phi(y))} & y \in \mathrm{Ch}(B) \setminus E \end{cases}$$

for all $f \in A$.

Proof. Define the map $S_*\colon B^* \to A^*$ by

$$S_*(\eta)(f) = \mathrm{Re}\eta(S(f)) - i\mathrm{Re}\eta(S(if)) \qquad (\eta \in B^*,\ f \in A).$$

Then S_* is a surjective, real-linear isometry with respect to the operator norm. Hence $S_*(Ext(B_1^*)) = Ext(A_1^*)$ and thus for each $y \in \mathrm{Ch}(B)$ there exist $\alpha \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$ such that $S_*(\delta_y) = \alpha\delta_x$. Such $\alpha \in \mathbb{T}$ and $x \in \mathrm{Ch}(A)$ are uniquely determined; in fact, if $S_*(\delta_y) = \alpha_0\delta_{x_0}$, then $\alpha f(x) = \alpha_0 f(x_0)$ for all $f \in A$. Taking $f = 1$, $\alpha = \alpha_0$, and thus $x = x_0$, since A separates the points of underlying space. As $i\delta_y \in Ext(B_1^*)$, $S_*(i\delta_y) = \beta\delta_{x'}$ for some $\beta \in \mathbb{T}$ and $x' \in \mathrm{Ch}(A)$. Therefore $S_*((1+i)\delta_y) = \alpha\delta_x + \beta\delta_{x'}$ by the real-linearity of S_*. Since $\sqrt{2}^{-1}(1+i)\delta_y \in Ext(B_1^*)$, there exist $\gamma \in \mathbb{T}$ and $x'' \in \mathrm{Ch}(A)$ such that $S_*(\sqrt{2}^{-1}(1+i)\delta_y) = \gamma\delta_{x''}$, and consequently $\sqrt{2}\,\gamma\delta_{x''} = \alpha\delta_x + \beta\delta_{x'}$. This implies that $\sqrt{2}\,\gamma f(x'') = \alpha f(x) + \beta f(x')$ for all $f \in A$. If we take $f = 1$, then $\sqrt{2}\,\gamma = \alpha + \beta$, and thus $\sqrt{2} = |1 + \overline{\alpha}\beta|$, where $\bar{\cdot}$ is the complex conjugate. As $\overline{\alpha}\beta \in \mathbb{T}$, we have $\overline{\alpha}\beta = \pm i$ and hence $\beta = \pm i\,\alpha$. This implies $\sqrt{2}\,\gamma = (1 \pm i)\alpha$, and therefore $(1 \pm i)f(x'') = f(x) \pm i f(x')$ for all $f \in A$. We assert that $x = x'$; for if $x \neq x'$, then $x'' \neq x$ and $x'' \neq x'$. By the triple separation of A, there exists $f_0 \in A$ such that $f_0(x'') = 1$ and $|f_0(x)|, |f_0(x')| < 1/\sqrt{2}$. Thus

$$\sqrt{2} = |1 + i| = |f_0(x) \pm i f_0(x')| \leq |f_0(x)| + |f_0(x')| < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2},$$

which is a contradiction. Hence, $x = x'$, and consequently $S_*(i\delta_y) = \pm i S_*(\delta_y)$.

Set $K = \{y \in \mathrm{Ch}(B) : S_*(i\delta_y) = iS_*(\delta_y)\}$. Then $S_*(i\delta_y) = iS_*(\delta_y)$ for $y \in K$, and $S_*(i\delta_y) = -iS_*(\delta_y)$ for $y \in \mathrm{Ch}(B) \setminus K$. The rest of proof is similar to the case when S is complex-linear, and we omit it. See for the detail [7]. □

As a direct consequence of the Mazur-Ulam theorem and Theorem **2.1**, we can describe all surjective isometries between triple separating function spaces, which extends [1, 2, 3, 9, 10].

Corollary 2.1. *Let A and B be be triple separating function spaces. If S is an isometry of A onto B, then there exist an open and closed subset K*

of Ch(B)*, a homeomorphism* $\phi\colon \mathrm{Ch}(B) \to \mathrm{Ch}(A)$ *and a unimodular continuous function* $u\colon \mathrm{Ch}(B) \to \mathbb{T}$ *such that*

$$S(f)(y) = S(0)(y) + \begin{cases} u(y)f(\phi(y)) & y \in E \\ \overline{u(y)f(\phi(y))} & y \in \mathrm{Ch}(B) \setminus E \end{cases}$$

for all $f \in A$.

If function spaces are not triple separating, there is another type of surjective isometry.

Example 2.3. Let $A = \{az+b \in C(\mathbb{T}) : a, b \in \mathbb{C}\}$. As seen in Example **2.2**, A is *not* triple separating. The following eight maps are all surjective, real-linear isometries on A;

$$az + b \mapsto \lambda az + \mu b, \quad \lambda az + \mu\bar{b}, \quad \lambda\bar{a}z + \mu b, \quad \lambda\bar{a}z + \mu\bar{b},$$
$$\lambda bz + \mu a, \quad \lambda bz + \mu\bar{a}, \quad \lambda\bar{b}z + \mu a, \quad \lambda\bar{b}z + \mu\bar{a},$$

where $\lambda, \mu \in \mathbb{T}$. This is easily verified, since $\|az + b\| = |a| + |b|$.

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ABOUT THE SOLUTION OF THE FRACTIONAL COULOMB EQUATION

Yanka Nikolova

Abstract

In this paper the Riemann-Liouville operator for fractional differentiation is applied to treat a generalization of the Coulomb wave equation. By means of the Frobenious method, the fractional Coulomb equation is solved and the solution is obtained in power series form. It is shown that this solution contains the regular solution of the classical Coulomb equation as a particular case.

MSC 2010: 26A33, 44A99, 35R11, 44A10, 44A15

Key Words and Phrases: Riemann-Liouville fractional derivative; Kummer functions; Kummer equation; differential equation of fractional order

1. Introduction

The equation

$$z\,\frac{d^2y}{dz^2} + (\gamma - z)\frac{dy}{dz} - \alpha y = 0 \tag{1.1}$$

is called the confluent hypergeometric equation, or the Kummer equation. By means of the power series method (Frobenious' method), two linearly independent solutions of (1.1) can be found (see [10]) as $y_1 = F(\alpha, \gamma, z)$ and $y_2 = (z)^{1-\gamma}F(\alpha - \gamma + 1, 2 - \gamma, z)$, where

$$F(\alpha, \gamma, z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!(\gamma)_n}\, z^n, \quad \gamma \neq 0, 1, 2, ... \tag{1.2}$$

is known as the confluent hypergeometric function, or simply, the Kummer function. The notation $(\alpha)_n$ in (1.2) is for the Pochhammer symbol

$$(\alpha)_n = \Gamma(\alpha + n)/\Gamma(\alpha) = \alpha(\alpha + 1)\cdots(\alpha + n - 1).$$

Since this function appears as a special case of the generalized hypergeometric function, it is often written in the form ${}_1F_1(\alpha, \gamma, z)$. The Kummer function is a single-valued entire (analytic in the whole z-plane) function.

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If we set $y = e^{z/2}z^{-\gamma/2}w(z)$ in the Kummer equation (1.1), it takes the form

$$\frac{d^2w}{dz^2} + \left[-\frac{1}{4} + \left(\frac{\gamma}{2} - \alpha\right)\frac{1}{z} + \frac{\gamma}{2}\left(1 - \frac{\gamma}{2}\right)\frac{1}{z^2}\right] w = 0. \tag{1.3}$$

Simple substitutions as $m = \frac{\gamma-1}{2}$ and $\alpha = \frac{1}{2} + m - k$, transform the equation (1.3) into

$$\frac{d^2w}{dz^2} + \left[-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right] w = 0, \tag{1.4}$$

known as the Whittaker equation. From (1.2) and the relation between the equations (1.1) and (1.4) it follows that, if $2m$ is not an integer, the two linearly independent solutions of (1.4) at the point $z = 0$ are (see e.g.[10]):

$$M_{k,m}(z) = z^{1/2+m}e^{-z/2}F(1/2 + m - k, 1 + 2m, z) \tag{1.5}$$

and

$$M_{k,-}\,m(z) = z^{1/2-m}e^{-z/2}F(1/2 - m - k, 1 - 2m, z) \tag{1.6}$$

known as Whittaker functions. The functions (1.5) and (1.6) are single-valued analytic functions for $-\pi < argz < \pi$.

In this paper we aim to solve the fractional order differential equation of the form

$$x^{2\alpha}{}_0D_x^{2\alpha}y(x) + (x^{2\alpha} + \alpha x^{\alpha} + b)y(x) = 0, \tag{1.7}$$

where $x > 0$, $0 < \alpha \le 1$, $a, b \in \mathbb{R}$ and ${}_0D_x^{2\alpha}$ is the Riemann-Liouville fractional derivative of order 2α. The reason to deal with the equation (1.7) is that it appears as a fractional generalization of the Whittaker equation (1.4). Indeed, if we set $\alpha = 1, a = -2\eta$ and $b = -m(m+1), m = 1, 2, ...$, the equation (1.7) reduces to the equation

$$\frac{d^2y}{dx^2} + \left[1 - \frac{2\eta}{x} - \frac{m(m+1)}{x^2}\right] y = 0, \tag{1.8}$$

also referred to as the Coulomb wave equation, see e.g. [2].

Because of the symmetry property of the solutions (1.5) and (1.6) with respect to the parameter m, we restrict ourself on determining the fractional analog of the regular solution of (1.8). For this purpose we apply the Frobenious method [6] for solving the equation (1.7). Some physical applications of its particular case - the regular solution of (1.8) can be also discussed.

2. Preliminaries

Fractional Calculus (differentiation and integration of arbitrary order) is nowadays one of the most intensively developing areas of the mathematical analysis. Its fields of application range from biology through physics and electrochemistry to economics, probability theory and statistics. The fractional derivatives provide an excellent instrument for modeling of the memory and hereditary properties of various materials and processes. The half-order derivatives and integrals (semi-derivatives and semi-integrals) prove to be more useful for the formulation of certain electrochemical problems than the classical methods [1]. The modeling of diffusion in a specific type of porous medium is one of the most significant applications of fractional derivatives [8],[4]. Recently, fractional differentiation and integration operators are also used for extension of the temperature field problem in oil strata [3],[6]. In special treats as [5],[7],[9] the mathematical aspects and applications of the fractional calculus are extensively discussed. For the purposes of this paper we adopt the Riemann-Liouville fractional derivative of a function $f(x)$ of order $\alpha > 0$, defined by

$$ {}_0D_x^\alpha f(x) = \frac{d^n}{dx^n}[{}_0D_x^{-(n-\alpha)} f(x)], $$

where $n > 0$ is positive integer such that $n - 1 \leq \alpha < n$ and

$$ {}_0D_x^{-(n-\alpha)} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-\tau)^{(n-\alpha-1)} f(\tau)\, d\tau $$

is the Riemann-Liouville fractional integral of $f(x)$ of order $n - \alpha$.

Let us mention that for $\alpha > 0$, $x > 0$ and $\beta > -1$, the fractional derivative of the power function x^β is given by

$$ {}_0D_x^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}\, x^{\beta-\alpha}, \tag{2.1} $$

that in the particular case $\beta = 0$ and $0 < \alpha < 1$, in contrast to the classical calculus, implies

$$ {}_0D_x^\alpha 1 = \frac{x^{-\alpha}}{\Gamma(1-\alpha)}. $$

3. Fractional Coulomb Equation

Theorem 3.1. *Let $0 < \alpha \leq 1$, $x > 0$, $a, b \in \mathbb{R}$ and ρ be such that $\rho - \alpha > -1$ and $\frac{\Gamma(\alpha+\rho+1)}{\Gamma(-\alpha+\rho+1)} + b = 0$. If the equation (1.7) has a solution given by convergent power series, then the solution has the form*

$$ y(x) = \sum_{k=1}^{\infty} a_k(\alpha, \rho)\, x^{k\alpha+\rho}, \tag{3.1} $$

where the coefficients satisfy the recurrence relations

$$ a_2(\alpha, \rho)\left[\frac{\Gamma(2\alpha+\rho+1)}{\Gamma(\rho+1)} + b\right] + a a_1(\alpha, \rho) = 0 \tag{3.2} $$

and for $k \geq 3$,

$$a_k(\alpha,\rho)\left[\frac{\Gamma(k\alpha+\rho+1)}{\Gamma((k-2)\alpha+\rho+1)}+b\right]+aa_{k-1}(\alpha,\rho)+a_{k-2}(\alpha,\rho)=0. \tag{3.3}$$

Proof. Following the basic idea of the power series method, we search the solution of the equation (1.7) in the form (3.1). Inserting (3.1) into (1.7) and taking into account (2.1), we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} a_k(\alpha,\rho)\frac{\Gamma(k\alpha+\rho+1)}{\Gamma((k-2)\alpha+\rho+1)}x^{k\alpha+\rho}\\ +\sum_{k=3}^{\infty} a_{k-2}(\alpha,\rho)x^{k\alpha+\rho}\\ +a\sum_{k=2}^{\infty} a_{k-1}(\alpha,\rho)x^{k\alpha+\rho}\\ +b\sum_{k=1}^{\infty} a_k(\alpha,\rho)x^{k\alpha+\rho}=0.\end{aligned}$$

Rearranging the terms in the above equality, we get

$$\begin{aligned}\sum_{k=3}^{\infty} a_k(\alpha,\rho)\frac{\Gamma(k\alpha+\rho+1)}{\Gamma((k-2)\alpha+\rho+1)}x^{k\alpha+\rho}\\ +[a_{k-2}(\alpha,\rho)+aa_{k-1}(\alpha,\rho)+ba_k(\alpha,\rho)]x^{k\alpha+\rho}\\ +[a_2(\alpha,\rho)\frac{\Gamma(2\alpha+\rho+1)}{\Gamma(\rho+1)}+aa_1(\alpha,\rho)+ba_2(\alpha,\rho)]x^{2\alpha+\rho}\\ +\left[\frac{\Gamma(\alpha+\rho+1)}{\Gamma(-\alpha+\rho+1)}+b\right]a_1(\alpha,\rho)x^{\alpha+\rho}=0.\end{aligned}$$

The latest equation implies directly the recurrence relations (3.2) and (3.3) and thus the statement is proved. □

4. Coulomb Wave Equation

Consider the Coulomb wave equation (1.8) that evidently is a particular case of the fractional equation (1.7) as $\alpha = 1$, $a = -2\eta$, $\rho = m$, $m = 1, 2, ...$, and $b = -m(m+1)$. If we choose $a_1(1,m) = 1$, then according to (3.2) it follows that

$$a_2(1,m)=\frac{2\eta\Gamma(m+1)}{\Gamma(m+3)-m(m+1)\Gamma(m+1)}=\frac{\eta}{m+1}. \tag{4.1}$$

In this particular case the recurrence formula (3.3) becomes

$$(k-1)(k+2m)a_k(1,m)=2\eta a_{k-1}(1,m)-a_{k-2}(1,m), \quad k\geq 3. \tag{4.2}$$

According to the notations $A_{k,m}$ used in [10] for the coefficients in the power series representation of the regular solution of (1.8), it becomes clear that $a_k(1,m) = A_{k+m,m}$, $k \geq 1$. Hence the solution (3.1) of the fractional equation (1.7) reduces in this particular case to the regular solution of the Coulomb wave equation of the form

$$y(x) = F_m(\eta, x) = C_m(\eta) x^{m+1} \sum_{k=m+1}^{\infty} A_{k,m}\, x^{k-m-1}, \tag{4.3}$$

where $A_{m+1,m} = 1$, $A_{m+2,m} = \frac{\eta}{m+1}$,

$$(k+m)(k-m-1)A_{k,m} = 2\eta A_{k-1,m} - A_{k-2,m},\ k > m+2 \tag{4.4}$$

and

$$C_m(\eta) = \frac{2^m e^{-\frac{\pi\eta}{2}} |\Gamma(m+1+i\eta)|}{\Gamma(2m+2)}. \tag{4.5}$$

The series representation (4.3) of the solution provides a convenient way for numerical evaluation of the solution. Another option for numerical evaluation of the solution of the equation (1.8) is to apply its expansion in terms of the Kummer confluent hypergeometric function (1.2),

$$F_m(\eta, x) = C_m(\eta) x^{m+1} e^{-ix} F(m+1-i\eta, 2m+2; 2ix). \tag{4.6}$$

5. Numerical study of the real and complex regular Coulomb wave equation

The real and complex Coulomb wave functions $F_m(\eta, x)$ and $G_m(\eta, x)$ initially defined for m - nonnegative integral, η - real and x - real, positive, are widely used in problems of nuclear physics. The solutions of the Coulomb wave equation (1.8), where $x > 0$, $-\infty < \eta < \infty$, $m = 0, 1, 2, \dots$, are the regular and irregular Coulomb functions $F_m(\eta, x)$ and $G_m(\eta, x)$, respectively. The physical problems for propagation in a ferrite or solid-plasma circular area call for the well-behaved function at the origin. We examine two cases: 1) $m = \pm 0, 6$, $-\infty < \eta < \infty$, $x > 0$; 2) $m = \pm 0, 6$, $\eta = ik$, $-\infty < k < \infty$, $\delta > 0$, $x = -i\delta$, $\delta > 0$ corresponding to fast and slow waved.

The numerical evaluation of the real function $F_m(\eta, x)$ and its zeros can be done through the expansion of $F_m(\eta, x)$ in terms of the Kummer confluent hypergeometric function (4.6).

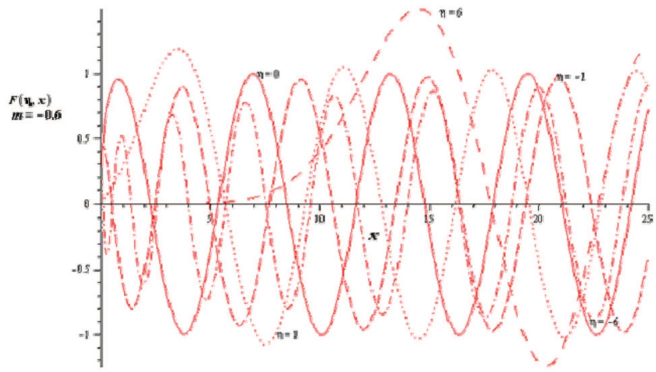

Fig. 5.1: Coulomb wave function $F_m(\eta, x)$ versus x for $\eta = 0, \pm 1, \pm 6$ in case $m = -0,6$

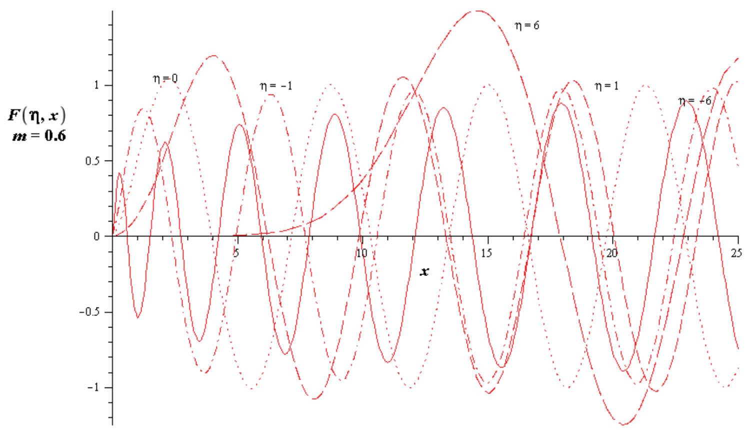

Fig. 5.2: Coulomb wave function $F_m(\eta, x)$ versus x for $\eta = 0, \pm 1, \pm 6$ in case $m = +0,6$

Fig. 5.1 and 5.2 show that the functions oscillate with the growth and depend strongly on the change of m and η. If m and η decrease (increase) the oscillation accelerates (slows down). Their values are restricted approximately within the interval $[-1, 1]$, for any x and η. If $\eta > 0$ ($\eta < 0$) the values of the initial extrema of $F_m(\eta, x)$ are larger (smaller) than 1 and tend to it when the number of extremum grows. Thus, $F_m(\eta, x)$ is a real function with twice smaller argument varying in a limited interval which is an advantage with respect to the complex Kummer function.

The Coulomb wave functions have applications to certain propagation problems.

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COMPARISON BETWEEN THE CONVERGENCE OF POWER AND GENERALIZED MITTAG-LEFFLER SERIES

Jordanka Paneva-Konovska

Abstract

In this paper we consider a family of the three-index generalizations of the classical Mittag-Leffler functions, introduced by Prabhakar. We consider series in such type of functions in the complex plane and study their convergence. More precisely, we determine where the series converges and where it does not, where the convergence is uniform, which the domain of convergence is, what the behaviour of the series is "near" the boundary of the domain of convergence, and on itself. Along with this, we state analogues of the Cauchy-Hadamard, Abel and Fatou theorems for the power series. Finally, we compare the obtained results with the classical ones for the widely used power series.

MSC 2010: 40A30, 33E12, 31A20, 30D15, 30B30, 30B10

Key Words and Phrases: Mittag-Leffler functions and generalizations, series in generalized Mittag-Leffler functions, convergence and divergence, Cauchy-Hadamard, Abel and Fatou type theorems

1. Introduction

Let $E^{\gamma}_{\alpha,\beta}$ denote the Prabhakar generalization (see [13]) of the Mittag-Leffler (M-L) functions E_α and $E_{\alpha,\beta}$, defined in the whole complex plane $\mathbb{C}$ by the power series:

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+\beta)} \frac{z^k}{k!}, \qquad \alpha,\ \beta,\ \gamma \in \mathbb{C}, \quad Re(\alpha) > 0, \tag{1.1}$$

where $(\gamma)_k$ is the Pochhammer symbol ([1], Section 2.1.1)

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\dots(\gamma+k-1).$$

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For $\gamma = 1$ this function coincides with M-L function $E_{\alpha,\beta}$, while for $\gamma = \beta = 1$ with E_α, i.e.:

$$E^1_{\alpha,1}(z) = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad E^1_{\alpha,\beta}(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{1.2}$$

with $\alpha,\ \beta \in \mathbb{C}, \quad Re(\alpha) > 0$.

Consider now Prabhakar's generalization for indices $\beta = n$ with integer $n = 0, 1, 2, ...,$ i.e.

$$E^\gamma_{\alpha,n}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+n)} \frac{z^k}{k!}, \qquad \alpha,\ \gamma \in \mathbb{C}, \quad Re(\alpha) > 0, \quad n \in \mathbb{N}_0. \tag{1.3}$$

Depending on γ and n, some coefficients in (1.3) may be equal to zero. This is possible only when $n = 0$ or γ is a non-positive integer. In the first case the coefficient with $k = 0$ is equal to zero, whereas in the second case (1.3) reduces to a finite sum, i.e. polynomial.

So, given a number γ, suppose that some of the coefficients in $E^\gamma_{\alpha,n}(z)$ are equal to zero, that is, there exist numbers $p, M \in \mathbb{N}_0$, such that the functions (1.3) can be written as follows:

$$E^\gamma_{\alpha,n}(z) = z^p \sum_{k=p}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k+n)} \frac{z^{k-p}}{k!} \ \text{ or } \ E^\gamma_{\alpha,n}(z) = z^p \sum_{k=p}^{M} \frac{(\gamma)_k}{\Gamma(\alpha k+n)} \frac{z^{k-p}}{k!}. \tag{1.4}$$

More precisely, as it is seen above, if γ is different from zero, then $p = 1$ for $n = 0$, whereas $p = 0$ for each positive integer n. In the case $\gamma = 0$, the following remark can be made.

Remark 1.1. If $\gamma = 0$, then the functions in (1.3) take the simplest form

1. $E^0_{\alpha,n}(z) = 0$ for $n = 0$,
2. $E^0_{\alpha,n}(z) = \dfrac{1}{\Gamma(n)}$ for $n \in \mathbb{N}$.

Furthermore, an asymptotic formula for "large" values of the indices n is valid as follows, for a proof it could be seen in [10].

Theorem 1.1. *Let $z, \alpha, \gamma \in \mathbb{C}, n \in \mathbb{N}_0, Re(\alpha) > 0, \gamma \neq 0$. Then there exist entire functions $\theta^\gamma_{\alpha,n}$ such that the generalized Mittag-Leffler function* (1.3) *has the following asymptotic formula*

$$E^\gamma_{\alpha,n}(z) = \frac{(\gamma)_p}{\Gamma(\alpha p+n)} z^p \left(1 + \theta^\gamma_{\alpha,n}(z)\right), \tag{1.5}$$

where $\theta^{\gamma}_{\alpha,n}(z) \to 0$ *as* $n \to \infty$, *with a corresponding* p, *depending on the index* n. *Moreover, on the compact subsets of the complex plane* $\mathbb{C}$, *the convergence is uniform and*

$$\theta^{\gamma}_{\alpha,n}(z) = O\left(\frac{1}{n^{Re(\alpha)}}\right) \quad (n \ \in \mathbb{N}). \tag{1.6}$$

Remark 1.2. According to the asymptotic formula (1.5), it follows that there exists a positive integer N_0 such that the functions (1.3) have no zeros for $n > N_0$, possibly except for the origin.

Remark 1.3. Each of the functions in (1.3) ($n \in \mathbb{N}$), being an entire function, not identically zero, has no more than a finite number of zeros in the closed and bounded set $|z| \le R$. Moreover, because of Remark **1.2**, no more than finite number of these functions have some zeros, possibly except for the origin.

2. Series in generalized M-L functions

In this section we recall briefly some results on the convergence in the complex plane of series in generalized M-L functions, like these in (1.3). These are results quite analogous to the ones for the classical power series. The same type convergence theorems have been earlier obtained for series in some other special functions, for example, for series in Laguerre and Hermite polynomials, by Rusev ([14]), and resp. by the author – for series in Bessel functions, their Wright's 2-, 3-, and 4-indices generalizations, and also more general multi-index (in a sense of [3], [2]) M-L functions (see e.g. [5]–[9]).

Setting

$$\widetilde{E}^{0}_{\alpha,0}(z) = 0, \quad \widetilde{E}^{0}_{\alpha,n}(z) = \Gamma(n)\, z^n\, E^{0}_{\alpha,n}(z), \quad n \in \mathbb{N},$$

$$\widetilde{E}^{\gamma}_{\alpha,n}(z) = \frac{\Gamma(\alpha p + n)}{(\gamma)_p}\, z^{n-p}\, E^{\gamma}_{\alpha,n}(z), \quad n \in \mathbb{N}_0 \quad (\gamma \ne 0), \tag{2.1}$$

(with the corresponding values of p), we consider the series in these functions, respectively of the form:

$$\sum_{n=0}^{\infty} a_n \widetilde{E}^{\gamma}_{\alpha,n}(z), \tag{2.2}$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$.

Finding their disks of convergence, we study the series behaviour inside the found disks and "near" their boundaries, as well as on the boundaries, giving theorems of Cauchy-Hadamard, Abel, as well as Fatou type.

3. Cauchy-Hadamard and Abel type theorems

In the beginning, we state a theorem of Cauchy-Hadamard type and a corollary for the series (2.2), considered above.

In what follows we use the notation $D(0;R)$ and $C(0;R)$ respectively for the open disk centered at the origin with a radius R and its boundary, i.e.

$$D(0;R) = \{z : |z| < R,\ z \in \mathbb{C}\}, \quad C(0;R) = \partial D(0;R) = \{z : |z| = R,\ z \in \mathbb{C}\}.$$

Theorem 3.1 (of Cauchy-Hadamard type)**.** *The domain of convergence of the series* (2.2) *with complex coefficients* a_n *is the disk* $D(0;R)$ *with a radius of convergence* R*, where*

$$R = \left(\limsup_{n\to\infty} (\ |a_n|\ )^{1/n} \right)^{-1}. \tag{3.1}$$

More precisely, the series (2.2) *is absolutely convergent on the disk* $D(0;R)$ *and divergent on the domain* $|z| > R$*. The cases* $R = 0$ *and* $R = \infty$ *fall in the general case.*

Thus, the considered series (2.2) converges in a disk, like in the theory of the widely used power series. Analogously, inside the disk, the convergence of the discussed series is uniform, i.e., the following corollary, similar to the classical Abel lemma, holds.

Corollary 3.1. *Let the series* (2.2) *converges at the point* $z_0 \neq 0$*. Then it is absolutely convergent on the disk* $D(0;|z_0|)$*. Inside the disk* $D(0;R)$*, i.e. on each closed disk* $|z| \leq r < R$ *(*R *defined by* (3.1)*), the convergence is uniform.*

The very disk of convergence is not obligatory a domain of uniform convergence and on its boundary the series may even be divergent.

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular domain with size $2\varphi < \pi$ and with a vertex at the point $z = z_0$, which is symmetric with respect to the straight line defined by the points 0 and z_0 and d_φ be the part of the angular domain g_φ, closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle. The next theorem refers to the uniform convergence of the series (2.2) on the set d_φ and its convergence at the point z_0, provided $z \in D(0;R) \cap g_\varphi$.

Theorem 3.2 (of Abel type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, R be the real number defined by* (3.1) *and $0 < R < \infty$. If $\widetilde{f}(z;\,\alpha,\,\gamma)$ is the sum of the series* (2.2) *on the domain $D(0;R)$, and this series converges at the point z_0 of the boundary $C(0;R)$, then:*

(i) *The following relation holds*

$$\lim_{z\to z_0} \widetilde{f}(z;\,\alpha,\,\gamma) = \sum_{n=0}^{\infty} a_n \widetilde{E}^{\gamma}_{\alpha,\,n}(z_0), \tag{3.2}$$

provided $z \in D(0;R) \cap g_\varphi$.

(ii) *The series* (2.2) *is uniformly convergent on the domain d_φ.*

The details of the proofs concerning the series (2.2) (including the equality (3.2)) could be seen in [10], except for the uniformity and Corollary. The ideas of the last ones go analogously to the [11].

4. Fatou type theorem

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers with $\limsup\limits_{n\to\infty} (|a_n|)^{1/n} = R^{-1}$, $0 < R < \infty$ and $f(z)$ be the sum of the power series $\sum\limits_{n=0}^{\infty} a_n z^n$ on the open disk $D(0;R)$, i.e.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in D(0;R). \tag{4.1}$$

Definition 4.1. A point $z_0 \in \partial D(0;R)$ is called regular for the function f if there exist a neighbourhood $U(z_0;\rho)$ and a function $f^*_{z_0} \in \mathcal{H}(U(z_0;\rho))$ (the space of complex-valued functions, holomorphic in the set $U(z_0;\rho)$), such that $f^*_{z_0}(z) = f(z)$ for $z \in U(z_0;\rho) \cap D(0;R)$.

By this definition it follows that the set of regular points of the power series is an open subset of the circle $C(0;R) = \partial D(0;R)$ with respect to the relative topology on $\partial D(0;R)$, i.e. the topology induced by that of $\mathbb{C}$.

In general, there is no relation between the convergence (divergence) of a power series at points on the boundary of its disk of convergence and the regularity (singularity) of its sum of such points. For example, the power series $\sum\limits_{n=0}^{\infty} z^n$ is divergent at each point of the unit circle $C(0;1)$ regardless of the fact that all the points of this circle, except for $z = 1$, are regular for its sum. The series $\sum\limits_{n=1}^{\infty} n^{-2}z^n$ is (absolutely) convergent at each point of the circle $C(0;1)$, but nevertheless one of them, namely $z = 1$, is a singular (i.e. not regular) for its sum. However, under additional conditions on the sequence $\{a_n\}_{n=0}^{\infty}$, such a relation does exists (see for details Fatou theorem in [4], Vol.1, Ch. 3, §7, 7.3, p. 357), namely, if the coefficients of the power series with the unit disk of convergence tend to the zero, i.e. $\lim\limits_{n\to\infty} a_n = 0$, then the power series converges, even uniformly, on each arc of the unit circle, all points of which (including the ends of the arc) are regular for the sum of the series.

Propositions referring to the properties discussed above have been established also for series in the Laguerre and Hermite polynomials, as well as in Mittag-Leffler systems (see e.g. [14], resp. [12]). Here we give such type of theorem for the Prabhakar systems as follows.

Theorem 4.1 (of Fatou type). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers satisfying the conditions*

$$\lim_{n\to\infty} a_n = 0, \qquad \limsup_{n\to\infty} (\ |a_n|\ )^{1/n} = 1, \tag{4.2}$$

and $F(z)$ be the sum of the series (2.2) *on the unit disk $D(0;1)$, i.e.*

$$F(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}^{\gamma}_{\alpha,\,n}(z), \quad z \in D(0;1).$$

Let σ be an arbitrary arc of the unit circle $C(0;1)$ with all its points (including the ends) regular to the function F. Then the series (2.2) *converges, even uniformly, on the arc σ.*

The **Proof** follows the lines of this one for the Mittag-Leffler functions, using the asymptotic formula (1.5) (for details, see [12]). ■

5. Special cases

In particular, as it has been discussed in the Introduction, for $\gamma = 1$ the Prabhakar function $E^{\gamma}_{\alpha,\,\beta}$, defined by (1.1), coincides with M-L function $E_{\alpha,\,\beta}$, i.e. $E^{1}_{\alpha,\,\beta}(z) = E_{\alpha,\,\beta}(z)$ (see (1.2)). So in this case the series (2.2) takes the form

$$\sum_{n=0}^{\infty} a_n \widetilde{E}^{1}_{\alpha,\,n}(z) = \sum_{n=0}^{\infty} a_n \widetilde{E}_{\alpha,\,n}(z), \tag{5.1}$$

with complex coefficients a_n $(n = 0, 1, 2, ...)$.

Such a kind of series were studied in details e.g. in [11] and [12], but all the obtained results concerning them follow as particular cases from the preceding sections, as well.

6. Conclusion

We emphasize that the results obtained for the series (2.2) are the same as these for the power series (4.1). As it is well seen, they have one and the same radius of convergence R, and are both absolutely and uniformly convergent on each closed disk $|z| \leq r$ $(r < R)$. More precisely, if each one of them converges at the point z_0 of the boundary of $D(0; R)$, then the theorems of Abel type hold for both series in one and the same angular domain. Finally, if $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers satisfying the conditions (4.2), and all the points (including the ends) of the arc σ of the unit circle $C(0; 1)$ are regular to the sums of both considered series, then the series (2.2) and (4.1) converge even uniformly, on the arc σ.

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CERTAIN CLASSES OF FUNCTIONS WITH NEGATIVE COEFFICIENTS, II

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Abstract

The aim of this paper is to obtain coefficient estimates, distortion theorem, and radii of close-to-convexity, starlikeness and convexity for functions belonging to the subclass $S_T(n,\alpha,\beta)$ with negative coefficients.

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Key Words and Phrases: univalent, convex, starlike functions

1. Introduction

Let S denote the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open unit disk $U = \{z : |z| < 1\}$. Let S^* and C be subclasses of S that are, respectively, starlike and convex.

A function

$$f(z) = \widetilde{C} \iff \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in U. \tag{1.2}$$

Let S_p be a class of starlike functions related to $\widetilde{C}$ defined as

$$f(z) \in S_p \iff \Re\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in U. \tag{1.3}$$

Note that

$$f \in \widetilde{C} \iff zf'(z) \in S_p. \tag{1.4}$$

A function f of the form (1.1) is in $S_p(\alpha)$ if it satisfies the analytic characterization:

$$\Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad -1 \leq \alpha < 1,\ z \in U. \tag{1.5}$$

The function $f \in \widetilde{C}(\alpha)$ if and only if $zf'(z) \in S_p(\alpha)$.

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By $\widetilde{C}_\beta$, $0 \le \beta < \infty$ we denote the class of all β-convex functions introduced by Kanas and Wisniowska [1]. It is know that [1] $f \in \widetilde{C}_\beta$ if and only if it satisfies the following condition:

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta\left|\frac{zf'(z)}{f'(z)}\right|, \quad z \in U,\ \beta \ge 0. \tag{1.6}$$

We consider the class S^*_β, $0 \le \beta < \infty$, of β-starlike functions [2], which are associated with the class $\widetilde{C}_\beta$ by the relation

$$f \in C^*_\beta \iff zf'(z) \in S^*_\beta. \tag{1.7}$$

Thus, the class S^*_p is the subclass of S, consisting of functions that satisfy

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta\left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in U,\ \beta \ge 0. \tag{1.8}$$

For a function $f \in S$, we define

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1 f(z) &= \frac{f(z) + zf'(z)}{2} = Df(z) \\ D^n f(z) &= D(D^{n-1} f(z)), \qquad n \in \mathbb{N} = \{1, 2, \dots\} \end{aligned} \tag{1.9}$$

It can be easily seen that

$$D^n f(z) = z + \sum_{n=2}^{\infty} \left(\frac{k+1}{2}\right)^n a_k z^k \quad (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}). \tag{1.10}$$

For $\beta \ge 0$, $-1 \le \alpha \le 1$ and $n \in \mathbb{N}_0$ let $S(n, \alpha, \beta)$ denote the subclass of S consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic condition

$$\Re\left\{\frac{z(D^n f(z))'}{D^n f(z)} - \alpha\right\} > \beta\left|\frac{z(D^n f(z))'}{D^n f(z)} - 1\right|. \tag{1.11}$$

We denote by T the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \quad a_k \ge 0. \tag{1.12}$$

Further, we define the class $S_T(n, \alpha, \beta)$ by

$$S_T(n, \alpha, \beta) = S(n, \alpha, \beta) \cap T. \tag{1.13}$$

2. Coefficient estimates

Theorem 2.1. *A necessary and sufficient condition for the function $f(z)$ of the form* (1.12) *to be in the class $S_T(n,\alpha,\beta)$ is that*

$$\sum_{k=1}^{\infty}[k(1+\beta)-(\alpha+\beta)]\left(\frac{k+1}{2}\right)^n a_k \le 1-\alpha, \tag{2.1}$$

where $-1 \le \alpha < 1$, $\beta \ge 0$ and $n \in \mathbb{N}_0$.

Proof. Let (2.1) holds true, then we have

$$\beta\left|\frac{z(D^n f(z))'}{D^n f(z)}-1\right|-\Re\left\{\frac{z(D^n f(z))'}{D^n f(z)}-1\right\} \le (1+\beta)\left|\frac{z(D^n f(z))'}{D^n f(z)}-1\right|$$

$$\le \frac{(1+\beta)\sum_{k=2}^{\infty}(k-1)\left(\frac{1+k}{2}\right)^n |a_k|}{1-\sum_{k=2}^{\infty}\left(\frac{1+k}{2}\right)^n |a_k|} \le 1-\alpha.$$

Then $f(z) \in S_T(n,\alpha,\beta)$.

Conversely, let $f(z) \in S_T(n,\alpha,\beta)$ and z be real, then

$$\frac{1-\sum_{k=2}^{\infty}k\left(\frac{k+1}{2}\right)^n a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{1+k}{2}\right)^n a_k z^{k-1}}-\alpha \ge \beta\left|\frac{\sum_{k=2}^{\infty}(k-1)\left(\frac{k+1}{2}\right)^n a_k z^{k-1}}{1-\sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^n a_k z^{k-1}}\right|,$$

Letting $z \to 1^-$ along the real axis, we obtain the desired inequality (2.1).

Remark 1. If $f(z) \in S(n,\alpha,\beta)$ the condition (2.1) is only sufficient.

Remark 2. Let the function $f(z)$ defined by (1.12) be in the class $S_T(n,\alpha,\beta)$. Then

$$a_k \le \frac{1-\alpha}{[k(1+\beta)-(\alpha+\beta)]\left(\frac{k+1}{2}\right)^n}, \qquad k \ge 2. \tag{2.2}$$

The result is sharp for the function

$$f(z) = z - \frac{1-\alpha}{[k(1+\alpha)-(\alpha+\beta)]\left(\frac{k+1}{2}\right)^n}\, z^k. \tag{2.3}$$

3. Growth and distortion theorems

Theorem 3.1. *Let the function $f(z)$ defined by (1.12) be in the class $S_T(n,\alpha,\beta)$. Then*

$$|D^i f(z)| \geq |z| - \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^{n-i}|z|^2 \tag{3.1}$$

and

$$|D^i f(z)| \leq z + \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^{n-i}|z|^2 \tag{3.2}$$

for $z \in U$, where $0 \leq i \leq n$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^{n} z^2. \tag{3.3}$$

Proof. Note that $f(z) \in S_T(n,\alpha,\beta)$ if and only if $D^i f(z) \in S_T(n,\alpha,\beta)$ and that

$$D^i f(z) = z - \sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^i a_k z^k. \tag{3.4}$$

Using Theorem 2.1 we know that

$$(2-\alpha+\beta)\left(\frac{3}{2}\right)^{n-i}\sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^i a_k \leq 1-\alpha \tag{3.5}$$

that is, that

$$\sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^i a_k \leq \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^{n-i}. \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$|D^i f(z)| \geq |z| - |z|^2\sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^i a_k \geq |z| - \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^{n-i}|z|^2 \tag{3.7}$$

and

$$|D^i f(z)| \leq |z| + |z|^2\sum_{k=2}^{\infty}\left(\frac{k+1}{2}\right)^i a_k \leq |z| + \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^{n-i}|z|^2 \tag{3.8}$$

Finally, we note that the bounds in (3.1) are attained for the function $f(z)$ defined by

$$D^i f(z) = z - \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^{n-i} z^2. \tag{3.9}$$

This completes proof of Theorem 3.1.

Corollary 3.1. *Let the function $f(z)$ defined by* (1.12) *be in the class $S_T(n,\alpha,\beta)$. Then*

$$|z| - \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^n |z|^2 \le |f(z)| \le |z| + \frac{1-\alpha}{2-\alpha+\beta}\left(\frac{2}{3}\right)^n |z|^2. \qquad (3.10)$$

The equalities in (3.10) *are attained for the function $f(z)$ given by* (3.3).

Proof. Taking $i = 0$ in Theorem 2.1, we immediately obtain (3.10).

4. Radii of close-to-convexity, starlikeness and convexity

A function $f(z) \in T$ is said to be close-to-convex of order ρ if it satisfies

$$\Re f'(z) > \rho, \quad 0 \le \rho < 1, \ z \in U. \qquad (4.1)$$

Theorem 4.1. *Let the function $f(z)$ defined by* (1.12) *be in the class $S_T(n,\alpha,\beta)$. Then $f(z)$ is close-to-convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_1$ where*

$$r_1 = r_1(n,\alpha,\beta,\rho) = \inf_k \left\{\frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)](k+1)^n}{2^n k(1-\alpha)}\right\}^{\frac{1}{k-1}},$$

$$k \ge 2. \qquad (4.2)$$

The result is sharp, with extremal $f(z)$ given by (2.3).

Proof. We must show that $|f'(z) - 1| \le 1-\rho$ for $|z| < r_1(n,\alpha,\beta,\rho)$ where $r_1(n,\alpha,\beta,\rho)$ is given by (4.2). Indeed we find from (1.12) that

$$|f'(z) - 1| \le \sum_{k=2}^{\infty} k a_k |z|^{k-1}.$$

Thus $|f'(z) - 1| \le 1-\rho$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho}\right) a_k |z|^{k-1} \le 1. \qquad (4.3)$$

But, by Theorem 2.1, (4.3) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \le \frac{[k(1+\beta)-(\alpha+\beta)](k+1)^n}{2^n(1-\alpha)}$$

that is, if

$$|z| \le \left\{\frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)](k+1)^n}{k(1-\alpha)2^n}\right\}^{\frac{1}{k-1}}, \quad k \ge 2. \qquad (4.4)$$

Theorem 4.1 follows easily from (4.4).

Theorem 4.2. *Let the function $f(z)$ defined by (1.12) be in the class $S_T(n,\alpha,\beta)$. Then the function $f(z)$ is starlike of order ρ $(0 \le \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = r_2(n,\alpha,\beta,\rho) = \inf_k \left\{ \frac{1-\rho)[k(1+\beta)-(\alpha+\beta)](k+1)^n}{(k-\rho)(1-\alpha)2^n} \right\}^{\frac{1}{k-1}}, \quad k \ge 2. \tag{4.5}$$

The result is sharp, with the extreme function $f(z)$ given by (2.3).

Proof. It is sufficient to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le 1-\rho \quad \text{for} \quad |z| < r_2(n,\alpha,\beta,\rho)$$

where $r_2(n,\alpha,\beta,\rho)$ is given by (4.5). Indeed we find again from (1.12) that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty} a_k z^{k-1}}.$$

Thus

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le 1-\rho$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k-\rho}{1-\rho} \right) a_k|z|^{k-1} \le 1 \tag{4.6}$$

But, by Theorem 2.1, (4.6) will be true if

$$\left(\frac{k-\rho}{1-\rho} \right) |z|^{k-1} \le \frac{[k(1+\beta)-(\alpha+\beta)](k+1)^n}{(1-\alpha)2^n},$$

that is, if

$$|z| \le \left\{ \frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)](k+1)^n}{(k-\rho)(1-\alpha)2^n} \right\}^{\frac{1}{k-1}}, \quad k \ge 2. \tag{4.7}$$

Corollary 4.1. *Let the function $f(z)$ defined by (1.12) be in the class $S_T(n,\alpha,\beta)$. Then $f(z)$ is convex of order ρ $(0 \le \rho < 1)$ in $|z| < r_3$, where*

$$r_3 = r_3(n,\alpha,\beta,\rho) = \inf_k \left\{ \frac{(1-\rho)[k(1+\beta)-(\alpha+\beta)](k+1)^n}{k(k-\rho)(1-\alpha)2^n} \right\}^{\frac{1}{k-1}}, \quad k \ge 2. \tag{4.8}$$

The result is sharp with extremal function $f(z)$ given by (2.3).

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THE CONTRIBUTIONS OF ACADEMICIAN LJUBOMIR ILIEV TO THE EDUCATION IN MATHEMATICS IN BULGARIA
(ПРИНОСИТЕ НА АКАД. ЛЮБОМИР ИЛИЕВ ЗА ОБУЧЕНИЕТО ПО МАТЕМАТИКА В БЪЛГАРИЯ)

Donka Pashkouleva (Донка Ж. Пашкулева)

Abstract

In 1936 Academician Ljubomir Iliev graduated in Mathematics from the Sofia University "St. Kliment Ohridski". After specialization for 2 years with Academician Nikola Obreshkov and defending a Doctorate dissertation in 1938, he became a teacher in Third Sofia Male Gymnasium. During the period 1941 – 1947 he was an Assistant Professor in the Sofia University, and during 1947 – 1952, an Associate Professor and a Deputy Dean of the Faculty of Natural Sciences and Mathematics there. He wrote 233 scientific publications and 9 publications for the education in Mathematics. Out of them 4 are textbooks for the secondary school, and 5 are for university students. Also he published 3 books with mathematical problems. He also translated 2 textbooks – one on Methodology of Geometry and another on Geometry. He set and solved new mathematical problems as well. He also published methodological articles in the journal of the Society in Mathematics and Physics. To the proposed report I enclose a full list of all the scientific publications of Academician Ljubomir Iliev, as well as a list of his publications on education in Mathematics.

MSC 2010: 01A70, 97-30, 97A30

Key Words and Phrases: education in mathematics, textbooks, elementary mathematics, higher education

1. Биографични данни

Акад. Илиев е роден в град Велико Търново на 07 (20) април 1913 г. Завършва Великотърновската мъжка гимназия. През 1936 г. завършва математика в СУ „Св. Кл. Охридски" като стипендиант на Фонда за даровити младежи. Две години специализира при акад. Никола Обрешков и защитава докторска дисертация (от стар тип) в София през 1938 г. През периода 1938-1940 г. работи като учител в Трета софийска мъжка гимназия. През

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1940 – 1941 г. е Хумболтов стипендиант в Мюнхен, където специализира при Перон и Каратеодори. През 1941-1947 е асистент в СУ, а през 1947-1949 г. – частен доцент в същия университет. През 1949-1952 г. е доцент, а през 1949 – 1951 изпълнява длъжността заместник-декан на Природоматематическия факултет на СУ. През 1952 г. е избран за професор и ръководител на катедрата по Висш анализ при Природоматематическия факултет на СУ. През 1957 г. става доктор на науките (от нов тип). През 1958 г. е избран за член-кореспондент на БАН, а през 1967 – за академик (действителен член) на БАН.

2. Публикации и друга научна дейност на акад. Л. Илиев

Акад. Илиев има общо 233 научни публикации, списък на които прилагам отделно. Първата публикация е през 1938 г. в немско списание. Написал е общо 3 монографии. Акад. Илиев създава няколко списания, които утвърждават българската математическа книжнина по света:

- сп. „Сердика“ (1974)
- тематична поредица „Плиска“ (1975)
- монографична поредица „Азбуки“ (1980)

Акад. Илиев е пионер и основател на компютърното дело у нас. Първият изчислителен център е създаден към Института по математика на БАН през 1961 г.

3. Университетски учебници и сравнение с други автори

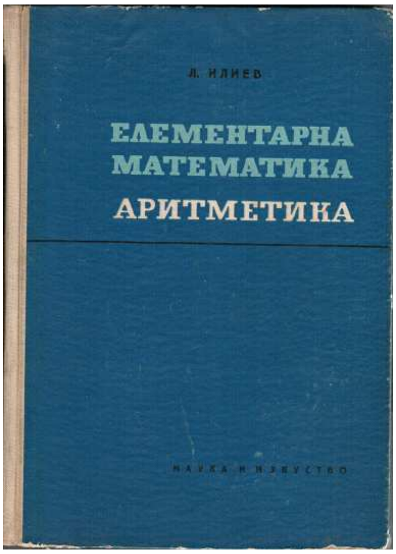
Акад. Илиев разработва университетски учебници в научните дисциплини аритметика, алгебра и тригонометрия. Прилагам отделно пълен списък на издадените учебници, на преведените учебници и други научни популярни публикации.

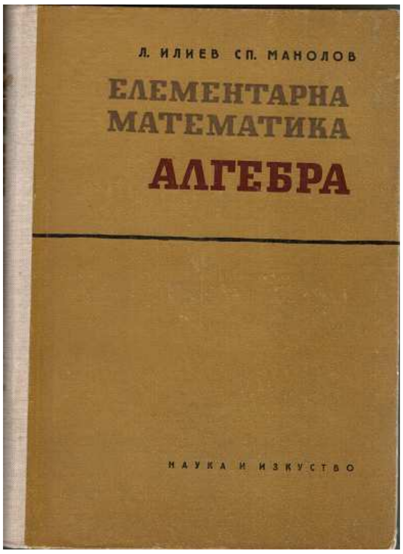
4. Анализ на учебника по алгебра

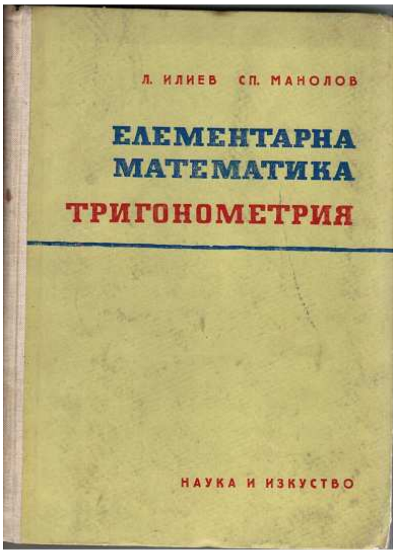
Този учебник е втора книга от една поредица помагала по елементарна математика. В тази поредица влизат още книгите „Аритметика”, „Планиметрия”, „Стереометрия” и „Тригонометрия”. Те се придружават и от Сборника от задачи по елементарна математика, който е издаден през 1955 г.

В учебника по алгебра са изложени систематично основните въпроси от елементарната алгебра:

- алгебрични изрази;
- уравнения и неравенства;
- системи уравнения;
- комбинаторика;
- прогресии;
- решаване на линейни уравнения в цели числа.







Не се съдържат главите за показателна и логаритмична функция. Те са изложени в книгата „Аритметика” след въвеждане на ирационалните числа. При написването на книгата „Алгебра” са използвани учебниците на Новоселов „Специален курс по елементарна алгебра”, издаден през 1951 г., и „Алгебра на елементарните функции”, издаден през 1952 г. Наред с въвеждането на алгебричните понятия, включително „пръстен” и „поле”, навсякъде се прокарва и развитието на понятието „функция”. Още от едночлените се започва разглеждането на изразите като функции.

Учебникът „Алгебра” се състои от въведение и 4 глави. Въведението е посветено на множествата. Разгледани са основните числови множества:

- естествени числа;
- цели числа;
- рационални числа;
- реални числа;

В множеството на реалните числа са дефинирани понятията „равно”, „по-голямо”, „по-малко”, които са понятия-релации. Обърнато е внимание на свойствата „разположеност на числови полета” и „плътност в полето на рационалните и реалните числа”. Разгледано е и съответствието и между числови и точкови множества.

Част Първа „Алгебрични изрази” се състои от 3 глави.

Глава Първа „Многочлени” започва с аналитични изрази, тъждествени преобразувания, графики на многочлени, теорема за тъждественост на многочлени, действия с многочлени.

В целия учебник „Алгебра” са следвани известни дотогава методики. В предговора авторът отбелязва, че съзнателно не е развита теорията върху полето на комплексните числа, както е при Новоселов, защото е искал да не се усложнява материалът.

Както бе посочено по-горе, глава Първа от учебника „Алгебра” е посветена на многочлените. Тук също методиките са стандартни. Теоремите за тъждественост на многочлени, действията с многочлени, формулите за съкратено умножение, някои основни тъждества и разлагането на многочлени на множители са изложени стегнато и коректно. Например в § 16 се систематизират различни начини за разлагане на един многочлен на множители. На практика разлагането на един многочлен на множители се извършва чрез различни комбинации на тези начини.

Ето кои основни начини за разлагане се предлагат в § 16:

(1) изнасяне на общ множител пред скоби;
(2) групиране на няколко члена или представянето на някои от членовете като суми;
(3) отделяне на точен квадрат;
(4) метод на неопределените коефициенти.

При доказване на теоремите за тъждественост на многочлени е използван методът на математическата индукция.

Глава Втора е посветена на дробни рационални изрази. Тук се включват видове рационални изрази и действия с тях.

В глава Трета „Ирационални изрази” освен преобразуване на изрази, съдържащи радикали, е дадено и изследване на ирационални функции.

Част Втора „Уравнения и неравенства"

Глава Четвърта е посветена на уравненията. Разглеждат се алгебрични, дробни и ирационални уравнения. Търсят се решения в полето на реалните числа. Изследва се въпроса за еквивалентност на две уравнения. Разглеждат се системи уравнения. При решаването на неравенствата са дадени много примери, в които неравенства са решени по чисто аритметичен начин. Обърнато е сериозно внимание на еквивалентността на неравенствата.

В глава Пета „Уравнения и неравенства от първа степен" отбелязваме, че при изследването на решенията на параметрични уравнения от първа степен е дадена и геометрична интерпретация. Застъпен е и въпросът за решаване на линейни уравнения в областта на целите числа. Разгледани са някои частни случаи на решаване на линейни системи уравнения. Също така е посочен метод за решаване на системи уравнения въз основа на понятието „еквивалентност".

Глава Шеста е озаглавена „Уравнения и неравенства от по-висока степен". Наред с квадратните уравнения се разглеждат и уравнения, свеждащи се към квадратни. Също така са решени и ирационални неравенства. В параграф 40 са посочени и някои забележителни неравенства.

Част Трета „Комбинаторика" включва както съединения без повторения, така и съединения с повторения.

Част Четвърта „Прогресии" включва аритметична, геометрична и безкрайно намаляща геометрична прогресия. Интерес представляват и аритметични редици от произволен ред.

5. Аритметични редици от произволен ред

Нека $f(x) = ax + b$, $f(0) = b \quad f(1) = a + b \quad f(2) = 2a + b$,...
$\div f(0),\ f(1),\ f(2),\ \ldots$ е аритметична прогресия с разлика a.
Като прави това наблюдение, Л. Илиев дава следната дефиниция:

Дефиниция 1. Ако $f(x) = ax + b$, $a \neq 0$, редицата $f(0)$, $f(1)$, $\ldots$, $f(n-1)$ наричаме аритметична прогресия.

Нека $f_k(x) = a_0x^k + a_1x^{k-1} + \ldots + a_{k-1}x$, $a \neq 0$ е произволен полином от степен k.

Дефиниция 2. Редицата от числа $f_k 0)$, $f_k(1)$, $\ldots$, $f_k(n-1)$ наричаме n-членна аритметична прогресия от ред k.

Нека a_1, a_2, ..., a_n е някаква редица от числа. Образуваме:

$$\begin{aligned} &\Delta a_1 = a_2 - a_1 \\ &\Delta a_2 = a_3 - a_2 \\ &\ldots\ldots\ldots\ldots \\ &\Delta a_{n-1} = a_n - a_{n-2} \end{aligned}$$

Числата от редицата Δa_1, Δa_2, ..., Δa_{n-1} наричаме първи разлики на редицата a_1, a_2, ..., a_n.

Аналогично числата от вида

$$\Delta^2 a_m = \Delta(\Delta a_m) = \Delta a_{m+1} - \Delta a_m$$

наричаме разлики от втори ред или втори разлики на редицата a_1, a_2, ..., a_n.

Разлика от k-ти ред или k-та разлика наричаме

$$\Delta^k a_m = \Delta^{k-1} a_{m+1} - \Delta^{k-1} a_m.$$

Теорема 1. *Редицата от първите разлики на една аритметична прогресия от k-ти ред е аритметична редица от $(k-1)$ ред.*

Следствие 1. *k-тите разлики на една аритметична редица от k-ти ред са постоянни.*

Следствие 2. *Разликите от $(k-1)$ ред на една аритметична редица от k-ти ред представят аритметична прогресия.*

Следствие 3.

$$\Delta a_1 + \Delta a_2 + \ldots + \Delta a_n = a_n + 1 - a_1. \qquad (*)$$

Приложения:

Пример 1. Ако $f_k(x) = (x+1)^{k+1}$, образуваме аритметичната редица от $(k+1)$-ви ред с $(n+1)$ члена

$$1^{k+1}, 2^{k+1}, \ldots, (n+1)^{k+1}.$$

Съгласно (*) имаме

$$\sum_{m=1}^{n} \Delta_m^{k+1} = (n+1)^{k+1} - 1.$$

Оттук се получава

$$(n+1)^{k+1} = C_{k+1}^1 S_k + C_{k+1}^2 S_{k-1} + \ldots + C_{k+1}^1 S_1 + n + 1.$$

Пример 2. Нека $\varphi(x)$ и $f(x)$ са две функции, така че $f(x) = \varphi(x+h) - \varphi(x)$ за фиксирано h и всяко x. Разглеждаме редицата $\varphi(x)$, $\varphi(x+h)$, ..., $\varphi(x+nh)$. Редицата от първите ѝ разлики е $f(x)$, $f(x+h$, ..., $f(x+\overline{n-1}h)$.

От (*) следва, че

$$f(x) + f(x+h) + \ldots + f(x+\overline{n-1}h) = \varphi(x+nh) - \varphi(x).$$

Да се намери сумата на

$$A = \sin x + \sin(x+h) + \ldots + \sin(x+\overline{n-1}h).$$

Полагаме $\varphi(x) = \cos\left(x - \dfrac{h}{2}\right)$, $f(x) = -2\sin x \sin\dfrac{h}{2}$. Лесно се стига до формулата

$$A = \frac{\sin\frac{nh}{2}\sin\left(x+\overline{n-1}\frac{h}{2}\right)}{\sin\frac{h}{2}}, \quad h \neq 2k\pi, \quad k \text{ – цяло.}$$

6. Заключение

Видно от гореизложеното, акад. Илиев има съществен принос за развитието на обучението по математика в България. От проф. Иван Тонов разбрах, че и досега той използва учебника „Алгебра“ в работата си със студенти. Дори и да не се чете в цялата пълнота или с осъвременени методики, 60 години след написването му, този учебник още носи полза на студенти и преподаватели.

Приложение 1 – Списък на научните публикации на акад. Илиев (Application 1 – List of Scientific Publications of Acad. L. Iliev)

1938

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1939

[2] Several theorems on the distribution of zeros of polynomials. Jubilee proc. of physical-mathematical society, part II, 1939, 60-64 (Bulgarian).

[3] Über die Nullstellen einiger Klassen von Polynomen. Tohoku Math. J., 45 (1939), 259-264 (German).

1940

[4] On the zeros of some classes of polynomials and entire functions. Sofia, 1940, 28 p. (Bulgarian). 1942

[5] Some elementary criteria for indecomposability of one polynomial of 3rd degree. J. Phys.-Math. Soc, 27 (1942), No 9-10, 294-298 (Bulgarian).

[6] Über trigonometrische Polynome mit monotoner Koeffizientenfolge. Annuaire Univ. Sofia, Fac. Sci., 38 (1942), No 1, 87-102 (Bulgarian, German summary).

1943

[7] Einige Probleme über nichtgleichmaessig gespannte ebene Membranen. Annuaire Univ. Sofia, Fac. Sci., 39 (1943), No 1, 251-286 (Bulgarian, German summary).

[8] Über das Gleichgewicht von elliptischen Membranen. Annuaire Univ. Sofia, Fac. Sci., 39 (1943), No 1, 409-426 (Bulgarian, German summary).

[9] Book of mathematical problems (with A. Mateev) Sofia, 1943 (Bulgarian).

[10] Über trigonometrische Polynome mit monotoner Koeffizientenfolge. Jahresber. Dtsch. Math.-Ver., 53 (1943), 12-23 (German).

1945

[11] Algebra. Textbook for 6th class of secondary school (with L. Chakalov and A. Mateev). 1st edition, Sofia, 1945 (Bulgarian).

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1946

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[17] Über ein Problem von D. Pompeiu. Annuaire Univ. Sofia, Fac. Sci., 42 (1946), No 1, 83-96. (Bulgarian, German summary).

[18] Book of mathematical problems with their solutions (with A. Mateev). Sofia, 1946 (Bulgarian).

1947

[19] On a boundary problem. Proceedings of National Assembly of Culture, 3 (1947) , No 1, 81-88 (Bulgarian).

[20] Über die in der Umgebung der Abzisse der absoluten Konvergenz einer Klasse Dirichletscher Reihen zugehörige singuläre Stellen. Annuaire Univ. Sofia, Fac. Sci., 43 (1947), No 1, 239-267 (Bulgarian).

[21] Book of mathematical problems with their solutions, part 1 – algebra, part 2 – geometry (with L. Chakalov and A. Mateev). 1st edition, Sofia, 1947 (Bulgarian).

1948

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1949

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1950

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Приложение 2 – Списък на публикациите на акад. Илиев в областта на обучението по математика (Application 2 – List of Acad. L. Iliev's Publications on Education in Mathematics)

Издадени учебници (Published Textbooks)

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PSEUDO-DIFFERENTIAL OPERATORS OF PRINCIPAL TYPE, SUBELLIPTIC ESTIMATES FOR SCALAR OPERATORS AND FOR THE $\bar{\partial}$-NEUMANN PROBLEM AND SOME APPLICATIONS

Peter R. Popivanov

Dedicated to the 100th anniversary
of my teacher Professor Ljubomir Iliev

Abstract

This survey talk deals with pseudodifferential operators of principal type including their local (non) solvability and subelliptic estimates. The main results are due to L. Hörmander, Y.V. Egorov, L. Nirenberg, F. Trèves, N. Lerner, N. Dencker. In the second part we discuss Catlin's results on subellipticity of the $\bar{\partial}$-Neumann problem for $(0,q), 1 \le q \le N-1$ forms in $\mathbb{C}^N$.

MSC 2010: 35S05, 35H20, 35A07, 32T27, 32W05, 32W25

Key Words and Phrases: pseudodifferential operator, operator of principal type, local solvability, regularity in Sobolev spaces, subelliptic estimates, $\bar{\partial}$-Neumann problem, loss and sharp loss of regularity

1. The starting point of the general theory of linear partial differential operators (PDO) and pseudodifferential operators (ψdo) is the famous Hans Lewy example

$$\frac{\partial u}{\partial x_1} + i\frac{\partial u}{\partial x_2} + i(x_1 + ix_2)\frac{\partial u}{\partial x_3} = f^{'}(x_3), \quad f \in C^1, \quad \text{near } 0. \tag{1}$$

Then a distribution solution $u \in D^{'}$ near 0 exists iff $f(x_3)$ is analytic near 0. Another interesting example is the Mizohata operator $P = \frac{\partial}{\partial x_1} + ib(x_1)\frac{\partial}{\partial x_2}$, where $b \in C^\infty$, $x_1 b(x_1) > 0$ for $x_1 \neq 0$. The operator P is locally nonsolvable in $D^{'}$. More precisely, put $B(x_1) = \int\limits_0^{x_1} b(s)ds$ and consider the Fourier integral operator (FIO)

$$(Kf)(x_2) = \int\limits_0^\infty \int\limits_{-\infty}^\infty \int\limits_{-\infty}^\infty \exp[i(x_2 - y + iB(s))]\ f(s,y)d\xi_2\ ds\ dy.$$

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Then the PDE $Pu = f$ is locally solvable at 0 in Schwartz distribution class iff $(Kf)(x_2)$ is real-analytic at x_2. Starting from those examples, we can sketch the theory of ψdo of principal type. Let p_m^0 be the corresponding principal symbol of P and $a = \Re(zp_m^0), b = \Im(zp_m^0)$ for $z \in \mathbb{C}$. Let $\gamma(t)$ be the nondegenerate bicharacteristic of a (i.e. it exists) passing through the characteristic point $\rho^0 = (x^0, \xi^0 \neq 0)$ of p_m^0. Denote $h(t) = b(x(t), \xi(t))$. Obviously, $a|_\gamma \equiv a(x^0, \xi^0)$.

Condition (ψ). For each $z \in \mathbb{C}^1 \setminus 0$ if $h(t_1) < 0$ for some t_1, then $h(t) \leq 0$ for $t \geq t_1$. It is sufficient to check (ψ) for only one value $z_0 \neq 0$. ((ψ) is due to Nirenberg-Trèves).

Theorem 1. ([11]) *Condition (ψ) is necessary for the local solvability of P at x^0.*

Condition (P). For each $z \in \mathbb{C}^1 \setminus 0$ the function b conserves its sign along the bicharacteristics γ of a.

Theorem 2. ([1]) *(P) is sufficient condition for the local solvability of P with sharp loss of regularity equal to 1.*

In 1994 Lerner [13] found counterexample to the sufficiency of (ψ) for local solvability with loss of one derivative in the Sobolev classes ($n \geq 3$).

Below we formulate the interesting result of Dencker of 2006.

Theorem 3. ([8]) *assume that the ψdo of principal type P satisfies (ψ). Then P is locally solvable at x^0 with loss of 2 derivatives in Sobolev classes. Thus,*

$$||u||_0 \leq C\left(||P^*u||_{2-m} + ||u||_{-1}\right), \ \forall u \in C_0^\infty(\omega), \tag{2}$$

$\omega \ni x^0$, where P^* is the L_2 adjoint operator to P.

Definition 1. A classical m-th order ψdo with principal symbol $p_m^0(x,\xi)$ is called subelliptic if there exists a constant $0 \leq \delta < 1$ such that for each compact set $K \subset\subset \Omega$ and for each real s one can find a positive constant $C_{K,s}$ for which one has:

$$||u||_{s-1} \leq C_{K,s}\left(||Pu||_{s-m+\delta} + u||_{s-1}\right), \ \forall u \in C_0^\infty(K). \tag{3}$$

Here H^s equipped with the norm $||.||_s$ is the classical Sobolev space.

One can see that $\delta = 0 \Rightarrow p_m^0 \neq 0$, while $0 < \delta < 1 \Rightarrow \nabla_{x,\xi}\, p_m^0(\rho^0) \neq 0$ for $p_m^0(\rho^0) = 0$. The number δ is called loss of regularity in comparison with the elliptic operators.

Theorem 4. ([9],[11]) *A classical ψdo with principal symbol p_m^0 is subelliptic with sharp loss of regularity $0 < \delta < 1$ iff for each $(x,\xi) \in T^* \setminus 0$ there exist no-negative integer $j(x,\xi)$ and $z \in \mathbb{C}^1 \setminus 0$ such that*

$$H^l_{\Re(zp^0)}\Im(zp^0)(x,\xi) = 0 \ \text{ for } \ 0 \leq l \leq j-1,$$

$$H^j_{\Re(zp^0)}\Im(zp^0)(x,\xi)\neq 0.$$

If j is odd then $H^j_{\Re(zp^0)}\Im(zp^0)>0$; if $p^0(x,\xi)\neq 0$ we put $j(x,\xi)=0$. Moreover, $\sup\limits_{(x,\xi)} j(x,\xi)=\dfrac{\delta}{1-\delta}$. The symbol H_a stands for the Hamiltonian vector filed of the real-valued function a.

Remark 1. Evidently, $\delta=\dfrac{k}{k+1}$ for some $k\in\mathbb{N}$.

Theorem 4 implies that if $\dfrac{k_0}{k_0+1}\le\delta<\dfrac{k_0+1}{k_0+2}$ for some $k_0\in\mathbb{N}$, then (3) holds with $\delta=\dfrac{k_0}{k_0+1}$. If condition (P) is satisfied, then k_0 is even, where $\delta=\dfrac{k_0}{k_0+1}$. Consequently, the losses of regularity form a sequence converging to 1.

Remark 2. Consider the linear DO with non-C^∞ smooth coefficients $P=\partial_{x_1}+iQ(x_1)\partial_{x_2}$, where

$$Q(x_1)=|x_1-a_1|^{\alpha_1}\cdots|x_1-a_{n-1}|^{\alpha_{n-1}}\,|x_1|^{\alpha_n}\,|x_1-b_1|^{\beta_1}\cdots|x_1-b_n|^{\beta_n},$$
$$a_1<\cdots a_{n-1}<0<b_1<\cdots<b_n,$$

and the powers α_j,β_j are either 0 or arbitrary numbers greater than 1. Put $M=\sum\alpha_j+\sum\beta_j$, $M>1$. Then one obtains local subelliptic estimate in L_2 with $\delta=\dfrac{M}{M+1}$. Therefore, the operator $R=\partial_{x_1}+|x_1|^\alpha\partial_{x_2}$ is subelliptic in L_2 with sharp loss of regularity $\delta=\dfrac{\alpha}{\alpha+1}$, $\alpha>1$. Certainly, δ is arbitrary number in $(1/2,1)$, i.e. in contrast with the previous case, the loss δ is "continuous".

2. An interesting application of the subelliptic estimates is that to the regularity of the $\bar\partial$-Neumann problem in pseudoconvex domains in $\mathbb{C}^N$. Thus, for $f\in C^1(\mathbb{C}^N)$ the operator

$$\bar\partial f=\sum_1^N\frac{\partial f}{\partial\bar z_j}\,d\bar z_j,\quad \frac{\partial f}{\partial\bar z_j}=\frac12\left(\frac{\partial f}{\partial x_j}+i\frac{\partial f}{\partial y_j}\right).$$

We shall work with $(0,q)$ forms and we shall use the operator $\bar\partial_q$. Certainly, if $q=1$ each smooth differential form is of the form

$$F=\sum_1^N\Phi_j\,d\bar z_j,\quad \bar\partial F=\sum\bar\partial\Phi_j\wedge d\bar z_j.$$

Let $\Omega\subset\mathbb{C}^N$ be a pseudoconvex domain with smooth boundary $\partial\Omega$. Then, for each $p\in\partial\Omega$ there exists a neighbourhood of p on which $\partial\Omega$ is given by

$r(z) = 0, dr(p) \neq 0$, r – smooth. $\partial\Omega$ is pseudoconvex if for each vector field $L = \sum_1^N a_j(z)\frac{\partial}{\partial z_j}$ tangential to $\partial\Omega$ (i.e. $\sum_1^N a_j(z)\frac{\partial r}{\partial z_j}(z) = 0$) we have that

$$\sum_{j,k=1}^{N} \frac{\partial^2 r}{\partial z_j\,\partial \bar{z}_k}\, a_j(z)\,\bar{a_k}(z) \geq 0.$$

J.J. Kohn studied the complex Laplacian

$$\Box_q = \bar{\partial}_{q-1}\,\bar{\partial}^*_{q-1} + \bar{\partial}^*_q\,\bar{\partial}_q : \; L^2_{(0,q)}(\Omega) \longrightarrow L^2_{(0,q)}(\Omega),$$

where $\bar{\partial}^*_q$ is the unbounded adjoint in L^2 operator of the non-bounded closed operator $\bar{\partial}_q$ in Hilbert space having dense domain in $L^2_{(0,q)}$, namely

$$D(\bar{\partial}_q) = \left\{ u \in L^2_{(0,q)}(\Omega),\ \bar{\partial}_q u \in L^2_{(0,q+1)}(\Omega) \right\}.$$

The problem of solving of the equation $\Box_q u = f,\quad u, f \in L^2_{(0,q)}(\Omega)$ is called $\bar{\partial}$-Neumann problem.

Definition 2. Let $p \in \partial\Omega$. The $\bar{\partial}$-Neumann problem verifies a subelliptic estimate on the compactly supported $(0, q)$ forms $D^{(0,q)}$ in $\bar{\Omega}$, $1 \leq q \leq N-1$, if there exist $\varepsilon > 0, C > 0$ and a neighbourhood $U \ni p$ such that

$$||u||_\varepsilon \leq C\left(||\bar{\partial}u||_0 + ||\bar{\partial}^* u||_0 + ||u||_0\right), \quad \forall u \in D^{(0,q)}(U \cap \bar{\Omega}),\ u \in D(\bar{\partial}^*_q). \tag{4}$$

Remark 3. The smooth (0,1) form $\sum_1^N \Phi_j d\bar{z}_j$ defined near p lies in $D(\bar{\partial}^*_1)$ iff $\sum \Phi_j \frac{\partial r}{\partial z_j} = 0$ on the set $r(z) = 0$. If u is compactly supported in Ω : $(\Box_q u, u) = ||\bar{\partial}u||^2 + ||\bar{\partial}^* u||^2$. The subelliptic theory for the $\bar{\partial}$-Neumann problem in $\mathbb{C}^2$ developed by J.J. Kohn in [12] is rather different from the same theory in $\mathbb{C}^N$, $N \geq 3$ developed by Catlin [2], [4]. D. Catlin proposed a geometrical N's condition for the validity of (4). It turns out that a crucial role in characterizing (4) is played by the notion "order of contact" at the point $p \in \partial\Omega$ among $\partial\Omega$ and the complex analytic varieties of dimension $1 \leq q \leq N-1$. The main novelty here is due to D'Angelo [6], [7], who considered not only the order of contact of $\partial\Omega$ with complex analytic manifolds of dimension q but with complex analytic varieties as well. The latter have singularities, of course. In the two above mentioned papers the so-called q-type d'Angelo $\Delta_q(p)$ was introduced and it was shown that $\Delta_q(p)$ is not semicontinuous function of p. D. Catlin (see [2],[3],[4],[5], mainly [3],[4]) proposed another version of the q-type D'Angelo, denoted by $D_q(p)$. *Here is the definition of* $D_q(p)$:

Let V_q be a complex analytic variety of dimension q and the point $p \in \partial\Omega$. Consider $S \bigcap V_q$, where $S \in G^{N-q+1}, 1 \leq q \leq N-1$, G^{N-q+1} being the complex Grassmanian manifold of dimension $N-q+1$. One can prove that in generic position $S \bigcap V_q$ consists of finitely many complex curves γ_k, $k = 1, ..., P$, such that $\max_k (\nu(r \circ \gamma_k)/\nu(\gamma_k))$ is a constant. The symbol $\nu(f)$ stands for the order of vanishing at 0 of the map f (in our case $\gamma_k(0) = p$). In generic position this number does not depend on the plane S and it is called generic order of contact between $\partial\Omega$ and V_q at the point p, denoted by $\tau(V_q, p) \geq 0$. Then $D_q(p) = \sup_{V_q} (\tau(V_q, p))$.

Theorem 5. ([3],[4]) *Consider the bounded pseudo convex, C^∞-smooth domain $\Omega \subset \mathbb{C}^N$. Then the $\bar{\partial}$-Neumann problem is subelliptic near the boundary point p for the (o, q) forms iff $D_q(p) < \infty$.*

Theorem 5 will be illustrated by the following example of D. Catlin [5], valid on the $(0, 1)$ forms in $\mathbb{C}^3$.

Example 1. Suppose that

$$r(z) = 2\Re z_3 + |z_2^{m_2}|^2 + |z_1^{m_1} - f(z_3)z_2|^2 + |z_2 g(z_3)|^2, \quad m_1 \geq 2, m_2 \geq 2, m_{1,2} \in \mathbb{N} \tag{5}$$

defines a pseudoconvex domain in $\mathbb{C}^3$, $f(0) = g(0) = 0$, f, g are analytic functions for $\Re z_3 < 0$. We can take:

a) $f(z_3) = z_3^p$, $g(z_3) = z_3^q$ with $\lambda = \frac{p}{q} \in (0, 1)$, $p, q \in \mathbb{N}$;

b) $f(\zeta) = \exp\left(\frac{-p}{\sqrt{-\zeta}}\right)$, $g(\zeta) = \exp\left(\frac{-q}{\sqrt{-\zeta}}\right)$;

f, q are flat at $\zeta = 0$ and analytic for $\Re\zeta < 0, 0 < p \leq q$. Evidently, $\log|f| = \lambda \log|g|$. Assume that $\Omega = \{r(z) < 0\}$, $0 \in \partial\Omega = \{r(z) = 0\}$.

In both cases the estimate (4) is valid near 0 with $\varepsilon \in \left[\frac{1}{2m_1 m_2}, \frac{1}{2m_1}\right]$ $\left(m_1 = 2, m_2 = 2 \to \varepsilon \in (0, \frac{1}{4})\right)$.

In case a) the optimal $\varepsilon > 0$ is rational number. In case b) for each $\varepsilon \in (0, \frac{1}{4})$ there exists a domain $\Omega \subset \mathbb{C}^3$ such that ε is the optimal loss of regularity in (4) for appropriate real positive p and q.

Moreover, if ε is rational, one can choose the domain Ω to be defined by (5), where $f = z_3^p, g = z_3^q$. There is a smooth pseudoconvex domain in $\mathbb{C}^3$ defined by (5) and such that (4) holds for all $0 \leq \varepsilon < \varepsilon_0$ but fails for $\varepsilon = \varepsilon_0$. J.J. Kohn proved (4) with $\varepsilon = 1/2$ for strictly pseudoconvex domains in 1963.

To complete this paper, we propose the following classical example. The operator of principal type:

a) with symbol $\xi_1 \pm i x_1^{2k}|\xi|$ is subelliptic with $\delta = \dfrac{2k}{2k+1}$;

b) with symbol $\xi_1 + i x_1^{2k+1}|\xi|$ is locally nonsolvable but hypoelliptic and subelliptic with $\delta = \dfrac{2k+1}{2k+2}$;

c) with symbol $\xi_1 - ix_1^{2k+1}|\xi|$ is neither subelliptic, nor hypoelliptic, but it is locally solvable having infinite dimensional kernel.

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THE INVARIANT SEQUENCES OF FEW DISCRETE TRANSFORMS

Predrag M. Rajković [1], Paul Barry [2], Nataša Savić [3]

Abstract

In this paper, we will expose the algorithms for constructing the invariant sequences for a few discrete transforms. Such sequences are their fixed points and therefore they are of the special concerning. Especially, we consider binomial, invert, Catalan and Hankel transform of a number sequence. Additionally, we include invariants of Riordan array. Indicated considerations are illustrated by examples.

MSC 2010: 44A55, 11B83, 35B06

Key Words and Phrases: sequences, discrete transforms, invariants, Hankel determinants, Riordan array

1. Introduction

An *invariant (fixed) point* of a transform is a point that is mapped to itself. It is such point that remains unchanged when transform is applied to it. Sometimes, there are a lot of such points which, in that case, form an *invariant set* of a transform. It is an interesting problem to determine such set.

The invariant sequences for inverse binomial transform were studied by Z.H. Sun in [7]. Here, we will discuss a few another transforms.

Let $\mathcal{S}$ be the linear space of real sequences, i.e.,

$$\mathcal{S} = \{\{a_n\} : \ a_n \in \mathbb{R} \quad (\forall n \in \mathbb{N}_0)\} \ .$$

A transform T on $\mathcal{S}$ is given by

$$T = \{T_n\}_{n\in\mathbb{N}_0} : \ \mathcal{S} \to \mathcal{S}$$

such that

$$T(a) = b \quad (a, b \in \mathcal{S}) : \quad T_n(a) = b_n \qquad (n \in \mathbb{N}_0).$$

Definition 1.1. A sequence e is *invariant* with respect to the transform T if

$$T(e) = e \ .$$

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Definition 1.2. A sequence f is *signed invariant* with respect to the transform T if

$$T_n(f) \in \{\pm f_n\} .$$

Especially, we say that f is *strictly signed invariant* if $T_n(f) = -f_n$ $(\forall n \in \mathbb{N})$.

Definition 1.3. The *ordinary* and the *exponential generating function* for a sequence $\{a_n\}$ are defined by

$$A(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad \mathcal{A}(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

2. Invariants of the binomial transform

The *binomial transform* of a sequence a is the sequence b given by

$$b = \mathbf{Bin}(a) : \; b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \qquad (n \in \mathbb{N}). \tag{2.1}$$

Theorem 2.1. *The sequences* $e = \{e_n\}$ *which are invariant of the binomial transform satisfy the recurrence relation*

$$e_n = \frac{-1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} e_k.$$

Example 2.1. Starting with $e_0 = 1$, we get that the sequence of Bernoulli numbers

$$\{\mathbf{B}_n\} = \{1, -1/2, 1/6, 0, -1/30, 0, 1/42, 0, -1/30, 0, 5/66, 0, -691/2730, \ldots\},$$

is invariant for the binomial transform.

The *inverse binomial transform* is defined by

$$b = \mathbf{Bin^{-1}}(a) : \; b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k. \tag{2.2}$$

Example 2.2. A few signed invariant sequences of the inverse binomial transform are

$$\{2^{-n}\}, \quad \{nF_{n-1}\},$$

where F_n is nth Fibonacci number (see [7]).

3. Invariants of the invert transform

The invert transform is given by

$$b = \mathbf{Invert}(a): \ b_n = \frac{1}{1-a_0}\Big(a_n + \sum_{k=1}^{n} b_{n-k}a_k\Big) \quad (n \in \mathbb{N}_0).$$

If $A(x)$ is the ordinary generating function of the sequence $\{a_n\}$, then the generating function of the transformed sequence $\{b_n\}$ is

$$B(x) = \frac{A(x)}{1-A(x)}.$$

Theorem 3.1. *The invert transform has only trivial invariant sequence.*

Proof. If $e = \{e_n\}$ is an invariant sequence for the invert transform, then it is valid

$$e_n = \frac{1}{1-e_0}\Big(e_n + \sum_{k=1}^{n} e_{n-k}e_k\Big) \qquad (n \in \mathbb{N}_0). \tag{3.3}$$

Hence

$$e_n = \frac{e_0}{1-e_0} \quad \Rightarrow \quad e_0 = 0.$$

Consequently, applying (3.3) for $n = 2$, we get $e_1 = 0$. By mathematical induction, we prove that this is the zero sequence. Really, for $n = 2m$, we have

$$e_{2m} = \frac{1}{1-e_0}\Big(e_{2m} + \sum_{k=1}^{2m} e_{2m-k}e_k\Big) \quad \Rightarrow \quad 2\sum_{k=1}^{m-1} e_{2m-k}e_k + e_m^2 = 0.$$

Since $e_0 = \cdots = e_{m-1} = 0$, we conclude that $e_m = 0$. □

4. Invariants of the Laguerre transform

The *Laguerre transform* of a sequence is defined by

$$b = \mathbf{Lag}(a): \ b_n = \sum_{k=0}^{n} \frac{n!}{k!}\binom{n}{k} a_k.$$

If $\mathcal{A}(x)$ is the exponential generating function of a sequence, the Laguerre transform of that sequence is the sequence whose exponential generating function is

$$\mathcal{B}(x) = \frac{1}{1-x}\mathcal{A}\Big(\frac{x}{1-x}\Big).$$

The *inverse Laguerre transform* of a sequence is defined by

$$b = \mathbf{Lag}^{-1}(a): \ b_n = \sum_{k=0}^{n} (-1)^{n-k}\frac{n!}{k!}\binom{n}{k} a_k.$$

Theorem 4.1. *A sequence $\{a_n\}$ is a signed inverse Laguerre invariant if and only if its exponential generating function $\mathcal{A}(x)$ satisfies the condition*

$$\mathcal{A}(x) = \frac{1}{1-x}\mathcal{A}\Big(\frac{x}{x-1}\Big).$$

Proof. By conditions, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_n \frac{x^n}{n!} &= \mathcal{A}(x) = \frac{1}{1-x}\mathcal{A}\Big(\frac{x}{x-1}\Big) = \frac{1}{1-x}\sum_{k=0}^{\infty}\frac{a_k}{k!}\Big(\frac{x}{x-1}\Big)^k \\ &= \sum_{k=0}^{\infty}\frac{a_k}{k!}x^k\frac{(-1)^k}{(1-x)^{k+1}} = \sum_{k=0}^{\infty}\frac{a_k}{k!}x^k\sum_{r=0}^{\infty}(-1)^k\binom{-1-k}{r}(-x)^r \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^n\frac{a_{n-k}}{(n-k)!}(-1)^k\binom{-1-n+k}{k}\right)x^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(-1)^k\frac{n!}{k!}\binom{n}{k}a_k\right)\frac{x^n}{n!}\ .\end{aligned}$$

Hence

$$a_n = \sum_{k=0}^{n}(-1)^k\frac{n!}{k!}\binom{n}{k}a_k = (-1)^n\Big(\mathbf{Lag^{-1}}(a)\Big)_n \qquad (n\in\mathbb{N}), \tag{4.4}$$

wherefrom the invariant property follows. □

Theorem 4.2. *A sequence $\{a_n\}$ is a signed inverse Laguerre invariant sequence if and only if $\{(2a_{n+1}/(n+1)-a_n)s\}$ $(s\in\mathbb{R})$ is the sequence of the same type.*

Proof. Suppose that $\{a_n\}$ is a signed inverse Laguerre invariant sequence. Let us consider a sequence defined by $b_n = r_{n+1}a_{n+1} - s_n a_n$. Now, it is

$$\mathcal{B}(x) = \sum_{n=0}^{\infty} b_n\frac{x^n}{n!} = \frac{1}{x}\sum_{n=0}^{\infty} r_{n+1}a_{n+1}\frac{x^{n+1}}{n!} - \sum_{n=0}^{\infty} s_n a_n\frac{x^n}{n!}.$$

We can easily recognize $\mathcal{A}(x)$ if we take $r_{n+1} = r/(n+1)$ and $s_n = s$, where r and s are two constants. Hence

$$\mathcal{B}(x) = \frac{r-sx}{x}\mathcal{A}(x) - \frac{a_0 r}{x}\ .$$

By a simple change $x\to x/(x-1)$, according to the Proposition **4.1**, we find

$$\mathcal{B}\left(\frac{x}{x-1}\right) = \frac{(r-s)x-r}{x}(1-x)\mathcal{A}(x) - a_0 r\frac{x-1}{x}\ .$$

If we are looking for an invariant sequence, there is no solution. But, if we take opposite sign, we get

$$-\left(\frac{r-sx}{x}\mathcal{A}(x)-\frac{a_0 r}{x}\right)=\frac{(r-s)x-r}{x}\mathcal{A}(x)+\frac{a_0 r}{x}\ ,$$

wherefrom we conclude that signed invariant sequence exists for $r = 2s$. □

Here is a practical method for the construction of such sequences. From (4.4), for $m \in \mathbb{N}$, we find

$$\sum_{k=0}^{2m-1}(-1)^k\frac{(2m)!}{k!}\binom{2m}{k}e_k=0,\quad e_{2m+1}=\frac{1}{2}\sum_{k=0}^{2m}(-1)^k\frac{(2m+1)!}{k!}\binom{2m+1}{k}e_k,$$

wherefrom

$$e_1=\frac{1}{2}a_0,\quad e_3=-\frac{3}{2}a_0+\frac{9}{2}a_2,\ldots\ .$$

By induction it can be proven that e_{2m+1} can be expressed by even indexed elements $e_0,\ e_2,\ldots,e_{2m}$. That is why we are free to choose those members.

Example 4.1. If we choose the initial subsequence $e_{2k} = F_k$ $(k \in \mathbb{N}_0)$ (F_k - the kth Fibonacci number), we find the signed inverse Laguerre-invariant sequence:

$$\{0,0,1,9/2,1,-275/2,2,49343/2,3,-25897293/2,5,15873708070,8,\ldots\}.$$

5. Invariants of the Catalan transform

The *Catalan transform* of a sequence is defined by

$$b=\mathbf{Cat}(a):\ b_n=\sum_{k=0}^{n}\frac{k}{2n-k}\binom{2n-k}{n-k}a_k\qquad(n\in\mathbb{N}).$$

The Catalan numbers, with general term C_n, are defined by

$$C_n=\frac{1}{n+1}\binom{2n}{n}\tag{5.5}$$

They have ordinary generating function

$$c(x)=\frac{1-\sqrt{1-4x}}{2x}.$$

If $\mathcal{A}(x)$ is the generating function of a sequence, the Catalan transform of that sequence is the sequence whose generating function is

$$\mathcal{B}(x)=\mathcal{A}(xc(x)).$$

Theorem 5.1. *A sequence e is the invariant sequence of Catalan transform if its elements satisfy the recurrence relation:*

$$e_{n+1} = \frac{-1}{n} \sum_{k=1}^{n-1} \frac{k}{2n+2-k} \binom{2n+2-k}{n+1-k} e_k,$$

with any inital value e_0.

Example 5.1. If we choose the initial value $e_0 = 1$, we find the sequence whose the first part is given by

$$\{1, -1/2, -5/6, 1/8, 7/60, 1/9, 5/56, 17/576, -539/6480, -197/900, \ldots\}.$$

6. Invariants of the Hankel transform

Hankel matrices are attached on a class of the structure matrices and they have a fundamental role in the different areas of mathematics and technics. They are often applied in numerical mathematics and theory of orthogonal polynomials (see [3], [4]). Vice versa, the orthogonal polynomials can be used for the computation of Hankel determinants in the closed form (see [1],[6]).

Definition 6.1. Let $a = \{a_0, a_1, a_2, \ldots\}$ be a sequence of reals. *Hankel matrix of the order* n, denoted by H_n, is the upper left $n \times n$ sub-matrix of H:

$$H_n = \left[a_{i+j}\right]_{0 \le i,j \le n}.$$

The determinant of matrix H_n is called *Hankel determinant of the order* n:

$$h_n = \det H_n.$$

Hankel transform of sequence $a = \{a_0, a_1, a_2, \ldots\}$ is the corresponding sequence of Hankel determinants $h = \{h_0, h_1, h_2, \ldots\}$.

Example 6.1. Hankel transform of Catalan sequence (5.5) is $\{1\}$.

Example 6.2. Hankel transform of the sequence of central binomial coefficients $\left\{\binom{2n}{n}\right\}_{n \in \mathbb{N}_0}$ is the sequence $h = \{2^n\}_{n \in \mathbb{N}_0}$.

It is known that the Hankel transform is invariant under the binomial and invert transform [5]. Here, we will consider quite different problem.

6.1. Invariant sequences of the Hankel transform

We will expose the algorithm for constructing nontrivial *Hankel–invariant* sequences, i.e. the sequences which satisfy

$$H(a) = a\ .$$

Let us start with a real number $h_0 = a_0 \neq 0$. Since

$$h_1 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix} = a_1 \quad \Leftrightarrow \quad a_0a_2 - a_1^2 = a_1 \quad \Leftrightarrow \quad a_2 = \frac{a_1(1+a_1)}{a_0}.$$

We choose $a_1 \notin \{-1, 0\}$, what provides $a_2 \neq 0$.

Further,

$$h_2 = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = a_2 \quad \Leftrightarrow \quad a_4 = \frac{1}{a_1}\Big(a_0a_3^2 - 2a_1a_2a_3 + a_2(1+a_2^2)\Big).$$

Here, we notice that a_4 is a quadratic function of a_3. To avoid $a_4 \neq 0$, we consider the equation

$$a_0a_3^2 - 2a_1a_2a_3 + a_2(1+a_2^2) = 0.$$

In order to get $a_4 \neq 0$, we choose

$$a_3 \notin \left\{0, \frac{a_1a_2 \pm \sqrt{(a_0+a_1a_2)a_2}}{a_0}\right\},$$

In general, we consider $a_n = H_n(a)$. Introducing the vector column by $c_{n,k} = [a_k\ a_{k+1}\ \cdots\ a_{k+n}]^T$, applying determinant expansion over the last row, we can write

$$h_n = a_{2n}h_{n-1} + \sum_{k=0}^{n-1} a_{n+k}(-1)^{n+k} \det_{\substack{0\le j\le n \\ j\neq k}} [c_{n-1,j}].$$

Hence

$$h_n = h_{n-1}a_{2n} - h_{n-2}a_{2n-1}^2 + p(a_0, a_1, \ldots, a_{2n-2})a_{2n-1} + q_1(a_0, a_1, \ldots, a_{2n-2}),$$

where p and q_1 are two polynomials of $a_0, a_1, \ldots, a_{2n-2}$. By invariant condition $h_k = a_k$ $(k = 0, 1, \ldots, n)$, we can write

$$a_{2n} = \frac{1}{a_{n-1}}\Big(a_{n-2}a_{2n-1}^2 - p(a_0, a_1, \ldots, a_{2n-2})a_{2n-1} - q(a_0, a_1, \ldots, a_{2n-2})\Big),$$

where $q = q_1 - a_n$. Choosing

$$a_{2n-1} \notin \left\{0,\ \frac{p \pm \sqrt{p^2 + 4qa_{n-2}}}{2a_{n-2}}\right\}, \tag{6.6}$$

we assure that $a_{2n} \neq 0$ for every $n \in \mathbb{N}$.

Remark 6.1. If we choose a priori the subsequence $\{a_{2n-1}\}$, the criteria (6.6) is hard for checking, so we are not able to guaranty that we can construct infinite Hankel-invarinat sequence in this way.

Example 6.3. Starting with initial values $f_0 = 1$ and $f_{2n-1} = F_n$ $(n \in \mathbb{N})$, we will get the Hankel-invariant sequence:

$$f = \{1, 1, 2, 1, 7, 2, \frac{111}{2}, 3, \frac{117533}{8}, 5, \frac{180219522060429}{224}, 8, \ldots\}.$$

Example 6.4. Starting with initial values $a_0 = 1$ and $a_{2n-1} = 1$ $(n \in \mathbb{N})$, we will get the Hankel-invariant sequence whose the first members are integers, but not all:

$$a = \{1, 1, 2, 1, 7, 1, 50, 1, 4592, 1, \frac{141243035489}{7}, \ldots\}.$$

Example 6.5. Taking $b_0 = 1$, $b_{2n-1} = 2^{-(2n-1)}$ $(n \in \mathbb{N})$, we will get the Hankel-invariant sequence whose the first members are:

$$b = \{1, \frac{1}{2}, \frac{3}{4}, \frac{1}{8}, \frac{35}{16}, \frac{1}{32}, \frac{1897}{192}, \ldots\}.$$

Remark 6.2. In the previous examples we have constructed a few Hankel-invariant sequences. So, we obtained

$$h_2(a) = 2, \qquad h_2(b) = \frac{3}{4}.$$

The matrix $H_2(a) + H_2(b)$ is of the Hankel type, but the sequence $a + b$ is not Hankel-invariant because of

$$h_2(a) + h_2(b) \neq h_2(a + b).$$

From the other hand, it is valid

$$\det(H_2(a) \cdot H_2(b)) = a_2 b_2,$$

but the matrix $H_2(a)H_2(b)$ is not of the Hankel type.

7. Riordan array and group

We shall use the notation a_n to denote the general term of the integer sequence $\{a_n\}_{n\geq 0}$ where $a_n \in \mathbb{Z}$. The ordinary generating function of the sequence a_n is the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad \text{with} \qquad a_n = [x^n]f(x), \tag{7.7}$$

where the operator $[x^n]$ extracts the coefficient of x^n.

Definition 7.1. If $f(x)$ is of the form

$$f(x) = \sum_{k=0}^{\infty} f_k\ x^k \quad (f_0 = 0,\ f_1 \neq 0),$$

then the *compositional inverse* of $f(x)$ is the power series

$$u = \sum_{k=0}^{\infty} u_k x^k = \bar{f}(x): \quad f(u) = x.$$

Obviously, then we have $f(\bar{f}(x)) = x$. The sequence $\{u_n\}$ with generating function $u = \bar{f}(x)$ we call the *series reversion* of the sequence with generating function $f(x)$.

The study of integer sequences often involves looking at transformations that send one integer sequence into another one.

Definition 7.2. For a pair of generating functions $g(x) = \sum_{k=0}^{\infty} g_k x^k$ and $f(x) = \sum_{k=0}^{\infty} f_k x^k$ where $f_0 = 0,\ f_1 \neq 0$, the *Riordan array* $\mathbf{R} = (g, f)$ is an infinite lower-triangular matrix whose k-th column is generated by $g(x)f(x)^k$, i.e.

$$\mathbf{R} = [r_{j,k}] \ : \quad r_{j,k} = [x^j]\big(g(x)f(x)^k\big) \qquad (j, k \in \mathbb{N}_0). \tag{7.8}$$

Example 7.1. The first columns are

$$r_{j,0} = g_j, \quad r_{j,1} = \sum_{i=0}^{j} g_{j-i} f_i, \quad r_{j,2} = \sum_{i=0}^{j} g_{j-i} \sum_{k=0}^{i} f_{i-k} f_k \quad (j \in \mathbb{N}_0).$$

Lemma 7.1. *For any fixed $k \in \mathbb{N}_0$, it is valid*

$$\sum_{i=0}^{\infty} r_{i,k} x^i = g(x)f(x)^k, \ \sum_{i=0}^{\infty} r_{k,i} x^i = \frac{g(x)}{1 - f(x)}, \ \sum_{i=0}^{\infty} r_{i,i} x^i = \frac{g(x)}{1 - x f(x)} \ . \tag{7.9}$$

Example 7.2. Putting $g_k = 0\ (k < 0)$, in more concrete case, we have

$$(g(x), x) = \mathbf{M} = [m_{j,k}] \ : \quad m_{j,k} = g_{j-k} \qquad (j \geq k)(\in \mathbb{N}_0). \tag{7.10}$$

Definition 7.3. The *Riordan group* is a set $\mathcal{R}$ of infinite lower-triangular matrices, where each matrix is defined by a pair of generating functions.

The group law is given by

$$(g,f)*(h,l)=\big(g\cdot(h\circ f),l\circ f\big), \tag{7.11}$$

where $\circ$ denotes the composition of functions.

Lemma 7.2. *The identity and the inverse in $\mathcal{R}$ are given by*

$$I=(1,x),\qquad (g,f)^{-1}=\left(\frac{1}{g\circ\hat f}\,,\hat f\right), \tag{7.12}$$

where $\hat f$ is the compositional inverse of f.

Lemma 7.3. *If $\mathbf{R}=(g,f)$, and $\mathbf{a}=(a_0,a_1,\dots)^T$ is an integer sequence with ordinary generating function $\mathcal{A}(x)$, then the sequence $\mathbf{Ra}$ has ordinary generating function $g(x)\mathcal{R}\big(f(x)\big)$.*

Example 7.3. The binomial matrix $\mathbf{B}$ is the following Roirdan array

$$\mathbf{B}=\Big[\binom{n}{k}\Big]_{n,k\ge 0}=\Big(\frac{1}{1-x}\,,\frac{x}{1-x}\Big).$$

More generally, $\mathbf{B}^m$ is the element

$$\mathbf{B}^m=\Big(\frac{1}{1-mx}\,,\frac{x}{1-mx}\Big)=\Big[\binom{n}{k}m^{n-k}\Big]_{n,k\ge 0}\,.$$

The inverse $\mathbf{B}^{-m}$ of $\mathbf{B}^m$ is given by

$$\Big(\frac{1}{1+mx}\,,\frac{x}{1+mx}\Big).$$

7.1. Invariants of the Riordan group

Definition 7.4. The *a-shifting transform* of a lower triangular matrix $\mathbf{M}$ is the matrix $\tilde{\mathbf{M}}_a=[\mu_{j,k}]$ such that

$$\mu_{j+1,j}=-a,\quad \mu_{j,k}=-m_{j,k+1}\ (j\ge k+2)\qquad (j,k\in\mathbb{N}_0).$$

Example 7.4. For $n=3$, we have

$$M=\begin{bmatrix} m_{00} & 0 & 0 & 0\\ m_{10} & m_{11} & 0 & 0\\ m_{20} & m_{21} & m_{22} & 0\\ m_{30} & m_{31} & m_{32} & m_{33}\end{bmatrix}\quad\Rightarrow\quad \tilde{M}_a=\begin{bmatrix} m_{00} & 0 & 0 & 0\\ -a & m_{11} & 0 & 0\\ -m_{10} & -a & m_{22} & 0\\ -m_{20} & -m_{21} & -a & m_{33}\end{bmatrix}$$

Theorem 7.1. *If $\mathbf{M}=\big(g(x),x\big)$, then $\tilde{\mathbf{M}}_a=\big(1-(a-1)x-xg(x),x\big)$.*

Proof. The first column of $\tilde{\mathbf{M}}_a$ has the following generating function:

$$\tilde{g}(x) = \sum_{j=0}^{\infty} \mu_{j,0} x^j = 1 - (a-1)x - xg(x).$$

In general, to the kth column, we can join the generating function

$$G_k(x) = \sum_{j=0}^{\infty} \mu_{j,k} x^j = x^k \tilde{g}(x) \qquad (k = 0, 1, \ldots).$$

□

Corollary 7.1. *The function*

$$g(x) = \frac{1-(a-1)x}{1+x}$$

has the property

$$\tilde{\mathbf{M}}_a = \mathbf{M}.$$

Corollary 7.2. *If the function $g(x)$ has the property*

$$g(x)\big(1 - (a-1)x - xg(x)\big) = 1,$$

then

$$\big(g(x), x\big)^{-1} = \big(g(x), x\big).$$

Definition 7.5. The (a,b)-*shifting transform* of a lower triangular matrix $\mathbf{M}$ is the matrix $\tilde{\mathbf{M}}_{a,b} = [\mu_{j,k}]$ such that

$$\mu_{j+1,j} = -a, \quad \mu_{j+2,j} = -b, \quad \mu_{j+3,k} = -m_{j+1,k}\ (k \le j) \qquad (j,k \in \mathbb{N}_0).$$

Theorem 7.2. *If $\mathbf{M} = \big(g(x), x\big)$, then*

$$\tilde{\mathbf{M}}_{a,b} = \big(1 - ax - (b-1)x^2 - x^2 g(x), x\big).$$

Corollary 7.3. *If the function $g(x)$ has the property*

$$g(x)\big(1 - ax - (b-1)x^2 - x^2 g(x)\big) = 1,$$

then

$$\big(g(x), x\big)^{-1} = \big(g(x), x\big).$$

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ON A GENERALIZATION OF CONTIGUOUS WATSON'S THEOREM FOR THE SERIES ${}_3F_2(1)$

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Abstract

The aim of this research paper is to establish explicit expressions of

$${}_3F_2\left[\begin{array}{ccc} a, & b, & c, \\ & & ; \quad 1 \\ \frac{1}{2}(a+b+i+1), & 2c+j & \end{array}\right]$$

for $i = 0, \pm1, \pm2, \ldots, \pm5;\ j = 0, \pm1, \pm2$.

For $i = j = 0$, we get the well known Watson's theorem for the series ${}_3F_2(1)$. Several new and known results are obtained as special cases of our main findings.

MSC 2010: 33C05, 33C20, 33C70

Key Words and Phrases: generalized hypergeometric functions; Watson's transformation theorem

1. Introduction

The generalized hypergeometric function with p numerator and q denominator parameters is defined by [8, P. 73, Eq. 2]

$${}_pF_q\left[\begin{array}{ccc} \alpha_1, & \ldots, & \alpha_p \\ & & ; \quad z \\ \beta_1, & \ldots, & \beta_q, \end{array}\right] = {}_pF_q\left[\ \alpha_1, \ \ldots, \ \alpha_p; \ \beta_1, \ \ldots, \ \beta_q; \ z\ \right]$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}\,, \tag{1.1}$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number α, by

$$(\alpha)_n = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+n-1); & n \in \mathbb{N} = \{1, 2, \ldots\} \\ 1; & n = 0 \end{cases}. \tag{1.2}$$

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Using the fundamental property $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written in the form

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},\ (n\in\mathbb{N}\cup\{0\}), \tag{1.3}$$

where $\Gamma()$ is the well known Gamma function.

It is interesting to mention here that whenever the generalized hypergeometric functions reduce to gamma functions, the results are very important from application point of view. Only a few summation theorems for the series ${}_3F_2$ with unit argument are available in the literature.

We start with the classical Watson's summation theorem for the generalized hypergeometric series ${}_3F_2$, [2, P. 16, Eq. 1], viz.

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}a, & b, & c\\ \frac{1}{2}(a+b+1), & 2c & \end{matrix};\ 1\right]\\ &=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)},\end{aligned} \tag{1.4}$$

provided $\mathrm{Re}(2c-a-b)>-1$.

The proof of this theorem when one of the parameters a or b is a negative integer was given by Watson in [12], and subsequently was established more generally in non-terminating case by Whipple in [13].

The standard proof of the general case given in [1, p. 149] and [10, p. 54] relies on the following transformation due to Thomae [11] viz.

$${}_3F_2\left[\begin{matrix}a, & b, & c\\ d, & e & \end{matrix};\ 1\right]=\frac{\Gamma(d)\Gamma(e)\Gamma(s)}{\Gamma(a)\Gamma(b+s)\Gamma(c+s)}\,{}_3F_2\left[\begin{matrix}d-a, & e-a, & s\\ b+s, & c+s & \end{matrix};\ 1\right] \tag{1.5}$$

where $s=d+e-a-b-c$ is the parametric excess, combined with Dixon's theorem for the evaluation of the sum on the right when $d=\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b$ and $e=2c$. An alternative and more involved proof [7, p. 363] exploits the quadratic transformations for the Gauss's hypergeometric functions. A third proof, due to Bhatt in [3] exploits a known relation between the F_2 and F_4 Appell's functions combined with a comparison of the coefficients in their series expansions.

Very recently Rathie and Paries [9] gave a very simple proof of Watson's theorem that relies on the well known Gauss's summation theorems for the ${}_2F_1$ function, namely [1, p. 556, 557]

$${}_2F_1\left[\begin{matrix}a, & b,\\ c & \end{matrix};\ 1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.6}$$

provided $\mathrm{Re}(c-a-b)>0$, and

$$ {}_2F_1\left[\begin{matrix} a, & b, \\ \frac{1}{2}(a+b+1) & \end{matrix} ; \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})}. \tag{1.7} $$

In 1987, Lavoie [5] obtained the following two summation formulas closely related to (1.4), viz.

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+1 & \end{matrix} ; 1\right] $$
$$ = \frac{2^{a+b-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} $$
$$ \times\left\{\frac{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}\right\}, \tag{1.8} $$

provided $\mathrm{Re}(2c-a-b)>-3$, and

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c-1 & \end{matrix} ; 1\right] $$
$$ = \frac{2^{a+b-2}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} $$
$$ \times\left\{\frac{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}{\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b-\frac{1}{2}\right)} - \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{1}{2}a\right)\Gamma\left(c-\frac{1}{2}b\right)}\right\}, \tag{1.9} $$

provided $\mathrm{Re}(2c-a-b)>1$.

Rathie and Paris [9] have given an interesting proof of the classical summation theorem (1.4).

In 1992, Lavoie, Grondin and Rathie [6] obtained explicit expression of

$$ {}_3F_2\left[\begin{matrix} a, & b, & c, \\ \frac{1}{2}(a+b+i+1), & 2c+j & \end{matrix} ; 1\right] \tag{1.10} $$

for $i,j=0,\pm1,\pm2$.

Very recently, Kim and Rathie [4] have obtained the result (1.10) by employing the same technique developed in [9] for $i=0,\pm1,\pm2,\dots,\pm5$ and $j=0$.

In this paper, we aim at establishing the results (1.10) for $i=0,\pm1,\pm2,\dots,\pm5$; $j=\pm1,\pm2$. Several known as well as new results have been mentioned. The results derived in this paper are simple, interesting, easily established and may be useful.

In order to derive our results, we shall use the following generalization of Watson's theorem due to Kim and Rathie [4]

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ & & & ; & 1 \\ \frac{1}{2}(a+b+i+1), & 2c & \end{matrix}\right] \\
&\quad = \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\left|i\right|+\frac{1}{2}\right)} \\
&\quad \times \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_m\left(\frac{1}{2}a+\frac{1}{2}\right)_m\left(\frac{1}{2}b\right)_m\left(\frac{1}{2}b+\frac{1}{2}\right)_m}{\left(c+\frac{1}{2}\right)_m m!}\ \alpha_i(a,b) \qquad (1.11)
\end{aligned}
$$

for $i = 0, \pm 1, \pm 2, \ldots, \pm 5$.

As usual, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. Also $\alpha_i(a,b)$ is given by

$$
\begin{aligned}
\alpha_i(a,b) &= \frac{A_i}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)\left(\frac{1}{2}a+\frac{1}{2}\right)_m\left(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\left[\frac{1+i}{2}\right]\right)_m} \\
&+ \frac{B_i}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)\left(\frac{1}{2}a\right)_m\left(\frac{1}{2}b+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)_m}. \qquad (1.12)
\end{aligned}
$$

The coefficients A_i and B_i are given in the following table:

i	A_i	B_i
-5	$(b+a+4m-4)^2-\frac{1}{4}(b-a-4)^2$ $-\frac{1}{2}(b+a+4m-4)(b-a-4)$ $+4(b+a+4m-4)-\frac{7}{2}(b-a-4)$	$(b+a+4m-4)^2-\frac{1}{4}(b-a-4)^2$ $+\frac{1}{2}(b+a+4m-4)(b-a-4)$ $+8(b+a+4m-4)-\frac{1}{2}(b-a-4)$
-4	$\frac{1}{2}(b+a+4m-3)(b+a+4m+1)$ $-\frac{1}{4}(b-a-3)(b-a+3)$	$2(b+a+4m-1)$
-3	$\frac{1}{2}(3a+b+8m-2)$	$\frac{1}{2}(3b+a+8m-2)$
-2	$\frac{1}{2}(b+a+4m-1)$	2
-1	1	1
0	1	0
1	-1	1
2	$\frac{1}{2}(b+a+4m-1)$	-2
3	$-\frac{1}{2}(3a+b+8m-2)$	$\frac{1}{2}(3b+a+8m-2)$
4	$\frac{1}{2}(b+a+4m+1)(b+a+4m-3)$ $-\frac{1}{4}(b-a+3)(b-a-3)$	$-2(b+a+4m-1)$
5	$-(b+a+4m+6)^2+\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4m+6)$ $+11(a+b+4m+6)-\frac{13}{2}(b-a+6)-20$	$(b+a+4m+6)^2-\frac{1}{4}(b-a+6)^2$ $+\frac{1}{2}(b-a+6)(b+a+4m+6)$ $-17(a+b+4m+6)-\frac{1}{2}(b-a+6)+62$

TABLE 1. Table for the coefficients A_i and B_i for $i=0,\pm1,\pm2,\ldots,\pm5$

2. Main Summation Formulas

In this section the following four general summation formulas for the series ${}_3F_2$ will be established.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+1 & \end{matrix} ; 1\right] \\
&= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)} \\
&\times \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_m\left(\frac{1}{2}a+\frac{1}{2}\right)_m\left(\frac{1}{2}b\right)_m\left(\frac{1}{2}b+\frac{1}{2}\right)_m}{\left(c+\frac{1}{2}\right)_m m!}\ \alpha_i(a,b) \\
&- \frac{ab}{(2c+1)(a+b+i+1)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)} \\
&\times \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_m\left(\frac{1}{2}a+1\right)_m\left(\frac{1}{2}b+\frac{1}{2}\right)_m\left(\frac{1}{2}b+1\right)_m}{\left(c+\frac{3}{2}\right)_m m!}\ \beta_i(a,b), \qquad (2.13)
\end{aligned}
$$

$i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$, where $\alpha_i(a,b)$ is the same as defined in (1.12) and $\beta_i(a,b)$ is given by

$$
\begin{aligned}
\beta_i(a,b) &= \frac{C_i}{\Gamma\left(\frac{1}{2}a+1\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}i+1-\left[\frac{1+i}{2}\right]\right)\left(\frac{1}{2}a+1\right)_m\left(\frac{1}{2}b+\frac{1}{2}i+1-\left[\frac{1+i}{2}\right]\right)_m} \\
&+ \frac{D_i}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\left[\frac{i}{2}\right]\right)\left(\frac{1}{2}a+\frac{1}{2}\right)_m\left(\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}-\left[\frac{i}{2}\right]\right)_m}. \qquad (2.14)
\end{aligned}
$$

The coefficients C_i and D_i can be obtained from the tables of A_i and B_i by simply changing a to $a+1$, and b to $b+1$ respectively.

$$
{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c-1 & \end{matrix} ; 1\right]
$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)}$$
$$\times \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_m \left(\frac{1}{2}a+\frac{1}{2}\right)_m \left(\frac{1}{2}b\right)_m \left(\frac{1}{2}b+\frac{1}{2}\right)_m}{\left(c-\frac{1}{2}\right)_m m!} \alpha_i(a,b)$$
$$+ \frac{ab}{(2c-1)(a+b+i+1)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)}$$
$$\times \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_m \left(\frac{1}{2}a+1\right)_m \left(\frac{1}{2}b+\frac{1}{2}\right)_m \left(\frac{1}{2}b+1\right)_m}{\left(c+\frac{1}{2}\right)_m m!} \beta_i\ (a,b) \tag{2.15}$$

for $i = 0, \pm 1, \pm 2, \ldots, \pm 5$. Here $\alpha_i(a,b)$, $\beta_i(a,b)$, A_i, B_i, C_i and D_i are same as defined as before.

$${}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+2 & \end{matrix} ;\ 1\right]$$
$$= \frac{2c+1}{c+1} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)}$$
$$\times \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_m \left(\frac{1}{2}a+\frac{1}{2}\right)_m \left(\frac{1}{2}b\right)_m \left(\frac{1}{2}b+\frac{1}{2}\right)_m}{\left(c+\frac{1}{2}\right)_m m!} \alpha_i(a,b)$$
$$- \frac{ab}{(i+1)(a+b+i+1)} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)}$$
$$\times \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_m \left(\frac{1}{2}a+1\right)_m \left(\frac{1}{2}b+\frac{1}{2}\right)_m \left(\frac{1}{2}b+1\right)_m}{\left(c+\frac{3}{2}\right)_m m!} \beta_i(a,b)$$
$$- \frac{c}{c+1} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)}$$
$$\times \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}a\right)_m \left(\frac{1}{2}a+\frac{1}{2}\right)_m \left(\frac{1}{2}b\right)_m \left(\frac{1}{2}b+\frac{1}{2}\right)_m}{\left(c+\frac{3}{2}\right)_m m!}\ \alpha_i(a,b). \tag{2.16}$$

Here, as usual, $\alpha_i(a,b)$, $\beta_i(a,b)$, A_i, B_i, C_i and D_i are same as defined as before.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c-2 & \end{matrix} ; 1\right] \\
&= \frac{2c-3}{c-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)} \\
&\times \sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}a\right)_m\left(\frac{1}{2}a+\frac{1}{2}\right)_m\left(\frac{1}{2}b\right)_m\left(\frac{1}{2}b+\frac{1}{2}\right)_m}{\left(c-\frac{3}{2}\right)_m m!}\ \alpha_i(a,b) \\
&+ \frac{ab}{(c-1)(a+b+i+1)}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{3}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)} \\
&\times \sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}a+\frac{1}{2}\right)_m\left(\frac{1}{2}a+1\right)_m\left(\frac{1}{2}b+\frac{1}{2}\right)_m\left(\frac{1}{2}b+1\right)_m}{\left(c-\frac{1}{2}\right)_m m!}\beta_i(a,b) \\
&- \frac{c-2}{c-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a-\frac{1}{2}b-\frac{1}{2}i+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}|i|+\frac{1}{2}\right)} \\
&\times \sum_{m=0}^{\infty}\frac{\left(\frac{1}{2}a\right)_m\left(\frac{1}{2}a+\frac{1}{2}\right)_m\left(\frac{1}{2}b\right)_m\left(\frac{1}{2}b+\frac{1}{2}\right)_m}{\left(c-\frac{1}{2}\right)_m m!}\ \alpha_i(a,b). \qquad (2.17)
\end{aligned}
$$

Here, as usual, $\alpha_i(a,b)$, $\beta_i(a,b)$, A_i, B_i, C_i and D_i are same as defined as before.

3. Derivations

In order to derive (2.13), we proceed as follows. It is just a simple exercise to prove the following relation involving three ${}_3F_2$ for $i = 0, \pm 1, \ldots, \pm 5$.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+1 & \end{matrix} ; 1\right] \\
&= {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c & \end{matrix} ; 1\right] \\
&- \frac{ab}{(2c+1)(a+b+i+1)}\,{}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+3+i), & 2c+2 & \end{matrix} ; 1\right]. \qquad (3.18)
\end{aligned}
$$

Now, it is easy to see that the two ${}_3F_2$ appearing in the right-hand side can be evaluated with the help of the generalized Watson's theorem (1.12) and after little simplification, we easily arrive at the right-hand side of (2.13). This completes the proof of (2.13).

In exactly the same manner, the result (2.15) can also be established by using the following relation involving three ${}_3F_2$, viz.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c-1 & \end{matrix}\ ;\ 1\right] \\
&= {}_3F_2\left[\begin{matrix} a, & b, & c-1 \\ \frac{1}{2}(a+b+i+1), & 2c-2 & \end{matrix}\ ;\ 1\right] \\
&+ \frac{ab}{(2c-1)(a+b+i+1)}\,{}_3F_2\left[\begin{matrix} a+1, & b+1, & c \\ \frac{1}{2}(a+b+3+i), & 2c & \end{matrix}\ ;\ 1\right] \qquad (3.19)
\end{aligned}
$$

for $i = 0, \pm 1, \ldots, \pm 5$.

In order to derive the result (3.19), we use the following relation, viz.

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+2 & \end{matrix}\ ;\ 1\right] \\
&= \frac{2c+1}{c+1}\,{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+1 & \end{matrix}\ ;\ 1\right] \\
&- \frac{c}{c+1}\,{}_3F_2\left[\begin{matrix} a, & b, & c+1 \\ \frac{1}{2}(a+b+i+1), & 2c+2 & \end{matrix}\ ;\ 1\right] \qquad (3.20)
\end{aligned}
$$

for $i = 0, \pm 1, \ldots, \pm 5$.

With the help of the relation (3.18), the formula (3.20) takes the following form

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+2 & \end{matrix}\ ;\ 1\right] \\
&= \frac{2c+1}{c+1}\,{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c & \end{matrix}\ ;\ 1\right] \\
&- \frac{ab}{(c+1)(a+b+i+1)}\,{}_3F_2\left[\begin{matrix} a+1, & b+1, & c+1 \\ \frac{1}{2}(a+b+i+3), & 2c+2 & \end{matrix}\ ;\ 1\right] \\
&- \frac{c}{c+1}\,{}_3F_2\left[\begin{matrix} a, & b, & c+1 \\ \frac{1}{2}(a+b+i+1), & 2c+2 & \end{matrix}\ ;\ 1\right] \qquad (3.21)
\end{aligned}
$$

for $i = 0, \pm 1, \ldots, \pm 5$.

Now, it is easy to see that all the three ${}_3F_2$ appearing in the right-hand side of (3.21) can now be evaluated with the help of (1.12) and after little simplification, we arrive at the right-hand side of (2.16).

In exactly the same manner, the result (2.17) can be established with the help of the following relation

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c-2 & \end{matrix}\ ;\ 1\right] \\
&= \frac{2c-3}{c-1}\,{}_3F_2\left[\begin{matrix} a, & b, & c-2 \\ \frac{1}{2}(a+b+i+1), & 2c-4 & \end{matrix}\ ;\ 1\right] \\
&+ \frac{ab}{(c-1)(a+b+i+1)}\,{}_3F_2\left[\begin{matrix} a+1, & b+1, & c-1 \\ \frac{1}{2}(a+b+i+3), & 2c-2 & \end{matrix}\ ;\ 1\right] \\
&- \frac{c-2}{c-1}\,{}_3F_2\left[\begin{matrix} a, & b, & c-1 \\ \frac{1}{2}(a+b+i+1), & 2c-2 & \end{matrix}\ ;\ 1\right] \qquad (3.22)
\end{aligned}
$$

for $i = 0, \pm 1, \ldots, \pm 5$.

4. Special Cases

In (2.13) to (2.17), if we take $i = 0, \pm 1, \pm 2$, we get known results recorded in [6] obtained by other means.

Concluding Remark

In this paper we have in all obtained 44 summation formulas closely related to classical Watson's theorem on the sum of ${}_3F_2$ out of which 20 results are known and obtained in [4]. Rest 24 results are believed to be new.

We remark by concluding this paper that the applications of newly established formulas are under investigations and will form a part of the subsequent paper in this direction.

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HOLOMORPHIC CLIFFORDIAN FUNCTIONS AS A NATURAL EXTENSION OF MONOGENIC AND HYPERMONOGENIC FUNCTIONS

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Abstract

This is an expository paper which aim is to defend the notion of holomorphic Cliffordian functions [11], [12]. The way to argue for is to exhibit non-trivial applications. Some of them were known earlier [13], [14]. The most recent interesting development was the contribution [4] in which clearly one see how holomorphic Cliffordian functions are able to solve a problem which was unsatisfactory solved before.

The organization of this paper is almost the same as in [20]. In section 1. we recall the fundamental definitions and properties of Clifford algebras, especially those of anti-euclidean type. Sections 2, 3 and 4 are devoted to a brief overview of the different theories of "hypercomplex" variables, namely the classical theory of monogenic functions, the holomorphic Cliffordian functions, and finally the hypermonogenic functions, respectively. A careful analysis of the connections between those three classes of functions argues for the holomorphic Cliffordian ones. This is a set which is endowed with many function theoretical tools that are also offered for complex holomorphic functions. Basically, they were introduced in order to contain the functions $x \mapsto x^n (n \in \mathbf{N}, x$ a paravector) and to be stable under any directional derivation. Consequently, they form a class of functions containing the two others.

They are two new sections. In Section 5, we deal with the problem how to construct analogues of the Weierstrass ζ and $\wp$ functions, as well as the Jacobi *cn*, *dn* and *sn*. We end with Section 6, deeply inspired of [4], where we are illustrating a new class of Clifford valued automorphic forms on arithmetic subgroups of the Ahlfors-Vahlen group.

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1. Clifford Algebras

Denote by $\mathbf{R}^{p+q}$ a real vector space of dimension $d = p+q$ provided with a non-degenerate quadratic form Q of signature (p, q).

Main definition: The Clifford algebra, we will denote by $\mathbf{R}_{p,q}$, of the quadratic form Q on the vector space $\mathbf{R}^{p+q}$ is an associative algebra over $\mathbf{R}$, generated by $\mathbf{R}^{p+q}$, with unit 1, if it contains $\mathbf{R}$ and $\mathbf{R}^{p+q}$ as distinct subspaces and
(1) $\forall v \in \mathbf{R}^{p+q}, v^2 = Q(v)$,
(2) the algebra is not generated by any proper subspace of $\mathbf{R}^{p+q}$.

Actually, if we consider the Clifford algebra $\mathbf{R}_{p,q}$ as a vector space, it has the splitting: $\mathbf{R}_{p,q} = \mathbf{R}^0_{p,q} \oplus \mathbf{R}^1_{p,q} \oplus ... \oplus \mathbf{R}^k_{p,q} \oplus ... \oplus \mathbf{R}^d_{p,q}$, where $\mathbf{R}^0_{p,q} = \mathbf{R}$ are the scalars, $\mathbf{R}^1_{p,q} = \mathbf{R}^{p+q}$ is the vector space, $\mathbf{R}^2_{p,q}$ is the vector space of the so-called bivectors corresponding to the planes in $\mathbf{R}^{p+q}$, and so on. Finally, $\mathbf{R}^d_{p,q}$ contains what we call the pseudoscalars. Moreover,

$$dim_{\mathbf{R}} \mathbf{R}^k_{p,q} = C^k_d, \quad dim_{\mathbf{R}} \mathbf{R}_{p,q} = 2^d.$$

Now, set $e_0 = 1$ as basis of $\mathbf{R}^0_{p,q} = \mathbf{R}$ and suppose $\{e_1, e_2, ..., e_d\}$ be an orthonormal basis for $\mathbf{R}^1_{p,q} = \mathbf{R}^{p+q}$. Thus, the corresponding vector spaces of the splitting will be provided with respective basis $\{e_0 = 1\}, \{e_1, e_2, ..., e_d\}, \{e_{ij} = e_i e_j, 1 \leq i < j \leq d\}, ..., \{e_{i_1...i_k} = e_{i_1} e_{i_2} ... e_{i_k}, 1 \leq i_1 < i_2 < ... < i_k \leq d\}, ..., \{e_{12...d} = e_1 e_2 ... e_d\}$, and the algebra will obey to the laws:

$$e_i^2 = 1, i = 1, ..., p, e_i^2 = -1, i = p+1, ..., d = p+q, e_i e_j = -e_j e_i, i \neq j.$$

This allows us to write down any Clifford number $a \in \mathbf{R}_{p,q}$ as a sum of its scalar part $< a >_0$, its vector part $< a >_1 \in \mathbf{R}^1_{p,q}$, its bivector part $< a >_2 \in \mathbf{R}^2_{p,q}$, up to its pseudoscalar part $< a >_d \in \mathbf{R}^d_{p,q}$, namely

$$a = < a >_0 + < a >_1 + ... + < a >_d,$$

where $< a >_k = \sum_{|J|=k} a_J e_J$, with $J = (j_1, ..., j_k)$ is a strictly increasing multi-index of lengh k and $e_J = e_{j_1} e_{j_2} ... e_{j_k}$, while $a_J \in \mathbf{R}$.

Some examples: The Clifford algebra $\mathbf{R}_{0,1}$ can be identified with the complex numbers $\mathbf{C}$. The algebra $\mathbf{R}_{0,2}$ is nothing else than the set of quaternions $\mathbf{H}$ if we identify $e_1 = i, e_2 = j, e_{12} = k$ using the traditional notations. Physicists are working very often with the algebras $\mathbf{R}_{1,3}$ or $\mathbf{R}_{3,1}$. It suffices to note the nature of the corresponding signatures $(+, -, -, -)$ and $(+, +, +, -)$, respectively. However, $\mathbf{R}_{1,3}$ and $\mathbf{R}_{3,1}$ are not isomorphic as algebras.

Recall that the main involution $'$, the reversion anti-automorphism $^\sim$ and the conjugation anti-automorphism $^-$ act on $a \in \mathbf{R}_{p,q}$ as follows:

$$a' = \sum_{k=0}^{d} (-1)^k < a >_k, \quad a^{\sim} = \sum_{k=0}^{d} (-1)^{\frac{k(k-1)}{2}} < a >_k$$

and

$$\bar{a} = \sum_{k=0}^{d} (-1)^{\frac{k(k+1)}{2}} < a >_k$$

We also need the following automorphism $* : \mathbf{R}_{0,n} \to \mathbf{R}_{0,n}$ defined by the relations: $e_n^* = -e_n, e_i^* = e_i$ for $i = 0, 1, ..., n-1$ and $(ab)^* = a^* b^*$ for $a, b \in \mathbf{R}_{0,n}$. These operations are not of algebraic type, they are of geometric type.

Remarks: For $\mathbf{C} = \mathbf{R}_{0,1}$, the usual complex conjugation is the main involution, as well as the conjugation. The reversion is useless.

For $\mathbf{H} = \mathbf{R}_{0,2}$, with the classical notations, we have: $a = \alpha + \beta i + \gamma j + \delta k, a' = \alpha - \beta i - \gamma j + \delta k, a^{\sim} = \alpha + \beta i + \gamma j - \delta k, \bar{a} = \alpha - \beta i - \gamma j - \delta k.$

Henceforth, we will consider Clifford algebras of antieuclidean type, namely $\mathbf{R}_{0,d}$, ([5], [2]). Note the first three, for $d = 0, 1, 2 : \mathbf{R}, \mathbf{C}$ and $\mathbf{H}$, are division algebras by the well known theorem of Frobenius.

Our aim is to survey different generalizations of the function theory of a complex variable which can be viewed as the study of those functions defined in a domain of $\mathbf{R}^2$ and taking their values in the Clifford algebra $\mathbf{R}_{0,1} = \mathbf{C}$.

The first key of the theory of holomorphic functions is, of course, the Cauchy-Riemann operator ($\partial/\partial\bar{z}$ in the classical notations), which can be written now as:

$$D = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1},$$

omitting the famous normalization constant $1/2$. It should be noted that the definition domain of f lies in a space whose elements are couples of a scalar and a vector, so that $\mathbf{R}^2$ should be identified to $\mathbf{R} \oplus i\mathbf{R}$.

2. Monogenic functions

Let $\mathbf{R}_{0,n}$ be the Clifford algebra of the real vector space V of dimension n provided with a quadratic form of negative signature, $n \in \mathbf{N}$. denote by S the set of the scalars in $\mathbf{R}_{0,n}$, identified with $\mathbf{R}$. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of V and set also $e_0 = 1$.

A point $x = (x_0, x_1, ..., x_n)$ of $\mathbf{R}^{n+1}$ will be considered as an element of $S \oplus V$, namely $x = \sum_{i=0}^{n} x_i e_i$. Such an element will be called a *paravector*. Obviously, it belongs to $\mathbf{R}_{0,n}$ and we can act on him with the conjugation: $\bar{x} = x_0 - \sum_{i=1}^{n} x_i e_i$. It is remarkable that:

$$x\bar{x} = \bar{x}x = \mid x \mid^2,$$

where $\mid x \mid$ denotes the usual euclidean norm of x in $\mathbf{R}^{n+1}$ and it shows that every non-zero paravector is invertible. Sometime, if necessary, we will resort to the notation $x = x_0 + \overrightarrow{x}$, , where $\overrightarrow{x}$ is the vector part of x, i.e. $\overrightarrow{x} = \sum_{i=1}^{n} x_i e_i$.

Let $f : \Omega \to \mathbf{R}_{0,n}$, where Ω is an open subset of $S \oplus V$. Introduce the Cauchy-Dirac-Fueter operator:

$$D = \sum_{i=0}^{n} e_i \frac{\partial}{\partial x_i}$$

Note D possesses a conjugate operator $\bar{D} = \dfrac{\partial}{\partial x_0} - \displaystyle\sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}$ and that $D\bar{D} = \bar{D}D = \Delta$, where Δ is the usual Laplacian.

Definition. ([2]) A function $\mathbf{R}_{0,n}$ be the Clifford algebra of the real vector space V of dimension n provided with a quadratic form of negative signature, $n \in \mathbf{N}$. Denote by S the set of the scalars in $\mathbf{R}_{0,n}$, identified with $\mathbf{R}$. Let $\{e_i\}_{i=1}^{n}$ be an orthonormal basis of V and set also $e_0 = 1$.

Obviously, in the case $n = 1$, we get the holomorphic functions of one complex variable.

Important remark: If $n > 1$, then the functions $x \mapsto x$ and $x \mapsto x^m, x \in S \oplus V, m \in \mathbf{N}$ are not monogenic.

Following R. Brackx, R. Delanghe and F. Sommen [2], recall that there exists a Cauchy kernel: $E(x) = (\omega_n^{-1})(\bar{x}/ \mid x \mid^{n+1})$ for $x \in S \oplus V - \{0\}$, where ω_n is the area of the unit sphere in $\mathbf{R}^{n+1}$. This kernel is really well adapted to the monogenic functions because it is himself a monogenic function with a singularity at the origin, i.e. $DE(x) = \delta$ for $x \in S \oplus V$.

Then, put $\omega(y) = dy_0 \wedge ... \wedge dy_n$ and $\gamma(y) = \sum_{i=0}^{n} (-1)^i e_i dy_0 \wedge ... \wedge \hat{dy_i} \wedge ... \wedge dy_n$. Thus, we have:

Integral representation formula: If f is monogenic in Ω and U is an oriented compact differentiable variety of dimension $n+1$ with boundary ∂U and $U \subset \Omega$, then

$$\int_{\partial U} E(y - x)\gamma(y) f(y) = f(x), \quad x \in \dot{U}.$$

Thanks to this, the analogous of the mean value theorem, the maximum modulus principle, Morera's theorem follow easily.

The depth and the wealth of the one complex variable theory come also thanks to the "duality": Cauchy-Riemann and Weierstrass, i.e. every holomorphic function is analytic and the reciprocal. How to understand what is the generalization of a power series?

In the frame of monogenic functions, an answer exists because, fortunately, the functions $w_k = x_k e_0 - x_0 e_k, k = 1, ..., n$ are monogenic. Omitting the details and roughly speaking one can expand every monogenic function in a series of polynomials which elementary monomials are the w_k and their powers. Just for an illustration let us show this phenomena is somehow natural: suppose $f : S \oplus V \to \mathbf{R}_{0,n}$ is real analytic on a neighborhood of the origin, so

$$f(h) = \sum_{k=0}^{\infty} (h_0 \frac{\partial}{\partial x_0} + ... + h_n \frac{\partial}{\partial x_n})^k f(0).$$

But at the same time f is monogenic, i. e.

$$\frac{\partial f}{\partial x_0} = - \sum_{i=1}^{n} e_i \frac{\partial f}{\partial x_i}.$$

Hence,

$$f(h) = \sum_{k=0}^{\infty} (\sum_{i=1}^{n} (h_i - e_i h_0) \frac{\partial}{\partial x_i})^k f(0).$$

3. Holomorphic Cliffordian functions

Here, consider functions $f : \Omega \to \mathbf{R}_{0,2m+1}$, where Ω is an open subset of $S \oplus V = \mathbf{R} \oplus \mathbf{R}^{2m+1} = \mathbf{R}^{2m+2}$. The paravectors of $S \oplus V$ will be written as $x = x_0 + \overrightarrow{x}, \quad x_0 \in \mathbf{R}, \quad \overrightarrow{x} = \sum_{i=1}^{2m+1} x_i e_i$.

Definition ([11], [12]).A function $f : \Omega \to \mathbf{R}_{0,2m+1}$ is said to be (left) *holomorphic Cliffordian* in Ω if and only if:

$$D \Delta^m f(x) = 0$$

for each $x \in \Omega$. Here Δ^m means the iterated Laplacian.

The set of holomorphic Cliffordian functions is wider than those of the monogenic ones: *every monogenic is also holomorphic Cliffordian, but the reciprocal is false.* Indeed, if $Df = 0$, then $D\Delta^m f = \Delta^m Df = 0$ because Δ^m is a scalar operator. The simplest example of a holomorphic Cliffordian function which is not monogenic is the identity $x \mapsto x$. Actually, one can prove that all entire powers of x are holomorphic Cliffordian, while they are not monogenic.

Note that f is holomorphic Cliffordian if and only if $\Delta^m f$ is monogenic.

There is a simple way to construct holomorphic Cliffordian functions which is based on the Fueter principle [7].

Lemma 1. *If* $u : \mathbf{R}^2 \to \mathbf{R}$ *is harmonic, then* $u(x_0, |\overrightarrow{x}|)$*, where* $x = x_0 + \overrightarrow{x}$ *and* $|\overrightarrow{x}|^2 = \sum_{i=1}^{2m+1} x_i^2$ *is* $(m+1)$*-harmonic, i. e.*

$$\Delta^{m+1} u(x_0, |\overrightarrow{x}|) = 0.$$

Lemma 2. *If* $f : (\xi, \eta) \mapsto f = u + iv$ *is a holomorphic function, then* $F(x) = u(x_0, |\overrightarrow{x}|) + \frac{\overrightarrow{x}}{|\overrightarrow{x}|} v(x_0, |\overrightarrow{x}|)$ *is a holomorphic Cliffordian function.*

If we summarize: from an usual holomorphic function f, with real part u, we construct the associated $(m+1)$-harmonic $u(x_0, |\overrightarrow{x}|)$ and then it suffices to take $\bar{D}u(x_0, |\overrightarrow{x}|)$ in order to get a holomorphic Cliffordian one. This receipt is very well adapted for the construction of trigonometric or exponential functions in $\mathbf{R}_{0,2m+1}$, [15]. *Note also the set of holomorphic Cliffordian functions is stable under any directional derivation.*

It is natural to ask for an integral representation formula but in this case, the operator $D\Delta^m$ being of order $2m+1$, such a formula would be much more complicated. Anyway, the first step is to exhibit an analogous to the Cauchy kernel. Remember the fundamental solution of the iterated Laplacian $\Delta^{m+1}h(x) = 0$ for $x \in S \oplus V - \{0\}$ is known: that is $h(x) = ln\,|x|$. Hence $\bar{D}(\frac{1}{2} ln(x\bar{x}))$ must be holomorphic Cliffordian. But:

$$\bar{D}(\frac{1}{2} ln(x\bar{x})) = \frac{1}{2} \frac{\bar{D}(|x|^2)}{|x|^2} = \frac{\bar{x}}{|x|^2} = x^{-1}.$$

By the way, we found the first holomorphic Cliffordian function with an isolated punctual singularity at the origin.

It is remarkable that, after computations, one get:

$$\Delta^m(x^{-1}) = (-1)^m 2^{2^m} (m!)^2 \omega_m E(x).$$

It becomes natural to introduce a new kernel:

$$N(x) = \varepsilon_m x^{-1},$$

with the suitable choice for the constant $\varepsilon_m = (-1)^m [2^{2m+1} m! \pi^{m+1}]^{-1}$.

So, N is the natural Cauchy kernel for holomorphic Cliffordian functions and we have:

$$D\Delta^m N(x) = DE(x) = \delta, \quad x \in S \oplus V.$$

Integral representation formula. Let B be the unit ball in $\mathbf{R}^{2m+2}$, x an interior point of B, $\frac{\partial}{\partial n}$ means the derivation in the direction of the outward normal.Thus, we have:

$$f(x) = \int_{\partial B} (\Delta^m N(y-x)\gamma(y) f(y)$$

$$-\sum_{k=1}^{m}\int_{\partial B}(\frac{\partial}{\partial n}\Delta^{m-k}N(y-x))D\Delta^{k-1}f(y)d\sigma_y$$

$$+\sum_{k=1}^{m}\int_{\partial B}(\Delta^{m-k}N(y-x))\frac{\partial}{\partial n}D\Delta^{k-1}f(y)d\sigma_y.$$

The above formula involves $2m+1$ integrals on ∂B, which means one can deduce the values of f inside B knowing the values on ∂B of $f, D\Delta^{k-1}f$ and $\frac{\partial}{\partial n}D\Delta^{k-1}f, \quad k=1,2,...,m$.

Recall that all integer powers of a paravector x are solutions of $D\Delta^m = 0$, and we saw also that $D\Delta^m(x^{-1}) = \delta$. Those facts can be proved directly following straightforward computations. Let $x = x_0 + \overrightarrow{x}$ be a paravector in a general Clifford algebra of antieuclidean type $\mathbf{R}_{0,d}, d \in \mathbf{N}$. Very fastidious calculations give:

$$D\Delta^m(x^{-1}) = (-1)2^m m!\prod_{j=0}^{m}(2j+1-d)(\mid x \mid)^{-(2m+2)}.$$

The right hand side is a scalar vanishing for $d = 1,3,5,...,2m+1$. Moreover, we have:

$$D\Delta^m(x^{2n+1}) = \prod_{j=0}^{m}(2j+1-d)\sum_{q=m}^{n}\prod_{k=1}^{m}(2q-2k+2)C_{2n+1}^{2q+1}x_0^{2n-2q}\overrightarrow{x}^{2q-2m},$$

and a similar formula for an even power x^{2n} of x. In both cases, the right hand sides are again scalars vanishing for $d = 1,3,5,...,2m+1$.

Polynomial solutions of $D\Delta^m = 0$. Set $\alpha = (\alpha_0, \alpha_1, ..., \alpha_{2m+1}), \alpha_j \in \mathbf{N}, \mid \alpha \mid = \sum_{j=0}^{2m+1}\alpha_j$. Consider the set

$$\{e_\nu\} = \{e_0,...,e_0,e_1,...,e_1,...,e_{2m+1},...,e_{2m+1}\},$$

where e_0 is written α_0 times, $e_i : \alpha_i$ times. Then set:

$$P_\alpha(x) = \sum_{\Theta}\prod_{\nu=1}^{|\alpha|-1}(e_{\sigma(\nu)}x)e_{\sigma(|\alpha|)},$$

the sum being expanded over all distinguishable elements σ of the permutation group Θ of the set $\{e_\nu\}$. The P_α are polynomials of degree $\mid \alpha \mid -1$. A straightforward calculation carried on them shows that P_α is equal up to a rational constant to $\partial^{|\alpha|}(x^{2|\alpha|-1})$. Thus, it follows that the P_α are holomorphic Cliffordian functions, which are left and right, thanks to the symmetrization process.

The classical way for getting Taylor's series of a holomorphic function is to expand the Cauchy kernel in the integral representation formula. The same procedure is available here:

$$(y-x)^{-1} = (y(1-y^{-1}x))^{-1} = (1-y^{-1}x)y^{-1}$$
$$= y^{-1} + y^{-1}xy^{-1} + y^{-1}xy^{-1}xy^{-1} + ... + (y^{-1}x)^n y^{-1} + ...$$

Obviously, we have $y^{-1} = \bar{y}(|\ y\ |)^{-2}$, and thus:

$$(y-x)^{-1} = \sum_{k=0}^{\infty} \frac{(\bar{y}x)^k \bar{y}}{|\ y\ |^{2k+2}}.$$

It is not difficult to observe the polynomials P_α appear again. Finally, as in the classical case, we can deduce the expansion in a "power series" of any holomorphic Cliffordian function f under the form:

$$f(x) = \sum_{|\alpha|=1}^{\infty} P_\alpha(x) C_\alpha,$$

where $C_\alpha \in \mathbf{R}_{0,2m+1}$.

Further, let us mention that a function f which is holomorphic Cliffordian in a punctured neighborhood of the origin possesses a Laurent expansion ([11], [12]). So, the set of meromorphic Cliffordian functions with isolated singularities is well defined.

What about the case $\mathbf{R}_{0,2m}$? More precisely, could we pass from holomorphic Cliffordian functions in $\mathbf{R}_{0,2m+1}$ to their restrictions in $\mathbf{R}_{0,2m}$? Roughly speaking, it is the same as between $\mathbf{C}$ and $\mathbf{R}$. In the general case of a Clifford algebra of type $\mathbf{R}_{0,d}$, we can observe that the set of homogeneous polynomials of degree n which are holomorphic Cliffordian is a right $\mathbf{R}_{0,d}$-module generated by the monomials $(ax)^n a$, where a is a paravector.

Now, let us consider $f : \mathbf{R} \oplus \mathbf{R}^d \to \mathbf{R}_{0,d}$.

Case 1: d is odd. The function f will be holomorphic Cliffordian if $D\Delta^{\frac{d-1}{2}} f = 0$.

Case 2: d is even. We say that f is *analytic Cliffordian* ([10]) if there is a holomorphic Cliffordian function of one more variable $F : \mathbf{R} \oplus \mathbf{R}^{d+1} \to \mathbf{R}_{0,d+1}$, which is even with respect to e_{d+1} and such that $F\ |_{x_{d+1}=0} = f$.

4. Hypermonogenic functions

In this part we will briefly discuss another class of functions, named *hypermonogenic*, which were introduced by H. Leutwiler ([16], [17]) and studied by himself and Sirkka-Liisa Eriksson-Bique ([6]).

Recall that any element $a \in \mathbf{R}_{0,n}$ may be uniquely decomposed as $a = b + ce_n$ for $b, c \in \mathbf{R}_{0,n-1}$. This should be compared with the classical decomposition of a complex number $a = b + ic$. Using the above decomposition, one introduces the projections $P : \mathbf{R}_{0,n} \to \mathbf{R}_{0,n-1}$ and $Q : \mathbf{R}_{0,n} \to \mathbf{R}_{0,n-1}$ given by $Pa = b, Qa = c$.

Now define the following modification of the Dirac operator D as follows:

$$Mf = Df + \frac{n-1}{x_n}(Qf)^*,$$

where * denotes the automorphism introduced above.

Definition. An infinetely differentiable function $f : \Omega \to \mathbf{R}_{0,n}, \Omega$ being an open subset of $\mathbf{R}^{n+1}$, such that $Mf = 0$ on $\Omega - \{x : x_n = 0\}$ is called a (left) *hypermonogenic* function.

Decomposing f into $f = Pf + (Qf)e_n$, its P-part satisfies the Laplace-Beltrami equation:

$$x_n \Delta(Pf) - (n-1)\frac{\partial(Pf)}{\partial x_n} = 0,$$

associated to the hyperbolic metric, defined on the upper half space $\mathbf{R}^{n+1}_+$ by $ds^2 = x_n^{-2}(dx_0^2 + dx_1^2 + ... + dx_n^2)$.

Its Q-part solves the eigenvalue equation:

$$x_n^2 \Delta(Qf) - (n-1)x_n \frac{\partial(Qf)}{\partial x_n} + (n-1)Qf = 0.$$

It turns out that hypermonogenic functions are stable by derivations in all possible directions *excepted this one on* x_n and that, for any $m \in \mathbf{N}$, the maps $x \mapsto x^m$ and $x \mapsto x^{-m}$ are hypermonogenic in $\mathbf{R}^{n+1}$, resp. $\mathbf{R}^{n+1} - \{0\}$.

Clearly, hypermonogenic functions generalize usual holomorphic functions of a complex variable. But what about the relations of this class with the class of holomorphic Cliffordian ?

One can prove that *every hypermonogenic function is also a holomorphic Cliffordian function.* Let us study this problem in the case $n = 3$.

Assuming $f : \Omega \to R_{0,3}$ is hypermonogenic, $Df = -\frac{2}{x_3}(Qf)^*$ and hence $D(\Delta f) = -2\Delta[\frac{(Qf)^*}{x_3}]$. An explicit calculation of the last expression combined with the above eigenvalue equation for the Q-part of f, allow us to conclude that

$$\Delta[\frac{(Qf)^*}{x_3}] = \frac{1}{x_3}[\Delta(Qf) - \frac{2}{x_3}\frac{\partial(Qf)}{\partial x_3} + 2\frac{Qf}{x_3^2}]^* = 0$$

forcing $D(\Delta f) = 0$.

Let say also the theory of hypermonogenic functions is provided with an integral representation formula and that the expansion in power series is generated by polynomials which are deeply related to the P_α above.

However, it should be noted an **important difference**: multiplication of e_n from the right to a hypermonogenic function *does not* in general give again a hypermonogenic function. In the larger class of holomorphic Cliffordian functions, this operation is allowed, even multiplication from the right with any Clifford number.

5. Elliptic Cliffordian functions

One of the most stimulating aspects of the theory of functions of a complex variable is the theory of elliptic functions. In [13], [14], we tried to put the foundations of their Cliffordian analogues. However, the main difficulty to overcome here is the lack of a satisfactory notion of product of two holomorphic Cliffordian functions. Thus, we had to adapt only the "additive" part of the theory in our case. Fortunately, the way drawn by Weierstrass, using the ζ and then the $\mathcal{P}$ function, was the right one. It was amazing that the analogues of the Jacobi functions *cn*, *dn* and *sn* could also be find on the same way.

Let us make a remark: If f is real-analytic in a neighborhood W of $a \in S \bigoplus V$ and is taking its values in $S \bigoplus V$, then f admits a Taylor expansion:

$$f(a+h) = \sum_{n=0}^{\infty} \frac{1}{n!}(h \mid \nabla_x)^n f(x) \mid_{x=a},$$

where $(h \mid \nabla_x)$ is the scalar product in $\mathbf{R}^{2m+2}$. Note also that $(h \mid \nabla_x)(x) = h$, $(h \mid \nabla_x)(x^{-1}) = -x^{-1}hx^{-1}$, and that

$$(h \mid \nabla_x)^q(x^{-1}) = (-1)^q q!(x^{-1}h)^q x^{-1},\ q \in N.$$

Now, let $N \in \{1, 2, ..., 2m+2\}$ and $\omega_\alpha \in S \oplus V$ be paravectors when $\alpha = 1, 2, ..., N$. Suppose the ω_α to be $\mathbf{R}$-independents. A function $f : \Omega \to \mathbf{R}_{0,2m+1}$ is said to be N-periodic if $f(x + 2\omega_\alpha) = f(x)$ for $x \in S \oplus V$ and $\alpha = 1, 2, ..., N$. The associated lattice is: $2\mathbf{Z}^N\omega = \{2k\omega, k \in \mathbf{Z}^N\}$, where $\omega = (\omega_1, ..., \omega_N)$ is the generic notation for a half period and $k\omega = \sum_{\alpha=1}^{N} k_\alpha \omega_\alpha$. Rewrite the lattice in $\{w_p\}_{p=0}^{\infty}$, where $w_0 = (0, ..., 0)$.

Definition. Introduce the ζ_N Weierstrass functions as:

$$\zeta_N(x) = x^{-1} + \sum_{p=1}^{\infty}\{(x-w_p)^{-1} + \sum_{\mu=0}^{N-1}(w_p^{-1}x)^\mu w_p^{-1}\}.$$

Thus $\zeta_N \ :\ S \oplus V \setminus 2\mathbf{Z}^N\omega \to S \oplus V$ is a holomorphic Cliffordian function and possesses simple poles on the vertices of the lattice.

The function ζ_N, is odd, is not itself a N-periodic function, but satisfies a property of "quasi-periodicity":

$$\zeta_N(x+2\omega) - \zeta_N(x) = 2 \sum_{p=0}^{[\frac{N+1}{2}]-1} \frac{(x+\omega \mid \nabla_y)^{2p}}{(2p)!} \zeta_N(y) \mid_{y=\omega}$$

which is equivalent to:

$$\zeta_N(x+\omega)-\zeta_N(x-\omega)=2\sum_{p=0}^{[\frac{N+1}{2}]-1}\frac{(x\mid\nabla_y)^{2p}}{(2p)!}\zeta_N(y)\mid_{y=\omega}.$$

In particular, for $N=2m+2$, one has:

$$\zeta_{2m+2}(x+\omega)-\zeta_{2m+2}(x-\omega)=2\sum_{p=0}^{m}\frac{(x\mid\nabla_y)^{2p}}{(2p)!}\zeta_{2m+2}(y)\mid_{y=\omega}$$

which is the natural generalization of the well-known:

$$\zeta(z+\omega)-\zeta(z-\omega)=2\zeta(\omega).$$

It is important to note that the right hand side of the previous equality is a holomorphic Cliffordian polynomial of degree $2m$.

As far as the Laurent expansion of ζ_N in a neighborhood of the origin, it is easy to get it:

$$\zeta_N(x)=x^{-1}+\sum_{k\geq[\frac{N}{2}]}\frac{1}{(2k+1)!}\sum_{p=1}(x\mid\nabla_w)^{2k+1}(w^{-1})\mid_{w=w_p}.$$

Note that in the special case $N=2m+2$, the first sum starts from $k=m+1$.

Now, in order to mimic the procedure of getting an elliptic function as $\mathcal{P}$ from ζ, where, in the case $m=0$, one need only one derivation, here, differentiating $2m+1$ times ζ_{2m+2}, we are able to produce analogous of the $\mathcal{P}$ function. It is remarkable, that now, we can differentiate in all the directions of $\mathbf{R}^{2m+2}$. The set of elliptic Cliffordian functions we are producing is quite larger than in the complex case.

Return now to the problem of the construction of analogous of the Jacobi functions. We have two problems to solve. Reduce the periods by half and eliminate the holomorphic Cliffordian polynomials appearing in the "quasi-periodicity" property.

Introduce the so-called translation operators E_j: for an arbitrary function $\varphi: S\oplus V\to\mathbf{R}_{0,2m+1}$, set:

$$E_j(\varphi)(x)=\varphi(x+\omega_j),\quad j=1,2,...,N.$$

When φ is N-periodic, we can write:

$$(I-E_j^2)(\varphi)(x)=0.$$

As far as the "quasi-periodicity" of ζ_{2m+2} is concerned, we can write it in the following form:

$$(I - E_j^2)(\zeta_{2m+2})(x) = -p_{2m}(x; \omega_j),$$

for $j = 1, 2, ..., 2m+2$, where p_{2m} is the polynomial appearing in the formula of "quasi-periodicity".

How eliminate such a polynomial? In numerical analysis there is a nice recipe which says: when one want annihilate a polynomial of degree d, it suffices to apply $d+1$ times operators of the form $I - E_j$ without being restricted to use always the same j.

Thus, for example, the following formula:

$$\prod_{j=1}^{2m+1} (I - E_j)(I - E_i^2)(\zeta_{2m+2})(x) = 0$$

is readable from the right to the left and says tat ζ_{2m+2} is quasi-periodic in the direction $2\omega_i$ and that after, we have proceeded to the elimination of the quasi-periodicity polynomial.

But the translation operators are commuting, so, the same formula can be written as:

$$(I - E_i^2)(\prod_{j=1}^{2m+1} (I - E_j)(\zeta_{2m+2})(x)) = 0$$

and it says that the function $\prod_{j=1}^{2m+1} (I - E_j)(\zeta_{2m+2})(x)$ is periodic with period $2\omega_i$.

Omitting some details, let set now:

$$C(x) = \prod_{j=1}^{2m+2} (I - E_j)(\zeta_{2m+2})(x)$$

$$S_i(x) = (I + E_i) \prod_{j=1, j\neq i}^{2m+2} (I - E_j)(\zeta_{2m+2})(x),$$

where $i = 1, 2, ..., 2m+2$.

Here, we have made use of the operator $I + E_i$ which reduces the periodicity from $2\omega_i$ by half to ω_i.

So, we have $2m+3$ elliptic Cliffordian functions whose periods are $\{\omega_i + \omega_k\}, i, k = 1, 2, ..., 2m+2$ for $C(x)$, and

$$2\omega_1, ...2\omega_{i-1}, \omega_i, 2\omega_{i+1}, ..., 2\omega_{2m+2}$$

for $S_i(x)$. The way they were obtained obeys to strong lays, they are no more, no less than $2m+3$ and they are obviously the natural generalizations in $\mathbf{R}^{2m+2}$ of the classical Jacobi functions cn, dn, sn (in this order!) we can get when $m = 0$.

6. Generalized automorphic forms

The story started with the problem: how to generalize the notion of modular forms in the case of functions with values in a Clifford algebra?

Historically, an enormous work has be done in some previous papers ([8], [3]), first in the case of monogenic functions and further, in the case of $k-$hypermonogenic functions. (The definition of such a function will be given below).

Although a technical virtuosity displayed in this situations, the results were finally unsatisfying in some sense: actually, the authors have not been able to propose a construction for non-vanishing $k-$hypermonogenic cusp forms for $k \neq 0$.

Recently, in [4], the authors decided to consider a larger class of functions that contains the class of $k-$hypermonogenic functions as a special subset.

When $k \in \mathbf{Z}$ is even, the class of considered functions, named $k-$holomorphic Cliffordian, are those which are annihilated by the operator$D\Delta^{\frac{k}{2}}$, where D is the Dirac operator, Δ is the Laplacian and $\Delta^{\frac{k}{2}}$ the iterated Laplacian. For $k = n-1$ and n odd this is the class of the holomorphic Cliffordian functions.

Finally, it turned out that the choice of the authors to consider $k-$holomorphic Cliffordian functions was fruitful. The main obstruction they met before, namely, multiplication of e_n from the right to a $k-$hypermonogenic function which does not in general give again a $k-$hypermonogenic function, was overcome in the new context: right multiplication of any Clifford number with a $k-$holomorphic Cliffordian function remains $k-$holomorphic Cliffordian. It is amazing that now all the machinery works perfectly, even elegantly.

The general Ahlfors-Vahlen group and some discrete arithmetic subgroups.

The set that consists of Clifford valued matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, whose coefficients satisfy the conditions below forms a group under matrix multiplication. It is called the *general Ahlfors-Vahlen group* $\mathcal{G}AV(\mathbf{R} \oplus \mathbf{R}^n)$. The action of this group on $\mathbf{R} \oplus \mathbf{R}^n$ is described by the associated Möbius transformation. We can also restrict this action on the upper half-space $H^+(\mathbf{R} \oplus \mathbf{R}^n) = \{x \in \mathbf{R} \oplus \mathbf{R}^n : x_n > 0\}$ in the following way:

$$\mathcal{G}AV(\mathbf{R} \oplus \mathbf{R}^n) \times H^+(\mathbf{R} \oplus \mathbf{R}^n) \to H^+(\mathbf{R} \oplus \mathbf{R}^n)$$

by $(M, x) \mapsto M < x >= (ax+b)(cx+d)^{-1}$. Here, the coefficients a, b, c, d from $\mathbf{R}_{0,n}$ satisfy:

(i) a, b, c, d are products of paravectors
(ii) $a\tilde{d} - b\tilde{c} \in \mathbf{R} \setminus \{0\}$
(iii) $ac^{-1}, c^{-1}d \in \mathbf{R}^{n+1}$ for $c \neq 0$ and $bd^{-1} \in \mathbf{R}^{n+1}$ for $c = 0$.

The subgroup consisting of those matrices from $\mathcal{G}AV(\mathbf{R} \oplus \mathbf{R}^n)$ that satisfy $ad^* - bc^* = 1$ is called the *special Ahlfors-Vahlen group*, denoted by $\mathcal{S}AV(\mathbf{R} \oplus \mathbf{R}^n)$.

The automorphism group of the upper half-space $H^+(\mathbf{R} \oplus \mathbf{R}^n)$ is the group $\mathcal{S}AV(\mathbf{R} \oplus \mathbf{R}^{n-1})$.

The rational Ahlfors-Vahlen group $\mathcal{S}AV(\mathbf{R} \oplus \mathbf{R}^{n-1}, \mathbf{Q})$ is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from $\mathcal{S}AV(\mathbf{R} \oplus \mathbf{R}^{n-1})$ that satisfy
(i) $a\bar{a}, b\bar{b}, c\bar{c}, d\bar{d} \in \mathbf{Q}$,
(ii) $a\bar{c}, b\bar{d} \in \mathbf{Q} \oplus \mathbf{Q}^n$,
(iii) $ax\bar{b} + b\bar{x}\bar{a}, cx\bar{d} + d\bar{x}\bar{c} \in \mathbf{Q}$,
(iv) $ax\bar{d} + b\bar{x}\bar{c} \in \mathbf{Q} \oplus \mathbf{Q}^n$.

The following definition provides us with a whole class of arithmetic subgroups of the Ahlfors-Vahlen group which act totally discontinuously on the upper half-space.

Definition:

$$\Gamma_{n-1}(\mathcal{I}) = \mathcal{S}AV(\mathbf{R} \oplus \mathbf{R}^{n-1}, \mathbf{Q}) \cap Mat(2, \mathcal{I}).$$

For $N \in \mathbf{N}$, set:

$$\Gamma_{n-1}(\mathcal{I})[N] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{n-1}(\mathcal{I}), a - 1, b, c, d - 1 \in N\mathcal{I} \right\}.$$

Here $\mathcal{I}$ is a $\mathbf{Z}$-order in the Clifford algebra which is roughly speaking a subring R such that the additive group of R is finitely generated and contains a $\mathbf{Q}$-basis of the algebra.

Notice that all the groups $\Gamma_{n-1}(\mathcal{I})[N]$ are discrete groups and act totally discontinuously on the upper half-space.
Concerning the above notions, see [1], [3], [4], [8] for the details.

Monogenic functions. An important property of the Dirac operator

$$D = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} e_1 + ... + \frac{\partial}{\partial x_n} e_n$$

is its quasi-invariance under Möbius transformations acting on the complete Euclidean space $\mathbf{R} \oplus \mathbf{R}^n$.

Let $M \in \mathcal{G}AV(\mathbf{R} \oplus \mathbf{R}^n)$ and f be a left monogenic function in the variable $y = M < x >= (ax+b)(cx+d)^{-1}$. Then the function:

$$g(x) = \frac{\overline{cx+d}}{\| cx+d \|^{n+1}} f(M < x >)$$

is again left monogenic on x for any $M \in \mathcal{G}AV(\mathbf{R} \oplus \mathbf{R}^n)$.

k-hypermonogenic functions. The class of monogenic functions belongs to the more general class of so-called k-hypermonogenic functions. They are defined as the null-solutions to the system:

$$Df + k\frac{(Qf)^*}{x_n} = 0,$$

where $k \in \mathbf{R}$.

Note that in the case of $k = 0$, we are dealing with left monogenic functions. The particular case of $k = n-1$ corresponds to hypermonogenic functions.

Now, if f is k-hypermonogenic in the variable $y = M < x >= (ax+b)(cx+d)^{-1}$, then:

$$F(x) = \frac{\overline{cx+d}}{\| cx+d \|^{n+1-k}} f(M < x >)$$

is k-hypermonogenic.

However, this invariance holds only for matrices from $\mathcal{S}AV(\mathbf{R} \oplus \mathbf{R}^{n-1})$ and not for all the matrices from $\mathcal{S}AV(\mathbf{R} \oplus \mathbf{R}^n)$.

This is due to the fact that a translation in the argument of a k-hypermonogenic function in the e_n-direction does not give a k-hypermonogenic function again.

As we said before, the authors [4] decided to consider a larger class of functions, containing in itself the set of k-hypermonogenic ones and possessing the previous property (being stable under right multiplication with e_n). Thus, this new class will have the extra property of being invariant under the whole group $\mathcal{S}AV(\mathbf{R} \oplus \mathbf{R}^n)$.

k-holomorphic Cliffordian functions

Definition. Let $n \in \mathbf{N}$ and suppose that k is an even positive integer. Let $U \subset \mathbf{R} \oplus \mathbf{R}^n$ be an open set. Then we call a function $f : U \to \mathbf{R}_{0,n}$ a *k-holomorphic Cliffordian function* if

$$D\Delta^{\frac{k}{2}} f = 0.$$

In the particular case of $k = n-1$ (n odd) we deal with the class of holomorphic Cliffordian functions.

One can also introduce k-holomorphic Cliffordian functions for negative even integers. This can be done through the Teodorescu transform which is the right inverse to D, i.e. $DTf = 0$. In view of the identity $\bar{D}D = \Delta$, one can formally express Δ^{-1} as $\bar{T}T$ on the upper half-space.

First of all, it is easy to prove that, for any even $k \in \mathbf{Z}$, every k-hypermonogenic function is also k-holomorphic Cliffordian.

As a consequence, the k-hypermonogenic kernel functions

$$G_k(x) = \frac{\overline{x}}{|| x ||^{n+1-k}}$$

are also k-holomorphic Cliffordian for all $k \in 2\mathbf{Z}$.

However, note that *not* every k-holomorphic Cliffordian function is k-hypermonogenic. Take for instance $k = n - 1$, with n odd. Then the functions xe_n and $x + e_n$, where x is a paravector, are both holomorphic Cliffordian but not hypermonogenic.

Theorem. *Let $k \in 2\mathbf{Z}$. Suppose that $M \in \mathcal{S}AV(\mathbf{R} \oplus \mathbf{R}^n)$. Let $y = M < x >= (ax + b)(cx + d)^{-1}$ be the image of a point x under such a Möbius transformation. Then such a function $f(y)$ that is k-holomorphic Cliffordian in the variable y is transformed to a function*

$$F(x) = \frac{\overline{cx + d}}{|| cx + d ||^{n+1-k}} f(M < x >)$$

which turns out to be k-holomorphic Cliffordian in the variable x.

Definition. Let $p \leq n - 1$ and suppose that $k \in 2\mathbf{Z}$. A left k-holomorphic Cliffordian function $f : H^+(\mathbf{R} \oplus \mathbf{R}^n) \to \mathbf{R}_{0,n}$ is called a *left k-holomorphic Cliffordian automorphic form on $\Gamma_p(\mathcal{I})[N]$*, if for all $x \in H^+(\mathbf{R} \oplus \mathbf{R}^n)$

$$f(x) = \frac{\overline{cx + d}}{|| cx + d ||^{n+1-k}} f(M < x >)$$

for all $M \in \Gamma_p(\mathcal{I})[N]$.

In the case $k = 0$ we re-obtain the class of left monogenic automorphic forms discussed in [8].

Moreover, all k-hypermonogenic automorphic forms discussed in [3] are included in this set.

The simplest examples of k-holomorphic Cliffordian automorphic forms on the groups $\Gamma_p(\mathcal{I})[N]$ are the generalized Eisenstein series given in [8], as well as the simplest examples of k-holomorphic Cliffordian automorphic forms for discrete translation groups for the special case of $k = n - 1$ with n odd, were given in [13] (the generalizations of the cotangent function, the Weierstrass ζ and $\wp$ functions).

Cusp forms

Definition: For even integers $k \leq 0$ a *left k-holomorphic Cliffordian cusp form on* $\Gamma_{n-1}(\mathcal{I})[N]$ is a left k-holomorphic Cliffordian automorphic form on $\Gamma_{n-1}(\mathcal{I})[N]$ that satisfies additionally:

$$\lim_{x_n \to +\infty} x_n^{-k} \frac{\overline{cx_n e_n + d}}{|| \, cx_n e_n + d \, ||^{n+1-k}} f(M < x_n e_n >) = 0$$

for all $M \in \Gamma_{n-1}(\mathcal{I})[N]$.

For positive even integers k, the factor x_n^{-k} is omitted.

In the last part of the paper [4], the authors establish a surprising result: a decomposition theorem of the spaces of k-holomorphic Cliffordian automorphic forms in terms of a direct orthogonal sum of the spaces of k-hypermonogenic Eisenstein series and of k-holomorphic Cliffordian cusp forms.

Let us end with a citation due to L. Ahlfors in [1] : *"The aim is not to prove new results, but to try to convince complex analysts of more traditional bent that the use of Clifford numbers is both natural, simple and useful."*

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TURÁN-ERŐD TYPE CONVERSE MARKOV INEQUALITIES FOR CONVEX DOMAINS ON THE PLANE

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Abstract

For a convex domain $K \subset \mathbb{C}$ the well-known general Bernstein-Markov inequality holds: a polynomial p of degree n must have $\|p'\| \leq c(K)n^2\|p\|$. However, for polynomials in general, $\|p'\|$ can be arbitrarily small, compared to $\|p\|$.

Turán investigated the situation under the condition that p have all its zeroes in the convex body K. With this assumption he proved $\|p'\| \geq (n/2)\|p\|$ for the unit disk D and $\|p'\| \geq c\sqrt{n}\|p\|$ for the unit interval $I := [-1, 1]$. Levenberg and Poletsky provided general lower estimates of order $\sqrt{n}$, and there were certain classes of domains with order n lower estimates.

We show that for *all* compact and convex *domains* K and polynomials p with all their zeroes in K $\|p'\| \geq c(K)n\|p\|$ holds true, while $\|p'\| \leq C(K)n\|p\|$ occurs for arbitrary compact connected sets $K \subset \mathbb{C}$. Moreover, the dependence on width and diameter of the set K is found up to a constant factor. Note that if K is *not* a domain ($\text{int}K = \emptyset$), then the order is only $\sqrt{n}$.

Erőd observed that in case the boundary of the domain is smooth and the curvature exceeds a constant $\kappa > 0$, then we can get an order n lower estimation with the curvature occurring in the implied constant. Elaborating on this idea several extensions of the result are given. Again, geometry is in focus, including a new, strong "discrete" version of the classical Blaschke Rolling Ball Theorem.

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Key Words and Phrases: Bernstein-Markov Inequality, Turán's lower estimate of derivative norm, logarithmic derivative, Chebyshev constant, convex sets and domains, width of a set, circular domains, convex curves, smooth convex bodies, curvature, osculating circle, Blaschke's rolling ball theorem, subdifferential or Lipschitz-type lower estimate of increase

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1. Introduction

On the complex plane $\mathbb{C}$ polynomials of degree n admit a Markov inequality[1] $\|p'\|_K \le c_K n^2 \|p\|_K$ on all convex, compact $K \subset \mathbb{C}$. Here the norm $\|\cdot\| := \|\cdot\|_K$ denotes sup norm over values attained on K.

In 1939 Paul Turán studied converse inequalities of the form $\|p'\|_K \ge c_K\, n^A \, \|p\|_K$. Clearly such a converse can hold only if further restrictions are imposed on the occurring polynomials p. Turán assumed that all zeroes of the polynomials must belong to K. So denote the set of complex (algebraic) polynomials of degree (exactly) n as $\mathcal{P}_n$, and the subset with all the n (complex) roots in some set $K \subset \mathbb{C}$ by $\mathcal{P}_n^{(0)}(K)$. The (normalized) quantity under our study is thus the "inverse Markov factor"

$$M_n(K) := \inf_{p \in \mathcal{P}_n^{(0)}(K)} M(p) \qquad \text{with} \qquad M := M(p) := \frac{\|p'\|}{\|p\|} \, . \tag{1.1}$$

Theorem 1.1 (**Turán,** [21, p. 90]). *If $p \in \mathcal{P}_n(D)$, where D is the unit disk, then we have*

$$\left\|p'\right\|_D \ge \frac{n}{2} \|p\|_D \, . \tag{1.2}$$

Theorem 1.2 (**Turán,** [21, p. 91]). *If $p \in \mathcal{P}_n(I)$, where $I := [-1,1]$, then we have*

$$\left\|p'\right\|_I \ge \frac{\sqrt{n}}{6} \|p\|_I \, . \tag{1.3}$$

Theorem **1**.1 is best possible, as the example of $p(z) = 1 + z^n$ shows. This also highlights the fact that, in general, the order of the inverse Markov factor cannot be higher than n. On the other hand, a number of positive results, started with J. Erőd's work, exhibited convex domains having order n inverse Markov factors (like the disk). We come back to this after a moment.

Regarding Theorem **1**.2, Turán pointed out by the example of $(1-x^2)^n$ that the $\sqrt{n}$ order is sharp. The slightly improved constant $1/(2e)$ can be found in [8], but the value of the constant is computed for all fixed n precisely in [6]. In fact, about two-third of the paper [6] is occupied by the rather lengthy and difficult calculation of these constants, which partly explains why later authors started to consider this achievement the only content of the paper. Nevertheless, the work of Erőd was much richer, with many important ideas occurring in the various approaches what he had presented.

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[1] Namely, to each point z of K there exists another $w \in K$ with $|w - z| \ge \operatorname{diam}(K)/2$, and thus application of Markov's inequality on the segment $[z,w] \subset K$ yields $|p'(z)| \le (4/\operatorname{diam}(K)) n^2 \|p\|_K$.

In particular, Erőd considered ellipse domains, which form a parametric family E_b naturally connecting the two sets I and D. Note that for the same sets E_b the best form of the Bernstein-Markov inequality was already investigated by Sewell, see [19].

Theorem 1.3 (**Erőd,** [6, p. 70]). *Let $0 < b < 1$ and let E_b denote the ellipse domain with major axes $[-1, 1]$ and minor axes $[-ib, ib]$. Then*

$$\|p'\| \geq \frac{b}{2} n \|p\| \tag{1.4}$$

for all polynomials p of degree n and having all zeroes in E_b.

Erőd himself provided two proofs, the first being a quite elegant one using elementary complex functions, while the second one fitting more in the frame of classical analytic geometry. In 2004 this theorem was rediscovered by J. Szabados, providing a testimony of the natural occurrence of the sets E_b in this context[2].

In fact, the key to Theorem **1**.1 was the following observation, implicitly already in [21] and [6] and formulated explicitly in [8].

Lemma 1.1 (**Turán, Levenberg-Poletsky**). *Assume that $z \in \partial K$ and that there exists a disc D_R of radius R so that $z \in \partial D_R$ and $K \subset D_R$. Then for all $p \in \mathcal{P}_n^{(0)}(K)$ we have*

$$|p'(z)| \geq \frac{n}{2R} |p(z)| \ . \tag{1.5}$$

So Levenberg and Poletsky [8] found it worthwhile to formally introduce the next definition.

Definition 1.1. A compact set $K \subset \mathbb{C}$ is called *R-circular*, if for any point $z \in \partial K$ there exists a disc D_R of radius R with $z \in \partial D_R$ and $K \subset D_R$.

With this they formulated various consequences. For our present purposes let us chose the following form, c.f. [8, Theorem 2.2].

Theorem 1.4 (**Erőd; Levenberg-Poletsky**). *If K is an R-circular set and $p \in \mathcal{P}_n^{(0)}(K)$, then*

$$\|p'\| \geq \frac{n}{2R} \|p\| \ . \tag{1.6}$$

[2]After learning about the overlap with Erőd's work, the result was not published.

Note that here it is not assumed that K be convex; a circular arc, or a union of disjoint circular arcs with proper points of join, satisfy the criteria. However, other curves, like e.g. the interval itself, do not admit such inequalities; as said above, the order of magnitude can be as low as $\sqrt{n}$ in general.

Erőd did not formulate the result that way; however, he was clearly aware of that. This can be concluded from his various argumentations, in particular for the next result.

Theorem 1.5 (**Erőd,** [6, p. 77])**.** *If K is a C^2-smooth convex domain with the curvature of the boundary curve staying above a fixed positive constant $\kappa > 0$, and if $p \in \mathcal{P}_n^{(0)}(K)$, then we have*

$$\left\|p'\right\| \geq c(K)n\|p\|. \tag{1.7}$$

From Erőd's argument one can not easily conclude that the constant is $c(K) = \kappa/2$; on the other hand, his statement is more general than that. Although the proof is slightly incomplete, let us briefly describe the idea[3].

Proof. The norm of p is attained at some point of the boundary, so it suffices to prove that $|p'(z)|/|p(z)| \geq cn$ for all $z \in \partial K$. But the usual form of the logarithmic derivative and the information that all the n zeroes $z_1, \dots, z_n$ of p are located in K allows us to draw this conclusion once we have for a fixed direction $\varphi := \varphi(z)$ the estimate

$$\Re\left(e^{i\varphi}\frac{1}{z - z_k}\right) \geq c > 0 \qquad (k = 1, \dots, n). \tag{1.8}$$

Choosing φ the (outer) normal direction of the convex curve ∂K at $z \in \partial K$, and taking into consideration that z_k are placed in $K \setminus \{z\}$ arbitrarily, we end up with the requirement that

$$\Re\left(e^{i\varphi}\frac{1}{z - w}\right) = \frac{\cos\alpha}{|z - w|} \geq c \qquad (w \in K \setminus \{z\},\ \alpha := \varphi - \arg(z - w))\ . \tag{1.9}$$

Now if K is strictly convex, then for $z \neq w$ we do not have $\cos\alpha = 0$, a necessary condition for keeping the ratio off zero. It remains to see if $|z - w|/\cos\alpha$ stays bounded when $z \in \partial K$ and $w \in K \setminus \{z\}$, or, as is easy to see, if only $w \in \partial K \setminus \{z\}$. Observe that $F(z, w) := |z - w|/\cos\alpha$ is a two-variate function on ∂K^2, which is not defined for the diagonal $w = z$, but under certain conditions can be extended continuously. Namely, for given z the limit, when $w \to z$, is the well-known geometric quantity $2\rho(z)$, where $\rho(z)$ is the radius of the osculating circle (i.e., the reciprocal of the curvature $\kappa(z)$). (Note here a gap in the argument for not taking into consideration also $(z', w') \to (z, z)$, which can be removed by

[3] For more about the life and work of János Erőd, see [15] and [16].

showing uniformity of the limit.) Hence, for smooth ∂K with strictly positive curvature bounded away from 0, we can define $F(z,z) := 2/\kappa(z) = 2\rho(z)$. This makes F a continuous function all over ∂K^2, hence it stays bounded, and we are done. □

We will return to this theorem and provide a somewhat different, complete proof giving also the value $c(K) = \kappa/2$ of the constant later in §**6**. For an analysis of the slightly incomplete, nevertheless essentially correct and really innovative proof of Erőd see [15].

From this argument it can be seen that whenever we have the property (1.9) for all given boundary points $z \in \partial K$, then we also conclude the statement. This explains why Erőd could allow even vertices, relaxing the conditions of the above statement to hold only piecewise on smooth Jordan arcs, joining at vertices. However, to have a fixed bound, either the number of vertices has to be bounded, or some additional condition must be imposed on them. Erőd did not elaborate further on this direction.

Convex domains (or sets) *not* satisfying the R-circularity criteria with any fixed positive value of R are termed to be *flat*. Clearly, the interval is flat, like any polygon or any convex domain which is not strictly convex. From this definition it is not easy to tell if a domain is flat, or if it is circular, and if so, then with what (best) radius R. We will deal with the issue in this work, aiming at finding a large class of domains having cn order of the inverse Markov factor with some information on the arising constant as well.

On the other hand a lower estimate of the inverse Markov factor of the same order as for the interval was obtained in full generality in 2002, see [8, Theorem 3.2].

Theorem 1.6 (**Levenberg-Poletsky**). *If $K \subset \mathbb{C}$ is a compact, convex set, $d := \operatorname{diam} K$ is the diameter of K and $p \in \mathcal{P}_n^{(0)}(K)$, then we have*

$$\|p'\| \geq \frac{\sqrt{n}}{20 \operatorname{diam}(K)} \|p\| . \tag{1.10}$$

Clearly, we can have no better order, for the case of the interval the $\sqrt{n}$ order is sharp. Nevertheless, already Erőd [6, p. 74] addressed the question: "For what kind of domains does the method of Turán apply?" Clearly, by "applies" he meant that it provides cn order of oscillation for the derivative.

The most general domains with $M(K) \gg n$, found by Erőd, were described on p. 77 of [6]. Although the description is a bit vague, and the proof shows slightly less, we can safely claim that he has proved the following result.

Theorem 1.7 (**Erőd**). *Let K be any convex domain bounded by finitely many Jordan arcs, joining at vertices with angles $< \pi$, with all the arcs being C^2-smooth and being either straight lines of length $\ell < \Delta(K)/4$, where $\Delta(K)$ stands for the transfinite diameter of K, or having positive curvature bounded away from 0 by a fixed constant. Then there is a constant $c(K)$, such that $M_n(K) \geq c(K)n$ for all $n \in \mathbb{N}$.*

To deal with the flat case of straight line boundary arcs, Erőd involved another approach, cf. [6, p. 76], appearing later to be essential for obtaining a general answer. Namely, he quoted Faber [7] for the following fundamental result going back to Chebyshev.

Lemma 1.2 (**Chebyshev**). *Let $J = [u, v]$ be any interval on the complex plane with $u \neq v$ and let $J \subset R \subset \mathbb{C}$ be any set containing J. Then for all $k \in \mathbb{N}$ we have*

$$\min_{w_1,\dots,w_k \in R} \max_{z \in J} \left| \prod_{j=1}^{k} (z - w_j) \right| \geq 2 \left(\frac{|J|}{4} \right)^k . \tag{1.11}$$

Proof. This is essentially the classical result of Chebyshev for a real interval, cf. [2, 9], and it holds for much more general situations (perhaps with the loss of the factor 2) from the notion of Chebyshev constants and capacity, cf. Theorem 5.5.4. (a) in [11]. □

The relevance of Chebyshev's Lemma is that it provides a quantitative way to handle contribution of zero factors at some properly selected set J. One uses this for comparison: if $|p(\zeta)|$ is maximal at $\zeta \in \partial K$, then the maximum on some J can not be larger. Roughly speaking, combining this with geometry we arrive at an effective estimate of the contribution, hence even on the location of the zeroes.

In his recent work [5], Erdélyi considered various special domains. Apart from further results for polynomials of some special form (e.g. even or real polynomials), he obtained the following.

Theorem 1.8 (**Erdélyi**). *Let Q denote the square domain with diagonal $[-1, 1]$. Then for all polynomials $p \in \mathcal{P}_n(Q)$ we have*

$$\|p'\| \geq C_0 n \|p\| \tag{1.12}$$

with a certain absolute constant C_0.

Note that the regular n-gon K_n is already covered by Erőd's Theorem **1**.7 if $n \geq 26$, but not the square Q, since the side length h is larger than the quarter of the transfinite diameter Δ: actually, $\Delta(Q) \approx 0.59017\ldots h$, while

$$\Delta(K_n) = \frac{\Gamma(1/n)}{\sqrt{\pi}2^{1+2/n}\Gamma(1/2+1/n)} h > 4h \quad \text{iff} \qquad n \geq 26,$$

see [11, p. 135]. Erdélyi's proof is similar to Erőd's argument[4]: sacrificing generality gives the possibility for a better calculation for the particular choice of Q.

Returning to the question of the order in general, let us recall that the term *convex domain* stands for a compact, convex subset of $\mathbb{C}$ *having nonempty interior*. Clearly, assuming boundedness is natural, since all polynomials of positive degree have $\|p\|_K = \infty$ when the set K is unbounded. Also, all convex sets with nonempty interior are *fat*, meaning that $\mathrm{cl}(K) = \mathrm{cl}(\mathrm{int}K)$. Hence taking the closure does not change the sup norm of polynomials under study. The only convex, compact sets, falling out by our restrictions, are the intervals, for what Turán has already shown that his $c\sqrt{n}$ lower estimate is of the right order. Interestingly, it turned out that among all convex compacta only intervals can have an inverse Markov constant of such a small order.

To study (1.1) some geometric parameters of the convex domain K are involved naturally. We write $d := d(K) := \mathrm{diam}\,(K)$ for the *diameter* of K, and $w := w(K) := \mathrm{width}\,(K)$ for the *minimal width* of K. That is,

$$w(K) := \min_{\gamma\in[-\pi,\pi]} \left(\max_{z\in K} \Re(ze^{-i\gamma}) - \min_{z\in K} \Re(ze^{-i\gamma}) \right). \tag{1.13}$$

Note that a (closed) convex domain is a (closed), bounded, convex set $K \subset \mathbb{C}$ with nonempty interior, hence $0 < w(K) \leq d(K) < \infty$. Our main result is the following.

Theorem 1.9. *Let $K \subset \mathbb{C}$ be any convex domain having minimal width $w(K)$ and diameter $d(K)$. Then for all $p \in \mathcal{P}_n^{(0)}(K)$ we have*

$$\frac{\|p'\|}{\|p\|} \geq C(K)n \qquad \text{with} \qquad C(K) = 0.0003\frac{w(K)}{d^2(K)}\ . \tag{1.14}$$

Then again, as regards the order of magnitude, (and in fact apart from an absolute constant factor), this result is sharp for all convex domains $K \subset \mathbb{C}$.

Theorem 1.10. *Let $K \subset \mathbb{C}$ be any compact, connected set with diameter d and minimal width w. Then for all $n > n_0 := n_0(K) := 2(d/16w)^2 \log(d/16w)$*

[4]Erdélyi was apparently not aware of the full content of [6] when presenting his rather similar argument.

there exists a polynomial $p \in \mathcal{P}_n^{(0)}(K)$ *of degree exactly* n *satisfying*

$$\|p'\| \le C'(K)\, n\, \|p\| \qquad \text{with} \qquad C'(K) := 600\, \frac{w(K)}{d^2(K)}\ . \tag{1.15}$$

Remark 1.1. Note that here we do not assume that K be convex, but only that it is a connected, closed (compact) subset of $\mathbb{C}$. (Clearly the condition of boundedness is not restrictive, $\|p\|$ being infinite otherwise.)

In the proof of Theorem **1**.9, due to generality, the precision of constants could not be ascertained e.g. for the special ellipse domains considered in [6]. Thus it seems that the general results are not capable to fully cover e.g. Theorem **1**.3.

However, even that is possible for a quite general class of convex domains with order n inverse Markov factors and a different estimate of the arising constants. This will be achieved working more in the direction of Erőd's first observation, i.e. utilizing information on curvature.

Since these results need some technical explanations, formulation of these will be postponed until §**6**. But let us mention the key ingredient, which clearly connects curvature and the notion of circular domains. In the smooth case, it is well-known as Blaschke's Rolling Ball Theorem, cf. [1, p. 116].

Lemma 1.3 (**Blaschke**). *Assume that the convex domain* K *has* C^2 *boundary* $\Gamma = \partial K$ *and that there exists a positive constant* $\kappa > 0$ *such that the curvature* $\kappa(\zeta) \ge \kappa$ *at all boundary points* $\zeta \in \Gamma$. *Then to each boundary points* $\zeta \in \Gamma$ *there exists a disk* D_R *of radius* $R = 1/\kappa$, *such that* $\zeta \in \partial D_R$, *and* $K \subset D_R$.

Again, geometry plays the crucial role in the investigations of variants when smoothness and conditions on curvature are relaxed. We will strongly extend the classical results of Erőd, showing that conditions on the curvature suffices to hold only almost everywhere (in the sense of arc length measure) on the boundary.

Theorem 1.11. *Assume that the convex domain* K *has boundary* $\Gamma = \partial K$ *and that the a.e. existing curvature of* Γ *exceeds* κ *almost everywhere, or, equivalently, assume the subdifferential condition* (3.6) *(or any of the equivalent formulations in* (3.1)-(3.6)*) with* $\lambda = \kappa$. *Then for all* $p \in \mathcal{P}_n^{(0)}(K)$ *we have*

$$\|p'\| \ge \frac{\kappa}{2} n \|p\|\ . \tag{1.16}$$

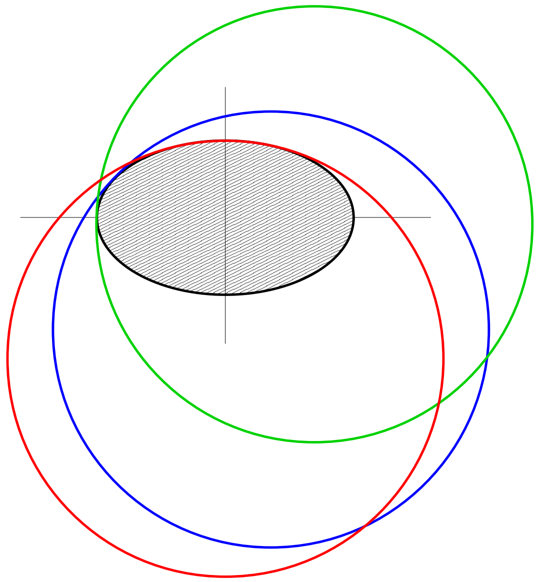

Figure 1. The Ellipse E_b is b-circular according to the Rolling Ball Theorem of Blaschke. Therefore, $\|p'\|/\|p\| \geq bn/2$.

This also hinges upon geometry, and we will have two proofs. One is essentially an application of a recent, quite far-reaching extension of the Blaschke Theorem by Strantzen. The other involves even more geometry: it hinges upon a new, discrete version of the Blaschke Rolling Ball Theorem, (which easily implies also Strantzen's Theorem), but which is suitable, at least in principle, to provide also some degree-dependent estimate of $M_n(K)$ by means of the *minimal oscillation* or *change* of the outer unit normal vector(s) along the boundary curve.

For applications to various domains, where yields of the different estimates can also be compared, see the later sections. Before that, in the next section we discuss the most general result, Theorem **1**.9, and its sharpness, as expressed by Theorem **1**.10.

In §**3** we start with describing the underlying geometry, and in §**5** we will describe variants and extensions on the theme of the Blaschke Rolling Ball Theorem. Finally, in §**6** we will formulate the resulting theorems and analyze the yields of them on various parametric classes of domains.

2. A discussion of the proof of the main theorem and its sharpness

Here we only comment on the ideas of the proofs, which otherwise are several pages long fine estimates and calculations. However, the point is not in the calculations themselves, but in the geometrical ideas behind. Those are what we try to explain a bit here. With the ideas clarified, it is still a matter of rather delicate, technical work, but still possible – without the proper insight it is rather improbable that one would just blindly compute them. In this regard we also thank a great deal to Prof. Gábor Halász, who provided us one of the key geometrical ingredients by suggesting a really insightful modification of our original argument. On the other hand, the proof is elementary in the sense that no special theoretical knowledge is required from the reader to fully follow the proofs, check the calculations in [13] or [17].

Let us recall that after the general result of Theorem **1**.6 of Levenberg and Poletsky it was widely felt that no better, than the $\sqrt{n}$ order, can be obtained for arbitrary convex bodies. Research was thus directed to *special* sets, still admitting better order Turán-Markov constants. It was a surprise when our preprint [14] surpassed $\sqrt{n}$ proving $n^{2/3}$ in general. In fact, there were serious people stating that this is the right order and that they have computed the then seemingly extremal, difficult to handle triangle having Turán-Markov constant of that order. As we could not reconstruct, could not conclude the allegedly working counterexample, we discussed the situation with Gábor Halász, who first also tried to fix the calculations, but then came up with the observation that our method, considering exactly normal lines to the selected maximal point $\zeta \in \partial K$ with $|p(\zeta)| = \|p\|_K$ is not optimal for the triangle.

He observed, that with ζ situated close (but not at) a vertex, the normal line provides a loss in the estimates, as the distance from zeroes lying possibly on (or close to) the (longer) part of the side of the triangle where ζ sits, grows, in view of the Pithagorean Theorem, only proportionally to δ^2, if δ is the distance, measured inward along the normal line, from ζ. For small δ and h (length of intersection of the normal line and K) this is a serious loss, compared to linear increase $c\delta$ if we can consider a *slanted* line, not normal, but tilted towards the short end of the side, where ζ sits (i.e towards the close vertex of the triangle). Actually, this observation gave almost immediately an order n Turán-Markov constant for the triangle, and finally proved to be equally powerful for the general case, too.

Let us thus go over the idea of the full proof now. Throughout we will assume, as we may, that K is also closed, hence a compact convex set with nonempty interior.

We start with picking up a boundary point $\zeta \in \partial K$ of maximality of $|p|$, and consider a supporting line at ζ to K. Our original argument of [14] then used a normal direction and compared values of p at ζ and on the intersection of K and this normal line.

Essential use is made of the fact that in case the length h of this intersection is small (relative to w), then, due to convexity, the normal line cuts K to halves unevenly: one part has to be small (of the order of h). That is, the situation in

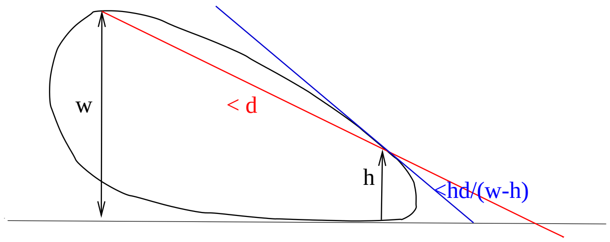

FIGURE 2. If the normal line is short, it cuts K to halves unevenly.

the general case is rather similar to the triangle – if the intersection of K with the normal is small, then one half of K, as divided by the normal, is altogether small. That was explicitly formulated in [14], and is used implicitly also in [17] through various calculations with the angles.

It is a major geometric feature at our help that when h is small, then one portion of K, cut into half by our slightly tilted line, is also small. This is the key feature which allows us to bend the direction of the normal a bit *towards the smaller portion* of K[5].

As said, in the proof written up in [13], we compare the values of p at ζ and *on a line slightly slanted off from the normal.* Comparing these calculations and the ones in [14] one can observe how this change led to a further, essential improvement of the result through improving the contribution of the factors belonging to zeroes z close to the supporting line. In [14] we could get a square term only (in terms of the distance $t < h$ we move away from the tangent point $\zeta = O$ along the normal), due to orthogonality and the consequent use of the Pythagorean Theorem in calculating the distances. However, here we obtain *linear dependence* in t via the general cosine theorem for the slanted segment J.

As a result of the improved estimates squeezed out this way, we do not need to employ the second usual technique, also going back to Turán, i.e. integration of $(p'/p)'$ over a suitably chosen interval. As pointed out already in [14], this part of the proof yields weaker estimates than cn, so avoiding it is not only a matter of convenience, but is an essential necessity.

The proof of the sharpness result Theorem **1**.10 also relies on the understanding of the geometry of the situation. Let us recall, how it starts.

Take $a, b \in K$ with $|a - b| = d$ and $m \in \mathbb{N}$ with $m > m_0$ to be determined later. Consider the polynomials $q(z) := (z-a)(z-b)$, $p(z) = (z-a)^m(z-b)^m =$

[5]If we try tilting the other way we would fail badly, even if the reader may find it difficult to distill from the proof where, and how. But if there were zeroes close to (or on) the supporting line and far from ζ *in the direction of the tilting*, then these zeroes were farther off from ζ, than from the other end of the intersecting segment. That would spoil the whole argument. However, since K is small in one direction of the supporting line, tilting towards this smaller portion does work.

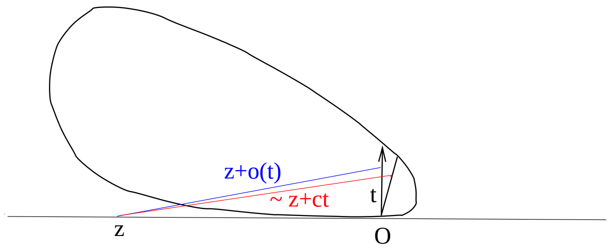

FIGURE 3. Tilting the normal line improves the growth of distance from zeroes z on or close to the supporting line to linear.

$q^m(z)$ and $P(z) = (z-a)^m(z-b)^{m+1} = (z-b)q^m(z)$. Clearly, $p, P \in \mathcal{P}_n^{(0)}(K)$ with $n = \deg p = 2m$ and $n = \deg P = 2m+1$, respectively. We claim that for appropriate choice of m_0 these polynomials satisfy inequality (1.15) for all $n > 2m_0$.

Without loss of generality we may assume $a = -1$, $b = 1$ and thus $d = 2$, as substitution by the linear function $\Phi(z) := \frac{2}{b-a}z - \frac{a+b}{b-a}$ shows. Indeed, if we prove the assertion for $\widetilde{K} := \Phi(K)$ and for $\widetilde{p}(z) = (z+1)^m(z-1)^m$, $\widetilde{P}(z) = (z+1)^m(z-1)^{m+1}$ defined on $\widetilde{K}$, we also obtain estimates for $p = \widetilde{p} \circ \Phi$ and $P = \widetilde{P} \circ \Phi$ on K. The homothetic factor of the inverse substitution Φ^{-1} is $\Lambda := \left|\frac{b-a}{2}\right| = d(K)/2$, and width changes according to $w(\widetilde{K}) = 2w(K)/d(K)$. Note also that under the linear substitution Φ the norms are unchanged but for the derivatives $\|p'\| = \Lambda^{-1}\|\widetilde{p}'\|$ and $\|P'\| = \Lambda^{-1}\|\widetilde{P}'\|$. So now we restrict to $a = -1$, $b = 1$, $d = 2$ and $q(z) := z^2 - 1$ etc.

First we make a few general observations. One obvious fact is that the imaginary axes separates $a = -1$ and $b = 1$, and as K is connected, it also contains some point $c = it$ of K. Therefore, $\|q\| \geq |q(c)| = 1 + t^2 \geq 1$. Also, it is clear that $q'(z) = 2z = (z-1) + (z+1)$: thus, by definition of the diameter

$$\|q'\| \leq \| |z-1| + |z+1| \| \leq 4 \ . \tag{2.1}$$

Let us put $w^+ := \sup_{z \in K} \Im z$ and $w^- := -\inf_{z \in K} \Im z$. We can estimate $w' := \max(w^+, w^-)$ from above by a constant times w. That is, we claim that for any point $\omega = \alpha + i\beta \in K$ we necessarily have $|\beta| \leq \sqrt{2}w$ and so the domain K lies in the rectangle $R := \mathrm{con}\{-1 - i\sqrt{2}w, 1 - i\sqrt{2}w, 1 + i\sqrt{2}w, -1 + i\sqrt{2}w\}$.

To see this first note that $\beta \leq \sqrt{3}$, since $d(K) = 2$ by assumption. Recalling (1.13), take $e^{i\gamma}$ be the direction of the minimal width of K: by symmetry, we may take $0 \leq \gamma < \pi$. Then there is a strip of width w and direction $ie^{i\gamma}$ containing K, hence also the segments $[-1, 1]$ and $[\alpha, \alpha + i\beta]$. It follows that $2|\cos\gamma| \leq w$ and $\beta \sin\gamma \leq w$. The second inequality immediately leads to $\beta \leq \sqrt{2}w$ if $\gamma \in [\pi/4, 3\pi/4]$. So let now $\gamma \in [0, \pi/4) \cup [3\pi/4, \pi)$, i.e. $|\cos\gamma| \geq 1/\sqrt{2}$.

Applying also $\beta \le \sqrt{3}$ now we deduce $\beta \le \sqrt{3} \le \sqrt{3/2}\,2|\cos\gamma| \le \sqrt{3/2}w$, whence the asserted $w^{\pm} \le \sqrt{2}w$ is proved.

The rest is a (tedious, delicate) computation of norms of the polynomials $\widetilde{p}, \widetilde{P}$ and their derivatives in the rectangle R. We spare the reader from details referring to [13] or [17].

3. Some geometrical notions

Let $\mathbb{R}^d$ be the usual Euclidean space of dimension d, equipped with the Euclidean distance $|\cdot|$. Our starting point is the following classical result of Blaschke [1, p. 116].

Theorem 3.1 (**Blaschke**). *Assume that the convex domain $K \subset \mathbb{R}^2$ has C^2 boundary $\Gamma = \partial K$ and that with the positive constant $\kappa_0 > 0$ the curvature satisfies $\kappa(\mathbf{z}) \le \kappa_0$ at all boundary points $\mathbf{z} \in \Gamma$. Then to each boundary points $\mathbf{z} \in \Gamma$ there exists a disk D_R of radius $R = 1/\kappa_0$, such that $\mathbf{z} \in \partial D_R$, and $D_R \subset K$.*

Note that the result, although seemingly local, does not allow for extensions to non-convex curves Γ. One can draw pictures of leg-bone like shapes of arbitrarily small upper bound of (positive) curvature, while at some points of touching containing arbitrarily small disks only. The reason is that the curve, after starting off from a certain boundary point $\mathbf{x}$, and then leaning back a bit, can eventually return arbitrarily close to the point from where it started: hence a prescribed size of disk cannot be inscribed.

On the other hand the Blaschke Theorem extends to any dimension $d \in \mathbb{N}$.

Also, the result has a similar, dual version, too, see [1, p. 116]. This was formulated already in Lemma **1**.3 above.

Now we start with introducing a few notions and recalling auxiliary facts. In §**4** we formulate and prove the two basic results – the discrete forms of the Blaschke Theorems. Then we show how our discrete approach yields a new, straightforward proof for a more involved sharpening of Theorem **3**.1, originally due to Strantzen.

Recall that the term planar *convex body* stands for a compact, convex subset of $\mathbb{C} \cong \mathbb{R}^2$ having nonempty interior. For a (planar) convex body K any interior point z defines a parametrization $\gamma(\varphi)$ – the usual polar coordinate representation of the boundary ∂K, – taking the unique point $\{z + te^{i\varphi} : t \in (0, \infty)\} \cap \partial K$ for the definition of $\gamma(\varphi)$. This defines the closed Jordan curve $\Gamma = \partial K$ and its parametrization $\gamma : [0, 2\pi] \to \mathbb{C}$. By convexity, from any boundary point $\zeta = \gamma(\theta) \in \partial K$, locally the chords to boundary points with parameter $< \theta$ or with $> \theta$ have arguments below and above the argument of the direction of any supporting line at ζ. Thus the tangent direction or argument function $\alpha_-(\theta)$

can be defined as e.g. the supremum of arguments of chords from the left; similarly, $\alpha_+(\theta) := \inf\{\arg(z-\zeta) \ : \ z=\gamma(\varphi),\ \varphi > \theta\}$, and any line $\zeta + e^{i\beta}\mathbb{R}$ with $\alpha_-(\theta) \le \beta \le \alpha_+(\theta)$ is a supporting line to K at $\zeta = \gamma(\theta) \in \partial K$. In particular the curve γ is differentiable at $\zeta = \gamma(\theta)$ if and only if $\alpha_-(\theta) = \alpha_+(\theta)$; in this case the tangent of γ at ζ is $\zeta + e^{i\alpha}\mathbb{R}$ with the unique value of $\alpha = \alpha_-(\theta) = \alpha_+(\theta)$. It is clear that interpreting $\alpha_\pm$ as functions on the boundary points $\zeta \in \partial K$, we obtain a parametrization-independent function. In other words, we are allowed to change parameterizations to arc length, say, when in case of $|\Gamma| = \ell$ ($|\Gamma|$ meaning the length of $\Gamma := \partial K$) the functions $\alpha_\pm$ map $[0,\ell]$ to $[0,2\pi]$.

Observe that $\alpha_\pm$ are nondecreasing functions with total variation $\mathrm{Var}\,[\alpha_\pm] = 2\pi$, and that they have a common value precisely at continuity points, which occur exactly at points where the supporting line to K is unique. At points of discontinuity $\alpha_\pm$ is the left-, resp. right continuous extension of the same function. For convenience, and for better matching with [3], we may even define the function $\alpha := (\alpha_+ + \alpha_-)/2$ all over the parameter interval.

For obvious geometric reasons we call the jump function $\beta := \alpha_+ - \alpha_-$ the *supplementary angle* function. In fact, β and the usual Lebesgue decomposition of the nondecreasing function α_+ to $\alpha_+ = \sigma + \alpha_* + \alpha_0$, consisting of the pure jump function σ, the nondecreasing singular component α_*, and the absolute continuous part α_0, are closely related. By monotonicity there are at most countable many points where $\beta(x) > 0$, and in view of bounded variation we even have $\sum_x \beta(x) \le 2\pi$, hence the definition $\mu := \sum_x \beta(x)\delta_x$ defines a bounded, non-negative Borel measure on $[0,2\pi)$. Now it is clear that $\sigma(x) = \mu([0,x])$, while $\alpha_*' = 0$ a.e., and α_0 is absolutely continuous. In particular, α or α_+ is differentiable at x provided that $\beta(x) = 0$ and x is not in the exceptional set of non-differentiable points with respect to α_* or α_0. That is, we have differentiability almost everywhere, and

$$\begin{aligned}&\int_x^y \alpha' = \alpha_0(y) - \alpha_0(x) = \lim_{z\to x-0} \alpha_0(y) - \alpha_0(z)\\ &= \lim_{z\to x-0} \{[\alpha_+(y) - \sigma(y) - \alpha_*(y))] - [\alpha_+(z) - \sigma(z) - \alpha_*(z)]\}\\ &= \alpha_+(y) - \beta(y) - \mu([x,y)) - \lim_{z\to x-0} \alpha_+(z) - \lim_{z\to x-0} [\alpha_*(y) - \alpha_*(z)] \le \alpha_-(y) - \alpha_+(x).\end{aligned} \tag{3.1}$$

It follows that

$$\alpha'(t) \ge \lambda \qquad \text{a.e.} \quad t \in [0,a] \tag{3.2}$$

holds true if and only if we have

$$\alpha_\pm(y) - \alpha_\pm(x) \ge \lambda(y-x) \qquad \forall x,y \in [0,a]\ . \tag{3.3}$$

Here we restricted ourselves to the arc length parametrization taken in positive orientation. Recall that one of the most important geometric quantities, curvature, is just $\kappa(s) := \alpha'(s)$, whenever parametrization is by arc length s.

Thus we can rewrite (3.2) as

$$\kappa(t) \geq \lambda \qquad \text{a.e.} \quad t \in [0,a] \ , \tag{3.4}$$

or, with radius of curvature $\rho(t) := 1/\kappa(t)$ introduced (writing $1/0 = \infty$),

$$\rho(t) \leq \frac{1}{\lambda} \qquad \text{a.e.} \quad t \in [0,a] \ . \tag{3.5}$$

Again, ρ is a parametrization-invariant quantity (describing the radius of the osculating circle). Actually, it is easy to translate all these conditions to arbitrary parametrization of the tangent angle function α. Since also curvature and radius of curvature are parametrization-invariant quantities, all the above hold for any parametrization.

Moreover, with a general parametrization let $|\Gamma(\eta,\zeta)|$ stand for the length of the counterclockwise arc $\Gamma(\eta,\zeta)$ of the rectifiable Jordan curve Γ between the two points $\zeta,\eta \in \Gamma = \partial K$. We can then say that the curve satisfies a Lipschitz-type increase or *subdifferential condition* whenever

$$|\alpha_{\pm}(\eta) - \alpha_{\pm}(\zeta)| \geq \lambda |\Gamma(\eta,\zeta)| \qquad (\forall \zeta, \eta \in \Gamma) \ , \tag{3.6}$$

here meaning by $\alpha_{\pm}(\xi)$, for $\xi \in \Gamma$, not values in $[0,2\pi)$, but a locally monotonously increasing branch of $\alpha_{\pm}$, with jumps in $(0,\pi)$, along the counterclockwise arc $\Gamma(\eta,\zeta)$ of Γ. Clearly, the above considerations show that all the above are equivalent.

In the paper we use the notation α (and also $\alpha_{\pm}$) for the tangent angle, κ for the curvature, and ρ for the radius of curvature. The counterclockwise taken right hand side tangent unit vector(s) will be denoted by $\mathbf{t}$, and the outer unit normal vectors by $\mathbf{n}$. These notations we will use basically in function of the arc length parametrization s, but with a slight abuse of notation also $\alpha_{-}(\varphi)$, $\mathbf{t}(\mathbf{x})$, $\mathbf{n}(\mathbf{x})$ etc. may occur with the obvious meaning.

Note that $\mathbf{t}(\mathbf{x}) = i\mathbf{n}(\mathbf{x}))$ and also $\mathbf{t}(\mathbf{x}) = \dot{\gamma}(s)$ when $\mathbf{x} = \mathbf{x}(s) \in \gamma$ and the parametrization/differentiation, symbolized by the dot, is with respect to arc length; moreover, with $\nu(s) : \arg(\mathbf{n}(\mathbf{x}(s))$ we obviously have $\alpha \equiv \nu + \pi/2 \mod 2\pi$ at least at points of continuity of α and ν. To avoid mod 2π equality, we can shift to the universal covering spaces and maps and consider $\widetilde{\alpha}, \widetilde{\nu}$, i.e. $\widetilde{\mathbf{t}}, \widetilde{\mathbf{n}}$ – e.g. in case of $\widetilde{\mathbf{n}}$ we will somewhat detail this right below. However, note a slight difference in handling α and $\widetilde{\mathbf{n}}$: the first is taken as a singlevalued function, with values $\alpha(s) := \frac{1}{2}\{\alpha_{-}(s) + \alpha_{+}(s)\}$ at points of discontinuity, while $\widetilde{\mathbf{n}}$ is a multivalued function attaining a full closed interval $[\widetilde{\mathbf{n}}_{-}(s), \widetilde{\mathbf{n}}_{+}(s)]$ whenever s is a point of discontinuity. Also recall that curvature, whenever it exists, is $|\ddot{\gamma}(s)| = \alpha'(s) = \widetilde{\mathbf{n}}'(s)$.

In this work we mean by a multi-valued function Φ from X to Y a (non-empty-valued) mapping $\Phi : X \to 2^Y \setminus \{\emptyset\}$, i.e. we assume that the domain of Φ is always the whole of X and that $\emptyset \neq \Phi(x) \subset Y$ for all $x \in X$. Recall the notions of modulus of continuity and minimal oscillation in the full generality of multi-valued functions between metric spaces.

Definition 3.1 (**modulus of continuity and minimal oscillation**)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. We call the *modulus of continuity* of the multivalued function Φ from X to Y the quantity

$$\omega(\Phi,\tau) := \sup\{d_Y(y,y') \ : \ x,x' \in X,\ d_X(x,x') \le \tau,\ y \in \Phi(x),\ y' \in \Phi(x')\}.$$

Similarly, we call *minimal oscillation* of Φ the quantity

$$\Omega(\Phi,\tau) := \inf\{d_Y(y,y') \ : \ x,x' \in X,\ d_X(x,x') \ge \tau,\ y \in \Phi(x),\ y' \in \Phi(x')\}.$$

If we are given a multi-valued *unit vector function* $\mathbf{v}(\mathbf{x}) : H \to 2^{S^{d-1}} \setminus \{\emptyset\}$, where $H \subset \mathbb{R}^d$ and S^{d-1} is the unit ball of $\mathbb{R}^d$, then the derived formulae become:

$$\omega(\tau){:=}\omega(\mathbf{v},\tau){:=}\sup\{\arccos\langle \mathbf{u},\mathbf{w}\rangle :\ \mathbf{x},\mathbf{y} \in H,\ |\mathbf{x}–\mathbf{y}| \le \tau,\ \mathbf{u}{\in} \ |!\mathbf{v}(\mathbf{x}), \mathbf{w}{\in}\mathbf{v}(\mathbf{y})\}, \tag{3.7}$$

and

$$\Omega(\tau){:=}\Omega(\mathbf{v},\tau){:=}\inf\{\arccos\langle \mathbf{u},\mathbf{w}\rangle : \mathbf{x},\mathbf{y} \in H,\ |\mathbf{x}-\mathbf{y}| \ge \tau,\ \mathbf{u}{\in}\mathbf{v}(\mathbf{x}),\ \mathbf{w}{\in}\mathbf{v}(\mathbf{y})\}. \tag{3.8}$$

For a *planar* multi-valued unit vector function $\mathbf{v} : H \to 2^{S^1} \setminus \{\emptyset\}$, where $H \subset \mathbb{R}^2 \simeq \mathbb{C}$ and S^1 is the unit circle in $\mathbb{R}^2$, we can parameterize the unit circle S^1 by the corresponding angle φ and thus write $\mathbf{v}(\mathbf{x}) = e^{i\Phi(\mathbf{x})}$ with $\Phi(\mathbf{x}) := \arg(\mathbf{v}(\mathbf{x}))$ being the corresponding angle. We will somewhat elaborate on this observation in the case when our multi-valued vector function is the outward normal vector(s) function $\mathbf{n}(\mathbf{x})$ of a closed convex curve.

Let γ be the boundary curve of a convex body in $\mathbb{R}^2$, which will be considered as oriented counterclockwise, and let the multivalued function $\mathbf{n}(\mathbf{x}) : \gamma \to 2^{S^1} \setminus \{\emptyset\}$ be defined as the set of all outward unit normal vectors of γ at the point $\mathbf{x} \in \gamma$. Observe that the set $\mathbf{n}(\mathbf{x})$ of the set of values of $\mathbf{n}$ at any $\mathbf{x} \in \gamma$ is either a point, or a closed segment of length less than π. Then there exists a unique lifting $\tilde{\mathbf{n}}$ of $\mathbf{n}$ from the universal covering space $\tilde{\gamma}(\simeq \mathbb{R}$, see below) of γ to the universal covering space $\mathbb{R} = \tilde{S^1}$ of S^1, with the respective universal covering maps $\pi_\gamma : \tilde{\gamma} \to \gamma$ and $\pi_{S^1} : \tilde{S^1} \to S^1$, with properties to be described below. Here we do not want to recall the concept of the universal covering spaces from algebraic topology in its generality, but restrict ourselves to give it in the situation described above. As already said, $\tilde{S^1} = \mathbb{R}$ and the corresponding universal covering map is $\pi_{S^1} : x \to (\cos x, \sin x)$ (We consider, as usual, S^1 as $\mathbb{R} \mod 2\pi$.) Similarly, for γ we have $\tilde{\gamma} = \mathbb{R}$, with universal covering map $\pi_\gamma : \mathbb{R} \to \gamma$ given in the following way. Let us fix some arbitrary point $\mathbf{x_0} \in \gamma$, (the following considerations will be independent of $\mathbf{x_0}$, in the natural sense). Let us denote by ℓ the length of γ. Then for $\lambda \in \mathbb{R} = \tilde{\gamma}$ we have that $\pi_\gamma(\lambda) \in \gamma$

is that unique point $\mathbf{x}$ of γ, for which the counterclockwise measured arc $\mathbf{x_0x}$ has a length $\lambda \mod \ell$.

Now we describe the postulates for the multivalued function $\tilde{\mathbf{n}} : \mathbb{R} = \tilde{\gamma} \to \tilde{S^1} = \mathbb{R}$, which determine it uniquely. First of all, we must have the equality $\pi_{S^1} \circ \tilde{\mathbf{n}} = \mathbf{n} \circ \pi_\gamma$, where $\circ$ denotes the composition of two multivalued functions. (In algebraic topology this is called *commutativity of a certain square of mappings.*) Second, the values of $\tilde{\mathbf{n}}$ must be either points or non-degenerate closed intervals (of length less than π; however this last property follows from the other ones). Third, $\tilde{\mathbf{n}}$ must be non-decreasing in the following sense: for $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 < \lambda_2$ we have $r_1 \in \tilde{\mathbf{n}}(\lambda_1), r_2 \in \tilde{\mathbf{n}}(\lambda_2) \implies r_1 \le r_2$. Further, $\tilde{\mathbf{n}}$ must be a non-decreasing multivalued function, continuous from the left, i.e., for any $\lambda \in \mathbb{R}$ we have that for any $\varepsilon > 0$ there exists a $\delta > 0$, such that $\cup_{\mu \in (\lambda-\delta,\lambda)} \tilde{\mathbf{n}}(\mu) \subset (\min \tilde{\mathbf{n}}(\lambda) - \varepsilon, \min \tilde{\mathbf{n}}(\lambda))$. Analogously, $\tilde{\mathbf{n}}$ must be a non-decreasing multi-valued function continuous from the right, i.e., for any $\lambda \in \mathbb{R}$ we have that for any $\varepsilon > 0$ there exists a $\delta > 0$, such that $\cup_{\mu \in (\lambda,\lambda+\delta)} \tilde{\mathbf{n}}(\mu) \subset (\max \tilde{\mathbf{n}}(\lambda), \max \tilde{\mathbf{n}}(\lambda) + \varepsilon)$. These are all the postulates for the multi-valued function $\tilde{\mathbf{n}}$. It is clear, that $\tilde{\mathbf{n}}$ exists and is uniquely determined, for fixed $\mathbf{x_0}$ (and, for $\mathbf{x_0}$ arbitrary, only the parametrization of $\mathbb{R} = \tilde{\gamma}$ changes, by a translation.)

The above listed properties imply still one important property of the multi-valued function $\tilde{\mathbf{n}}$: we have for any $\lambda \in \mathbb{R}$ that $\tilde{\mathbf{n}}(\lambda + \ell) = \tilde{\mathbf{n}}(\lambda) + 2\pi$.

Definition 3.2. We define the modulus of continuity of the multi-valued normal vector function $\mathbf{n}(\mathbf{x})$ *with respect to arc length* as the (ordinary) modulus of continuity of the multi-valued lift-up function $\tilde{\mathbf{n}} : \mathbb{R} \to \mathbb{R} \setminus \{\emptyset\}$, i.e. as

$$\begin{aligned}\tilde{\omega}(\tau) &:= \tilde{\omega}(\mathbf{n}, \tau) := \omega(\tilde{\mathbf{n}}, \tau) \\ &:= \sup\{|r_1 - r_2| \mid r_1 \in \tilde{\mathbf{n}}(\lambda_1), r_2 \in \tilde{\mathbf{n}}(\lambda_2),\ \lambda_1, \lambda_2 \in \mathbb{R}, |\lambda_1 - \lambda_2| \le \tau\}.\end{aligned} \tag{3.9}$$

Similarly, we define the minimal oscillation of the multi-valued normal vector function $\mathbf{n}(\mathbf{x})$ *with respect to arc length* as the (ordinary) minimal oscillation function of $\tilde{\mathbf{n}}$, i.e. as

$$\begin{aligned}\tilde{\Omega}(\tau) &:= \tilde{\Omega}(\mathbf{n}, \tau) := \Omega(\tilde{\mathbf{n}}, \tau) \\ &:= \inf\{|r_1 - r_2| \mid r_1 \in \tilde{\mathbf{n}}(\lambda_1), r_2 \in \tilde{\mathbf{n}}(\lambda_2),\ \lambda_1, \lambda_2 \in \mathbb{R}, |\lambda_1 - \lambda_2| \ge \tau\}.\end{aligned} \tag{3.10}$$

By writing "modulus of continuity" we do not mean to say anything like continuity of $\tilde{\mathbf{n}}$. In fact, if for some $\lambda \in \mathbb{R}$ $\tilde{\mathbf{n}}(\lambda)$ is a non-degenerate closed segment, then the left-hand side and right-hand side limits of $\tilde{\mathbf{n}}$ at λ - in the sense of the definition of continuity from the left or right, respectively - are surely different.

We evidently have that the modulus of continuity of $\tilde{\mathbf{n}}$ is subadditive, meaning $\tilde{\omega}(\tau_1+\tau_2) \le \tilde{\omega}(\tau_1)+\tilde{\omega}(\tau_2)$, and similarly, that the minimal oscillation of $\tilde{\mathbf{n}}$ is superadditive, meaning $\tilde{\Omega}(\tau_1+\tau_2) \ge \tilde{\Omega}(\tau_1)+\tilde{\Omega}(\tau_2)$. In fact, a standard property of the modulus of continuity of *any (non-empty valued) multivalued function from* $\mathbb{R}$ *(or from any convex set, in the sense of metric intervals) to* $\mathbb{R}$ is subadditivity, and similarly, minimal oscillation of such a function is superadditive. These properties with non-negativity and non-decreasing property also imply that $\tilde{\omega}(\tau)/\tau$ and $\tilde{\Omega}(\tau)/\tau$ have limits when $\tau \to 0$; moreover, $\lim_{\tau\to 0}\tilde{\omega}(\tau)/\tau = \sup \tilde{\omega}(\tau)/\tau$ and $\lim_{\tau\to 0}\tilde{\Omega}(\tau)/\tau = \inf \tilde{\Omega}(\tau)/\tau$. Note that metric convexity is essential here, so e.g. it is not clear if in $\mathbb{R}^d$ any proper analogy could be established.

Observe that if the curvature of γ exists at $\mathbf{x}_0$, then for the non-empty valued multi-valued function $\mathbf{n}(\mathbf{x})$:="set of values of all outer unit normal vectors of γ at $\mathbf{x}$", we necessarily have $\#\mathbf{n}(\mathbf{x}_0)=1$ and the curvature can be written as

$$\kappa(\mathbf{x}_0) = \lim_{\mathbf{y}\to\mathbf{x}_0\ \mathbf{v}\in\mathbf{n}(\mathbf{y})} \frac{\arccos\langle \mathbf{n}(\mathbf{x}_0), \mathbf{v}\rangle}{|\mathbf{x}_0-\mathbf{y}|}, \tag{3.11}$$

where the limit in (3.11) exists with arbitrary choice of $\mathbf{v} \in \mathbf{y}$ and is independent of this choice.

The next two propositions are well-known. We, however, detail their proof in [12] and also in [17] for self-contained presentation, which we do not aim at here.

Proposition 3.1. *Let γ be a planar convex curve. Recall that* (3.7) *and* (3.8) *is the modulus of continuity and the minimal oscillation of the multi-valued normal vector function* $\mathbf{n}(\mathbf{x})$ *with respect to chord length, and that* (3.9) *and* (3.10) *stand for the modulus of continuity and the minimal oscillation of* $\mathbf{n}(\mathbf{x})$ *with respect to arc length. Then for all* $\mathbf{x}\in\gamma$ *with curvature* $\kappa(\mathbf{x})\in[0,\infty]$ *we have*

$$\lim_{\tau\to 0}\frac{\Omega(\tau)}{\tau} = \lim_{\tau\to 0}\frac{\tilde{\Omega}(\tau)}{\tau} \le \kappa(\mathbf{x}) \le \lim_{\tau\to 0}\frac{\tilde{\omega}(\tau)}{\tau} = \lim_{\tau\to 0}\frac{\omega(\tau)}{\tau}. \tag{3.12}$$

In the following proposition arccos will denote the branch with values in $[0,\pi]$.

Proposition 3.2. *Let γ be a closed convex curve, and* (3.7) *and* (3.8) *be the modulus of continuity and the minimal oscillation of the (in general, multi-valued) unit normal vector function* $\mathbf{n}(\mathbf{x})$*.*

(i) If the curvature exists and is bounded from above by κ_0 all over γ, then there exists a bound $\tau_0 > 0$ so that for any two points $\mathbf{x},\mathbf{y}\in\gamma$ with $|\mathbf{x}-\mathbf{y}| \le$

$\tau \le \tau_0$ we must have $\omega(\mathbf{n},\tau) < \pi/2$ and $\arccos\langle \mathbf{n}(\mathbf{x}), \mathbf{n}(\mathbf{y})\rangle \le \kappa_0\tau/\cos(\omega(\mathbf{n},\tau))$*. Thus we also have $\omega(\mathbf{n},\tau) \le \kappa_0\tau/\cos(\omega(\mathbf{n},\tau))$ for $\tau \le \tau_0$.*

(ii) If the curvature $\kappa(\mathbf{x})$ exists (linearly, that is, according to arc length parametrization) almost everywhere, and is bounded from below by κ_0 (linearly) almost everywhere on γ, then for any two points $\mathbf{x}, \mathbf{y} \in \gamma$ with $|\mathbf{x}-\mathbf{y}| \ge \tau$ and for all $\mathbf{u} \in \mathbf{n}(\mathbf{x})$,$\mathbf{v} \in \mathbf{n}(\mathbf{y})$ we have $\arccos\langle \mathbf{u}, \mathbf{v}\rangle \ge \kappa_0\tau$ *and hence $\Omega(\mathbf{n},\tau) \ge \kappa_0\tau$.*

Rotations of $\mathbb{C} = \mathbb{R}^2$ about the origin O by the counterclockwise measured (positive) angle φ will be denoted by U_φ, that is,

$$U_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}. \tag{3.13}$$

We denote T the reflection to the y-axis, i.e. the linear mapping defined by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

Definition 3.3 (**Mangled n-gons**). Let $2 \le k \in \mathbb{N}$ and put $n = 4k-4$, $\varphi^* := \frac{\pi}{2k}$. We define the *standard mangled n-gon* as the convex n-gon

$$M_k := \operatorname{con}\{A_1, \dots, A_{k-1}, A_{k+1}, \dots, A_{2k-1}, A_{2k+1}, \dots, A_{3k-1}, A_{3k+1}, \dots, A_{4k-1}\}, \tag{3.14}$$

of $n = 4k-4$ vertices with

$$A_m := \left(\sum_{j=1}^{m} \cos(j\varphi^*) - \sum_{\ell=1}^{\lfloor m/k \rfloor} \cos(\ell k \varphi^*), \sum_{j=1}^{m} \sin(j\varphi^*) - \sum_{\ell=1}^{\lfloor m/k \rfloor} \sin(\ell k \varphi^*) \right), \tag{3.15}$$

where $m \in \{1, \dots, 4k\} \setminus \{k, 2k, 3k, 4k\}$. That is, we consider a regular $4k$-gon of unit sides, but cut out the middle "cross-shape" (i.e., the union of two rectangles which are the convex hulls of two opposite sides of the regular $4k$-gon, these pairs of opposite sides being perpendicular to each other) and push together the left over four quadrants (i.e., shift the vertices $A_{\ell k}$ to the position of $A_{\ell k-1}$ consecutively to join the remaining sides of the polygon. Observe that taking $A_0 := O$, the same formula (3.15) is valid also for $A_0 := O = A_{4k} = A_{4k-1}$ and $A_{\ell k} = A_{\ell k-1}$, $\ell = 1, 2, 3, 4$, showing how the vertices of the regular $4k$-gon were moved into their new positions.)

Now let $\tau > 0$, $\alpha \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^2$ and $\varphi \in (0, \pi/4]$ be arbitrary. Take $k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor$, so that $\varphi^* := \frac{\pi}{2k} \ge \varphi$.

Then we write $M(\varphi) := M_k$, and, moreover, we also define

$$M(\mathbf{x}, \alpha, \varphi, \tau) := M(\mathbf{x}, \alpha, \varphi^*, \tau) := U_\alpha(\tau M_k) + \mathbf{x}, \tag{3.16}$$

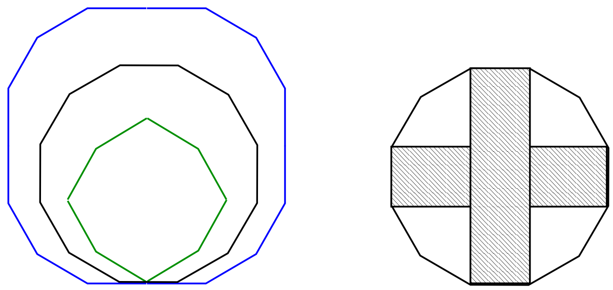

FIGURE 4. Left: The regular, mangled and fattened n-gons with $n = 12$. For $n \to \infty$ the sizes are all closer and closer to each other and the same circle. Right: The shaded "middle cross" of the regular n-gon is either cut out (in the mangled) or doubled (in the fattened) n-gon.

that is, the copy shifted by $\mathbf{x}$ of the $4k-4$-gon obtained by dilating $M(\varphi) = M_k$ from $O = A_0 = A_{4k-1}$ with τ and rotating it counterclockwise about O by the angle α.

E.g. if $\varphi \in (\pi/6, \pi/4]$, then $k = 2$, $\varphi^* = \pi/4$, $n = 4$, and M_2 is just a unit square, its side lines having direction tangents ± 1 and having its lowest vertex at O. It is the left over part, pushed together, of a regular octagon of unit side length, when the middle cross-shape is removed from its middle.

It is easy to see that the inradius $\rho(\varphi)$ and the circumradius $R(\varphi)$ of $M(\varphi) = M(\varphi^*) = M_k$ are

$$\begin{cases} r(\varphi) & = \frac{1}{2}\left\{\cot\frac{\pi}{4k} - \sqrt{2}\cos\left(\frac{1-(-1)^k}{8k}\pi\right)\right\}, \\ R(\varphi) & = \frac{1}{2}\left\{\cot\frac{\pi}{4k} - 1\right\}, \end{cases} \qquad \left(k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor\right), \tag{3.17}$$

respectively.

Similarly to the *mangled n-gons* M_k, we also define the *fattened n-gons* F_k.

Definition 3.4 (Fattened n-gons). Let $k \in \mathbb{N}$ and put $n = 4k$, $\varphi^* := \frac{\pi}{2k}$. We first define the *standard fattened n-gon* as the convex n-gon

$$F_k := \operatorname{con}\{A_1, \dots, A_{k-1}, A_k, A_{k+1}, \dots, A_{4k-1}, A_{4k}\}, \tag{3.18}$$

of $n = 4k$ vertices with

$$A_m := \left(\sum_{j=1}^{m} \cos(j\varphi^*) + \sum_{\ell=0}^{\lfloor m/k \rfloor} \cos(\ell k \varphi^*), \sum_{j=1}^{m} \sin(j\varphi^*) + \sum_{\ell=0}^{\lfloor m/k \rfloor} \sin(\ell k \varphi^*) \right). \tag{3.19}$$

That is, we consider a regular $4k$-gon , but fatten the middle "cross-shape" to twice as wide, and move the four quadrants to the corners formed by this width-doubled cross (i.e., shift the vertices $A_{\ell k}$ to the position of $A_{\ell k-1}+2(A_{\ell k}-A_{\ell k-1})$ consecutively to join the remaining sides of the polygon). Observe that $A_{4k-1} = (-1,0)$ and $A_{4k} = (1,0)$.

Let $\tau > 0$, $\alpha \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^2$ and $\varphi \in (0,\pi)$ be arbitrary. Now we take $k := \left\lceil \frac{\pi}{2\varphi} \right\rceil$, whence $\varphi^* := \frac{\pi}{2k} \leq \varphi$.

Then we write $F(\varphi) := F_k$, and, moreover, we also define

$$F(\mathbf{x},\alpha,\varphi,\tau) := F(\mathbf{x},\alpha,\varphi^*,\tau) := U_\alpha\left(\tau F_k\right) + \mathbf{x}, \tag{3.20}$$

that is, the copy shifted by $\mathbf{x}$ of the $4k$-gon obtained by dilating $F(\varphi) = F_k$ from O with τ and rotating it counterclockwise about O by the angle α.

E.g. if $\varphi \geq \pi/2$, then $k = 1$, $\varphi^* = \pi/2$, $n = 4$, and F_4 is just the square spanned by the vertices (1,0), (1,2), (-1,2), (-1,0) and having sides of length 2.

Observe that using the usual Minkowski addition, we can represent the connections of these deformed n-gons and the regular n-gon easily. Write Q_n for the regular n-gon placed symmetrically to the y-axis but above the x-axis with $O \in \partial Q_n$ a midpoint (hence not a vertex) of a side of Q_n. (This position is uniquely determined.) Also, denote the standard square as $S := Q_4 := \operatorname{con}\{(1/2,0);(1/2,1);(-1/2,1);(-1/2,0)\}$. Then we have $M_k + S = Q_{4k}$ and $Q_{4k} + S = F_k$.

It is also easy to see that the inradius $\mathfrak{r}(\varphi)$ and the circumradius $\mathfrak{R}(\varphi)$ of $F(\varphi) = F(\varphi^*)$ are

$$\mathfrak{r}(\varphi) = \frac{1}{2}\cot\frac{\pi}{4k} + \frac{1}{2} \qquad \left(k := \left\lceil \frac{\pi}{2\varphi} \right\rceil\right), \tag{3.21}$$

and

$$\mathfrak{R}(\varphi) = \begin{cases} \frac{1}{2\sin\frac{\pi}{4k}} + \frac{1}{\sqrt{2}} & \text{if } 2 \mid k \\ \sqrt{\frac{1}{2} + \frac{1}{4\sin^2\frac{\pi}{4k}} + \frac{1}{\sqrt{2}}\cot\frac{\pi}{4k}} & \text{if } 2 \mid k \end{cases} \qquad \left(k := \left\lceil \frac{\pi}{2\varphi} \right\rceil\right), \tag{3.22}$$

respectively.

The actual values of the above in- and circumradii in (3.17), (3.21), (3.22) are not important, but observe that for $\varphi \to 0$, or, equivalently, for $k \to \infty$, we have the asymptotic relation $r(\varphi) \sim R(\varphi) \sim \mathfrak{r}(\varphi) \sim \mathfrak{R}(\varphi) \sim \frac{1}{\varphi}$.

4. Discrete versions of the Blaschke Rolling Ball Theorems

Our further results will all be derived from various extensions and strengthening of the Blaschke Theorem. In this section we skip the tedious, elaborate proofs, to be found in [12] and also in [17]. However, we list the geometry results, which may be of independent interest. At least we know of other useful

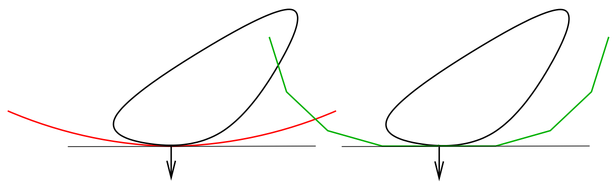

FIGURE 5. Outscribed circles and n-gons at a boundary point as provided by the smooth and discrete dual Blaschke Theorems.

applications of them in analytic problems, so it may prove to be useful elsewhere for others, too.

Theorem 4.1. *Let $K \subset \mathbb{C}$ be a convex body and $0 < \varphi < \pi/4$. Denote $\mathbf{n}$ the (multivalued) function of outer unit normal(s) to the closed convex curve $\gamma := \partial K$ and assume that $\omega(\mathbf{n}, \tau) \leq \varphi < \pi/4$. Put $k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor$. If $\mathbf{x} \in \partial K = \gamma$, and $\mathbf{n}_0 = (\sin\alpha, -\cos\alpha) \in \mathbf{n}(\mathbf{x})$ is outer unit normal to γ at $\mathbf{x}$, then $M(\mathbf{x}, \alpha, \varphi, \tau) \subset K$.*

An even stronger version can be proved considering the modulus of continuity $\tilde{\omega}$ with respect to arc length. We thank this sharpening to Endre Makai, who kindly called our attention to this possibility and suggested the crucial Lemma for the proof.

Theorem 4.2. *Let $K \subset \mathbb{C}$ be a planar convex body and $0 < \varphi < \pi/4$. Denote $\mathbf{n}$ the (multivalued) function of outer unit normal(s) to the closed convex curve $\gamma := \partial K$ and assume that $\tilde{\omega}(\tau) \leq \varphi < \pi/4$. Put $k := \left\lfloor \frac{\pi}{2\varphi} \right\rfloor$. If $\mathbf{x} \in \partial K = \gamma$, and $\mathbf{n}_0 = (\sin\alpha, -\cos\alpha) \in \mathbf{n}(\mathbf{x})$ is outer unit normal to γ at $\mathbf{x}$, then $M(\mathbf{x}, \alpha, \varphi, \tau) \subset K$.*

Finally, as in case of the classical Blaschke theorems, there is a dual version of all these considerations. The result is this.

Theorem 4.3. *Let $K \subset \mathbb{C}$ be a (planar) convex body and $\tau > 0$. Denote $\mathbf{n}$ the (multivalued) function of outer unit normal(s) to the closed convex curve $\gamma := \partial K$ and assume that $\Omega(\mathbf{n}, \tau) \geq \varphi$. Take $k := \left\lceil \frac{\pi}{2\varphi} \right\rceil$. If $\mathbf{x} \in \partial K = \gamma$, and $\mathbf{n}_0 = (\sin\alpha, -\cos\alpha) \in \mathbf{n}(\mathbf{x})$ is normal to γ at $\mathbf{x}$, then $F(\mathbf{x}, \alpha, \varphi, \tau) \supset K$.*

This is the version we actually make use of in this work, see Figure **4**.

5. Extensions of the Blaschke Rolling Ball Theorem

As the first corollaries, we can immediately deduce the classical Blaschke theorems. We denote by $D(\mathbf{x}, r)$ the closed disc of center $\mathbf{x}$ and radius r.

Proof of Theorem 3.1. Let τ_0 be the bound provided by (i) of Proposition **3**.2. Under the condition, we find (with $\omega(\mathbf{n}, \tau) < \pi/2$)

$$\omega(\mathbf{n}, \tau) \leq \frac{\kappa_0 \tau}{\cos(\omega(\mathbf{n}, \tau))} =: \varphi(\tau) \qquad (\tau \leq \tau_0). \tag{5.1}$$

Let us apply Theorem **4**.1 for the boundary point $\mathbf{x} \in \gamma$ with normal vector $\mathbf{n}(\mathbf{x}) = (\sin\alpha, -\cos\alpha)$. If necessary, we have to reduce τ so that the hypothesis $\varphi(\tau) \leq \pi/4$ should hold. We obtain that the congruent copy $U_\alpha(\tau M_k) + \mathbf{x}$ of τM_k is contained in K, where $k = \lfloor \pi/2\varphi(\tau) \rfloor$. Note that $U_\alpha(\tau M_k) + \mathbf{x} \supset D(\mathbf{z}, \tau r(\varphi(\tau)))$, where $\mathbf{z} = \mathbf{x} - \tau R(\varphi(\tau))\mathbf{n}(\mathbf{x})$. When $\tau \to 0$, also $\varphi(\tau) \to 0$, therefore also $\omega(\mathbf{n}, \tau) \to 0$ in view of (5.1), and we see

$$\lim_{\tau\to 0} (\tau R(\varphi(\tau))) = \lim_{\tau\to 0} (\tau r(\varphi(\tau))) = \lim_{\tau\to 0} \frac{\tau}{\varphi(\tau)} = \lim_{\tau\to 0} \frac{\cos(\omega(\mathbf{n}, \tau))}{\kappa_0} = \frac{1}{\kappa_0}.$$

Note that we have made use of $\omega(\mathbf{n}, \tau) \to 0$ in the form $\cos(\omega(\mathbf{n}, \tau)) \to 1$. It follows that $D(\mathbf{x} - \frac{1}{\kappa_0}\mathbf{n}(\mathbf{x}), \frac{1}{\kappa_0}) \subset K$, whence the assertion. □

Note that in the above proof of Theorem **3**.1 we did not assume C^2-boundary, as is usual, but only the existence of curvature and the estimate $\kappa(\mathbf{x}) \leq \kappa_0$. So we found the following stronger corollary (still surely well-known).

Corollary 5.1. *Assume that $K \subset \mathbb{R}^2$ is a convex domain with boundary curve γ, that the curvature κ exists all over γ, and that there exists a positive constant $\kappa_0 > 0$ so that $\kappa \leq \kappa_0$ everywhere on γ. Then to all boundary point $\mathbf{x} \in \gamma$ there exists a disk D_R of radius $R = 1/\kappa_0$, such that $\mathbf{x} \in \partial D_R$, and $D_R \subset K$.*

Similarly, one can deduce also the "dual" Blaschke theorem, i.e. Lemma**1**.3, in a similarly strengthened form. In fact, the conditions can be relaxed even further, as was shown by Strantzen, see [3, Lemma 9.11]. Our discrete approach easily implies Strantzen's strengthened version, originally obtained along different lines.

Corollary 5.2 (Strantzen). *Let $K \subset \mathbb{R}^2$ be a convex body with boundary curve γ. Assume that the (linearly) a.e. existing curvature κ of γ satisfies $\kappa \geq \kappa_0$ (linearly) a.e. on γ. Then to all boundary point $\mathbf{x} \in \gamma$ there exists a disk D_R of radius $R = 1/\kappa_0$, such that $\mathbf{x} \in \partial D_R$, and $K \subset D_R$.*

Proof. Now we start with (ii) of Proposition **3**.2 to obtain $\Omega(\tau) \geq \kappa_0 \tau$ for all τ. Put $\varphi := \varphi(\tau) := \kappa_0 \tau$. Clearly, when $\tau \to 0$, then also $\varphi(\tau) \to 0$ and $k := \lceil \pi/(2\varphi(\tau)) \rceil \to \infty$. Take $\mathbf{n}(\mathbf{x}) = (\cos\alpha, \sin\alpha)$ and apply Theorem **4**.3 to obtain $U_\alpha(\tau F_k) + \mathbf{x} \supset K$ for all $\tau > 0$. Observe that $D_\varphi := D((0, \mathfrak{r}(\varphi)), \mathfrak{R}(\varphi)) \supset F_k$, hence $U_\alpha(\tau D_\varphi) + \mathbf{x} \supset K$. In the limit, since $\mathfrak{r}(\varphi(\tau)) \sim \mathfrak{R}(\varphi(\tau)) \sim 1/(\varphi(\tau)) = 1/(\kappa_0 \tau)$, we find $D(\mathbf{x} - (1/\kappa_0)\mathbf{n}, 1/\kappa_0) \supset K$, for any $\mathbf{n} \in \mathbf{n}(\mathbf{x})$, that implies the statement. □

6. Further results for non-flat convex domains

The above Theorem **1**.3 was formulated with very precise constants. In particular, it gives a good description of the "inverse Markov factor"

$$M(E_b) := \inf_{p \in \mathcal{P}_n(E_b)} M(p),$$

when n is fixed and $b \to 0$. In this section we aim at a precise generalization of Theorem **1**.3 using appropriate geometric notions. Our argument stems out of the notion of "circular sets", used in [8] and going back to Turán's work. This approach can indeed cover the full content of Theorem **1**.3. Moreover, the geometric observation and criteria we present will cover a good deal of different, not necessarily smooth domains. First let us have a recourse to Theorem **1**.5.

Theorem 6.1. *Let $K \subset \mathbb{C}$ be any convex domain with C^2-smooth boundary curve $\partial K = \Gamma$ having curvature $\kappa(\zeta) \geq \kappa$ with a certain constant $\kappa > 0$ and for all points $\zeta \in \Gamma$. Then $M(K) \geq (\kappa/2)n$.*

Proof. The proof hinges upon geometry in a large extent. For this smooth case we use Blaschke's Rolling Ball Theorem, i.e. Lemma **1**.3. This means, with our definition above, that if the curvature of the boundary curve of a twice differentiable convex body exceeds $1/R$, then the convex body is R-circular. From this an application of Theorem **1**.4 yields the assertion. □

So now it is worthy to calculate the curvature of ∂E_b.

Lemma 6.1. *Let E_b be the ellipse with major axes $[-1, 1]$ and minor axes $[-ib, ib]$. Consider its boundary curve Γ_b. Then at any point of the curve the curvature is between b and $1/b^2$.*

Proof. Now we depart from arc length parameterization and use for $\Gamma_b := \partial E_b$ the parameterization $\gamma(\varphi) := (\cos(\varphi), b\sin(\varphi))$. Then we have

$$\kappa(\gamma(\varphi)) = \frac{|\dot\gamma(\varphi) \times \ddot\gamma(\varphi)|}{|\dot\gamma(\varphi)|^3} ,$$

that is,

$$\begin{aligned}\kappa(\gamma(\varphi)) &= \frac{|(-\sin\varphi, b\cos\varphi)\times(-\cos\varphi, -b\sin\varphi)|}{|(-\sin\varphi, b\cos\varphi)|^3}\\ &= \frac{b\sin^2\varphi + b\cos^2\varphi}{(\sin^2\varphi + b^2\cos^2\varphi)^{3/2}}\\ &= \frac{b}{(\sin^2\varphi + b^2\cos^2\varphi)^{3/2}}\ .\end{aligned}$$

Clearly, the denominator falls between $(b^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2} = b^3$ and $(\sin^2\varphi+\cos^2\varphi)^{3/2} = 1$, and these bounds are attained, hence $\kappa(\gamma(\varphi)) \in [b, 1/b^2]$ whenever $b \le 1$. □

Proof of Theorem 1.3. The curvature of Γ_b at any of its points is at least b according to Lemma **6**.1. Hence $M(E_b) \ge (b/2)n$ in view of Theorem **6**.1, and Theorem **1**.3 follows. □

However, not only smooth convex domains can be proved to be circular. E.g. it is easy to see that if a domain is the intersection of finitely many R-circular domains, then it is also R-circular. The next generalization is not that simple, but is still true.

Lemma 6.2 (**Strantzen**). *Let the convex domain K have boundary $\Gamma = \partial K$ with angle function $\alpha_\pm$ and let $\kappa > 0$ be a fixed constant. Assume that $\alpha_\pm$ satisfies the curvature condition $\kappa(s) = \alpha'(s) \ge \kappa$ almost everywhere. Then K is $R = 1/\kappa$-circular.*

Proof. This result is essentially the far-reaching, relatively recent generalization of Blaschke's Rolling Ball Theorem by Strantzen, i.e. Corollary **5**.2 above. The only slight alteration from the standard formulation in [3], suppressed in the above quotations, is that Strantzen's version assumes $\kappa(t) \ge \kappa$ wherever the curvature $\kappa(t) = \alpha'(t)$ exists (so almost everywhere for sure), while above we stated the same thing for almost everywhere, but not necessarily at every points of existence. This can be overcome by reference to the subdifferential version, too. □

Now we are in an easy position to prove Theorem **1**.11.

Proof of Theorem 1.11. The proof follows from a combination of Theorem **1**.4 and Lemma **6**.2. □

Let us illustrate the strengths and weaknesses of the above results on the following instructive examples, suggested to us by J. Szabados (personal communication). Consider for any $1 < p < \infty$ the ℓ_p unit ball

$$B^p := \{(x,y) : |x|^p + |y|^p \le 1\}, \qquad \Gamma^p := \partial B^p = \{(x,y) : |x|^p + |y|^p = 1\}. \tag{6.1}$$

Also, let us consider for any parameter $0 < b \le 1$ the affine image (”ℓ_p-ellipse”)

$$B^p_b := \{(x,y) : |x|^p + |y/b|^p \le 1\}, \qquad \Gamma^p_b := \partial B^p_b = \{(x,y) : |x|^p + |y/b|^p = 1\}. \tag{6.2}$$

By symmetry, it suffices to analyze the boundary curve $\Gamma := \Gamma^p_b$ in the positive quadrant. Here it has a parametrization $\Gamma(x) := (x, y(x))$, where $y(x) = b\,(1-x^p)^{1/p}$. As above, the curvature of the general point of the arc in the positive quadrant can be calculated and we get

$$\kappa(x) = \frac{(p-1)bx^{p-2}(1-x^p)^{1/p-2}}{\left(1 + b^2x^{2p-2}(1-x^p)^{2/p-2}\right)^{3/2}} \tag{6.3}$$

For $p > 2$, the curvature is continuous, but it does not stay off 0: e.g. at the upper point $x = 0$ it vanishes. Therefore, neither Theorem **6**.1 nor Theorem **1**.11 can provide any bound, while Theorem **1**.9 provides an estimate, even if with a small constant: here $d(B) = 2$, $w(B) = 2b$, and we get $M(B) \ge 0.00015bn$.

When $p = 2$, we get back the disk and the ellipses: the curvature is minimal at $\pm ib$, and its value is b there, hence $M(B) \ge (b/2)n$, as already seen in Theorem **1**.3. On the other hand Theorem **1**.9 yields only $M(B) \ge 0.00015bn$ also here.

For $1 < p < 2$ the situation changes: the curvature becomes infinite at the ”vertices” at $\pm ib$ and ± 1, and the curvature has a positive minimum over the curve Γ. When $b = 1$, it is possible to explicitly calculate it, since the role of x and y is symmetric in this case and it is natural to conjecture that minimal curvature occurs at $y = x$; using geometric-arithmetic mean and also the inequality between power means (i.e. Cauchy-Schwartz), it is not hard to compute $\min \kappa(x,y) = (p-1)2^{1/p-1/2}$, (which is the value attained at $y = x$). Hence Theorem **1**.11 (but not Theorem **6**.1, which assumes C^2-smoothness, violated here at the vertices!) provides $M(B^p) \ge (p-1)2^{1/p-3/2}n$, while Theorem **1**.9 provides, in view of $w(B^p) = 2^{3/2-1/p}$, something like $M(B^p) \ge 0.0003\,2^{-1/2-1/p}n \ge 0.0001n$, which is much smaller until p comes down very close to 1.

For general $0 < b < 1$ we obviously have $d(B) = 2$, $(\sqrt{2}b <)2b/\sqrt{1+b^2} < w(B) < 2b$, and Theorem **1**.9 yields $M(B) \ge 0.0001bn$ independently of the value of p.

Now $\min \kappa$ can be estimated within a constant factor (actually, when $b \to 0$, even asymptotically precisely) the following way. On the one hand, taking $x_0 := 2^{-1/p}$ leads to $\kappa(x_0) = (p-1)b2^{1+1/p}/(1+b^2)^{3/2} < b(p-1)2^{1+1/p}$, hence

$\min\kappa(x < b(p-1)2^{1+/p}$. Note that when $b \to 0$, we have asymptotically $\kappa(x-0) \sim b(p-1)2^{1+/p}$. On the other hand denoting $\xi := x^p$ and $\beta := 2/p-1 \in (0,1)$, from (6.3) we get

$$\begin{aligned}\frac{(p-1)b}{\kappa(x)} &= [\xi(1-\xi)]^\beta \left[\xi^{1-\beta} + b^2(1-\xi)^{1-\beta}\right]^{3/2} \\ &\le 2^{-2\beta}\left[(\xi+(1-\xi))^{1-\beta}(1+(b^2)^{1/\beta})^\beta\right]^{3/2},\end{aligned}$$

with an application of geometric-arithmetic mean inequality in the first and Hölder inequality in the second factor. In general we can just use $b < 1$ and get

$$\kappa(x) \ge (p-1)b2^{2\beta}\left[1+b^{2/\beta}\right]^{-3\beta/2} \ge (p-1)b2^{\beta/2} = (p-1)b2^{1/p-1/2},$$

within a factor $2^{3/2}$ of the upper estimate for $\min\kappa$.

Therefore, inserting this into Theorem **1**.11 as above, we derive $M(B_b^p) \ge (p-1)b2^{1/p-3/2}n$.

In all, we see that Theorems **6**.1 (essentially due to Erőd) and **1**.11 usually (but not always, c.f. the case $p \approx 1$ above !)) give better constants, when they apply. However, in cases the curvature is not bounded away from 0, we can retreat to application to the fully general Theorem **1**.9, which, even if with a small absolute constant factor, but still gives a precise estimate even regarding dependence of the constant on geometric features of the convex domain. According to Theorem **1**.10, this latter phenomenon is not just an observation on some particular examples, but is a general fact, valid even for not necessarily convex domains.

7. Further remarks and open problems

In the case of the unit interval also Turán type L^p estimates were studied, see [23] and the references therein. It would be interesting to consider the analogous question for convex domains on the plane. Note that already Turán remarked, see the footnote in [21, p.141], that on D an L^p version holds, too. Also note that for domains there are two possibilities for taking integral norms, one being on the boundary curve and another one of integrating with respect to area. It seems that the latter is less appropriate and convenient here.

In the above we described a more or less satisfactory answer of the problem of inverse Markov factors for convex domains. However, Levenberg and Poletsky showed that star-shaped domains already do not admit similar inverse Markov factors. A question, posed by V. Totik, is to determine exact order of the inverse Markov factor for the "cross" $C := [-1,1] \cup [-i,i]$; clearly, the point is not in the answer for the cross itself, but in the description of the inverse Markov factor for some more general classes of sets.

Another question, still open, stems from the Szegő extension of the Markov inequality, see [20], to domains with sector condition on their boundary. More precisely, at $z \in \partial K$ K satisfies the *outer sector condition* with $0 < \beta < 2$, if there exists a small neighborhood of z where some sector $\{\zeta : \arg(\zeta - z) \in (\theta, \beta\pi + \theta)\}$ is disjoint from K. Szegő proved, that if for a domain K, bounded by finitely many smooth (analytic) Jordan arcs, the supremum of β-values satisfying outer sector conditions at some boundary point is $\alpha < 2$, then $\|P'\| \ll n^{\alpha}\|P\|$ on K. Then Turán writes: "Es ist sehr wahrscheinlich, daß auch den Szegőschen Bereichen $M(p) \geq cn^{1/\alpha}$...", that is, he finds it rather likely that the natural converse inequality, suggested by the known cases of the disk and the interval (and now also by any other convex domain) holds also for general domains with outer sector conditions.

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QUANTIZATION OF UNIVERSAL TEICHMÜLLER SPACE

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Abstract

In the first part of the paper we describe the complex geometry of the universal Teichmüller space $\mathcal{T}$ which may be realized as an open subset in the complex Banach space of holomorphic quadratic differentials in the unit disc. The quotient $\mathcal{S}$ of the diffeomorphism group of the circle modulo Möbius transformations is treated as a regular part of $\mathcal{T}$. In the second part we consider the quantization of universal Teichmüller space $\mathcal{T}$. We explain first how to quantize the regular part $\mathcal{S}$ by embedding it into a Hilbert–Schmidt Siegel disc. This quantization method, however, does not apply to the whole universal Teichmüller space $\mathcal{T}$. For its quantization we use an approach, similar to the "quantized calculus" of Connes and Sullivan.

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Introduction

The universal Teichmüller space $\mathcal{T}$, introduced by Ahlfors and Bers, plays a key role in the theory of quasiconformal maps and Riemann surfaces. It can be defined as the space of quasisymmetric homeomorphisms of the unit circle S^1 (i.e. homeomorphisms of S^1, extending to quasiconformal maps of the unit disc Δ) modulo Möbius transformations. The space $\mathcal{T}$ has a natural complex structure, induced by embedding of $\mathcal{T}$ into the complex Banach space $B_2(\Delta)$ of holomorphic quadratic differentials in the unit disc Δ. It also contains all classical Teichmüller spaces $T(G)$, where G is a Fuchsian group, as complex submanifolds. The space $\mathcal{S} := \text{Diff}_+(S^1)/\text{Möb}(S^1)$ of normalized diffeomorphisms of the circle may be considered as a "regular" part of $\mathcal{T}$.

Our motivation to study $\mathcal{T}$ comes from the string theory. Physicists have noticed that the space $\Omega_d := C_0^\infty(S^1, \mathbb{R}^d)$ of smooth loops in the d-dimensional vector space $\mathbb{R}^d$ may be identified with the phase space of the theory of smooth bosonic closed strings. By this identification the standard symplectic form (of type "$dp \wedge dq$") on the phase space translates into a natural symplectic form

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ω on Ω_d. This form has a remarkable property that it can be extended to the Sobolev completion of Ω_d, coinciding with the space $V_d := H_0^{1/2}(S^1, \mathbb{R}^d)$ of half-differentiable vector-functions on S^1. Moreover, V_d is the largest space among all Sobolev spaces $H_0^s(S^1, \mathbb{R}^d)$ on which ω can be correctly defined. In other words, V_d is a natural phase space, "chosen" by the form ω itself. From that point of view, it seems more reasonable to consider V_d as the phase space of bosonic string theory, rather than Ω_d. In these lectures we set $d = 1$ for simplicity and study the space $V := V_1 = H_0^{1/2}(S^1, \mathbb{R})$.

According to Nag–Sullivan [7], there is a natural group, attached to the space $V = H_0^{1/2}(S^1, \mathbb{R})$, namely the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle. Again one can say that the space V itself chooses the "right" group to be acted on. The group $\mathrm{QS}(S^1)$ acts on V by reparametrization of loops and this action is symplectic with respect to the form ω. The universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$ can be identified by this action with the space of complex structures on V which can be obtained from a reference complex structure by the action of reparametrization group $\mathrm{QS}(S^1)$. It is well known that such a space plays a crucial role in quantization which is the main subject of the second part of our lectures.

In these lectures we try to define what is the quantum counterpart of the space $\mathcal{T}$, provided with the action of the group $\mathrm{QS}(S^1)$. In order to explain the arising difficulties we consider first an analogous problem for the regular part $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ of $\mathcal{T}$, provided with the action of the group $\mathrm{Diff}_+(S^1)$. This space can be quantized, using an embedding of $\mathcal{S}$ into the Hilbert-Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$. Under this embedding the diffeomorphism group $\mathrm{Diff}_+(S^1)$ is realized as a subgroup of the Hilbert–Schmidt symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$, acting on the Siegel disc by operator fractional-linear transformations. There is a holomorphic Fock bundle $\mathcal{F}$ over $\mathcal{D}_{\mathrm{HS}}$, provided with a projective action of $\mathrm{Sp}_{\mathrm{HS}}(V)$, which covers its action on $\mathcal{D}_{\mathrm{HS}}$. The infinitesimal version of this action is a projective representation of the Hilbert–Schmidt symplectic Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the fibre F_0 of the Fock bundle $\mathcal{F}$. This defines the Dirac quantization of the Siegel disc $\mathcal{D}_{\mathrm{HS}}$. Its restriction to $\mathcal{S}$ gives a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$ of the group $\mathrm{Diff}_+(S^1)$ in the Fock space F_0 which defines the Dirac quantization of the space $\mathcal{S}$.

However, the described quantization procedure does not apply to the whole universal Teichmüller space $\mathcal{T}$. By this reason we choose another approach to this problem, based on Connes quantization. Briefly, the idea is the following. The $\mathrm{QS}(S^1)$-action on the Sobolev space V, mentioned above, cannot be differentiated in the classical sense (in particular, there is no Lie algebra, associated to $\mathrm{QS}(S^1)$). However, one can define a quantized infinitesimal version of this action by associating with any quasisymmetric homeomorphism $f \in \mathrm{QS}(S^1)$ a quantum differential $d^q f$ which is an integral operator on V with kernel, given essentially by the finite-difference derivative of f. In these terms the quantization of $\mathcal{T}$ is given by a representation of the algebra of derivations of V, generated by quantum differentials $d^q f$, in the Fock space F_0.

I. UNIVERSAL TEICHMÜLLER SPACE

1. Definition of universal Teichmüller space

1.1. Quasiconformal maps

Let $w : D \to w(D)$ be a homeomorphism of the domain $D \subset \overline{\mathbb{C}}$ in the extended complex plane (Riemann sphere) $\overline{\mathbb{C}}$ onto domain $w(D) \subset \overline{\mathbb{C}}$ which has locally integrable derivatives (in generalized sense).

Definition 1.1. The homeomorphism w is called *quasiconformal* if there exist a function $\mu \in L^\infty(D)$ with norm $\|\mu\|_\infty =: k < 1$ such that the following *Beltrami equation*

$$w_{\bar{z}} = \mu w_z \tag{1}$$

is satisfied for almost all $z \in D$. The function μ is called the *Beltrami differential* of w and the constant k is often indicated in the name of k-*quasiconformal* maps.

Remark 1.1. For $k = 0$ the equation (1) reduces to the Cauchy–Riemann equation and so determines a conformal map $w : D \to w(D)$. Such a map sends infinitesimally small circles, centered at a point $z \in D$, again into infinitesimally small circles, centered at $w(z)$. While in the case of a general smooth quasiconformal map w such a map sends infinitesimally small circles, centered at $z \in D$, into infinitesimally small ellipses, centered at $w(z)$, with eccentricity (the ratio of the large axis to the small one) being uniformly bounded (w.r. to $z \in D$) by a common constant $K < \infty$. This constant K is related to the above constant $k = \|\mu\|_\infty$ by the formula

$$K = \frac{1+k}{1-k} \geq 1 \ .$$

The least possible constant K is called the *maximal dilatation* of w and is also sometimes indicated in the name of K-*quasiconformal* maps.

Remark 1.2. The term "Beltrami differential" for μ is motivated by the behavior of μ under conformal changes of variable. Namely, according to (1), the function μ should transform under a conformal change $z \mapsto f(z)$ as

$$\mu\left(f(z)\right) = \mu(z) \frac{\overline{f'(z)}}{f'(z)},$$

i.e. as a $(-1,1)$-differential.

Remark 1.3. Quasiconformal maps $w : D \to D$ form a group, i.e. the composition of quasiconformal maps and the inverse of a quasiconformal map are again quasiconformal.

Theorem 1.1 (uniqueness theorem)**.** *Suppose that quasiconformal maps $w_1, w_2 : D \to D'$ satisfy the same Beltrami equation in D (i.e. have the same Beltrami differential in D). Then the maps*

$$w_1 \circ w_2^{-1} \quad \textit{and} \quad w_2 \circ w_1^{-1}$$

are conformal. The composition $f \circ w$ of a quasiconformal map $w : D \to D'$ with a conformal map $f : D' \to D''$ satisfy the same Beltrami equation in D as w.

Remark 1.4. A quasiconformal map $w : D \to D'$ is always extended to a homeomorphism $w : \overline{D} \to \overline{D'}$ of the closures which is Hölder-continuous up to the boundary.

Theorem 1.2 (existence theorem)**.** *For any function $\mu \in L^\infty(\overline{\mathbb{C}})$ with $\|\mu\|_\infty < 1$ there exists a solution w of the Beltrami equation in $\overline{\mathbb{C}}$. Any other solution $\tilde{w}$ of this equation has the form $\tilde{w} = w \circ f$ where f is a fractional-linear transform.*

Remark 1.5. In Theorem **1.2** we have restricted ourselves to the case $D = \overline{\mathbb{C}}$ since the case of a general domain $D \subset \overline{\mathbb{C}}$ is easily reduced to the case of the extended complex plane. Indeed, given a Beltrami differential $\mu \in L^\infty(D)$ with norm $\|\mu\|_\infty < 1$ we can always extend it (e.g. by zero outside D) to the whole $\overline{\mathbb{C}}$, preserving the inequality $\|\mu\|_\infty < 1$, and then apply the above theorem to get a solution of Beltrami equation in $\overline{\mathbb{C}}$. Its restriction to D yields a solution of Beltrami equation in D, defined up to conformal maps, according to the uniqueness theorem.

1.2. Quasisymmetric homeomorphisms

Definition 1.2. A homeomorphism $f : S^1 \to S^1$ of the unit circle S^1, preserving its orientation, is called *quasisymmetric* if it extends to a quasiconformal homeomorphism $w : \Delta \to \Delta$ of the unit disc Δ. The set of all quasisymmetric homeomorphisms of S^1 is a group, denoted by $\mathrm{QS}(S^1)$.

Definition 1.3. The *universal Teichmüller space* $\mathcal{T}$ is the quotient

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$$

where Möb(S^1) denotes the Möbius group of fractional-linear automorphisms of the unit disc Δ, restricted to the unit circle S^1.

Remark 1.6. One can avoid taking the quotient by Möbius group in the definition of $\mathcal{T}$ by considering only *normalized* quasisymmetric homemorphisms, leaving three fixed points in the circle, say $\pm 1, i$, invariant.

Remark 1.7. Any orientation-preserving diffeomorphism in $\mathrm{Diff}_+(S^1)$ extends to a diffeomorphism of the closed unit disc $\overline{\Delta}$ which is quasiconformal, according to Remark **1.1**. So $\mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1)$, and we have the following chain of embeddings

$$\mathrm{M\ddot{o}b}(S^1) \subset \mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1) \subset \mathrm{Homeo}_+(S^1)\ .$$

Hence,

$$\mathcal{S} := \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1) \hookrightarrow \mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1).$$

The space $\mathcal{S}$ can be otherwise defined as the space of normalized diffeomorphisms of S^1 and will be considered as a "regular" part of $\mathcal{T}$.

Since quasisymmetric homeomorphisms of S^1 were defined via quasiconformal maps of Δ, i.e. in terms of solutions of Beltrami equation in Δ, one can expect that there should be a definition of $\mathcal{T}$ directly in terms of Beltrami differentials.

Denote by $B(\Delta)$ the set of Beltrami differentials in the unit disc Δ. It can be identified (as a set) with the unit ball in the complex Banach space $L^\infty(\Delta)$. Given a Beltrami differential $\mu \in B(\Delta)$, we can extend it to a Beltrami differential $\check{\mu}$ on the extended complex plane $\overline{\mathbb{C}}$ by setting $\check{\mu}$ equal to zero outside the unit disc Δ. Then we can apply the existence Theorem **1.2** for quasiconformal maps on the extended complex plane $\overline{\mathbb{C}}$ and obtain a normalized quasiconformal homeomorphism w^μ, satisfying Beltrami equation (1) on $\overline{\mathbb{C}}$ with potential $\check{\mu}$. This homeomorphism is conformal on the exterior $\Delta_- := \overline{\mathbb{C}} \setminus \overline{\Delta}$ of the closed unit disc $\overline{\Delta}$ on $\overline{\mathbb{C}}$ and fixes the points $\pm 1, -i$.

Introduce an equivalence relation between Beltrami differentials in Δ by identifying two Beltrami differentials μ and ν for which the corresponding conformal maps coincide: $w^\mu|_{\Delta_-} \equiv w^\nu|_{\Delta_-}$. The universal Teichmüller space $\mathcal{T}$ coincides with the quotient

$$\mathcal{T} = B(\Delta)/\sim$$

of the space $B(\Delta)$ of Beltrami differentials modulo the introduced equivalence relation.

2. Complex structure of universal Teichmüller space

We introduce a complex structure on the universal Teichmüller space $\mathcal{T}$, using its embedding into the space of holomorphic quadratic differentials.

Consider an arbitrary point $[\mu]$ of $\mathcal{T}$, represented by the quasiconformal map w^μ. Its restriction to Δ_- is a conformal map so we can take its Schwarzian $S(w^\mu|_{\Delta_-})$.

Digression 2.1. Recall that the *Schwarzian* of a conformal map f is defined by

$$S(f) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

A characteristic property of Schwarzian is its invariance under fractional-linear maps

$$S\left(\frac{af+b}{cf+d}\right) = S(f).$$

By taking the Schwarzian $S(w^\mu|_{\Delta_-})$, we get a holomorphic quadratic differential in the disc Δ_- (the latter fact follows from the transformation properties of Beltrami differentials, prescribed by Beltrami equation (1)). Moreover, the image of this map does not depend on the choice of Beltrami differential μ in the class $[\mu]$. Composing this map with a standard fractional-linear isomorphism $\Delta_- \to \Delta$, we obtain an embedding

$$\Psi : \mathcal{T} \longrightarrow B_2(\Delta) \ , \quad [\mu] \longmapsto \psi(\mu), \tag{2}$$

having its image in the space $B_2(\Delta)$ of holomorphic quadratic differentials in the unit disc Δ.

The space $B_2(\Delta)$ of holomorphic quadratic differentials in Δ is a complex Banach space, provided with a natural hyperbolic norm, given by

$$\|\psi\|_2 := \sup_{z\in\Delta}(1-|z|^2)^2|\psi(z)|$$

for a quadratic differential ψ. It can be proved (cf. [5]) that $\|\psi[\mu]\|_2 \leq 6$ for any Beltrami differential $\mu \in B(\Delta)$.

The constructed map $\Psi : \mathcal{T} \to B_2(\Delta)$, called the *Bers embedding*, is a homeomorphism of $\mathcal{T}$ onto an open bounded connected contractible subset in $B_2(\Delta)$, containing the ball of radius $1/2$, centered at the origin (cf. [5]).

Using the constructed embedding (2), we can introduce a complex structure on the universal Teichmüller space $\mathcal{T}$ by pulling it back from the complex Banach space $B_2(\Delta)$. It provides $\mathcal{T}$ with the structure of a complex Banach manifold.

3. Classical Teichmüller spaces

The universal Teichmüller space $\mathcal{T}$ contains all classical Teichmüller spaces $T(G)$ as complex submanifolds. In particular, it is true for all Teichmüller spaces of compact Riemann surfaces of genus g. This property motivates the use of the term "universal" in the name of $\mathcal{T}$.

Let X be a compact Riemann surface of genus $g > 1$, uniformized by the unit disc Δ. Such a surface can be represented as the quotient

$$X = \Delta/G$$

where G is a discrete (Fuchsian) subgroup of $\text{Möb}(\Delta)$.

Definition 3.1. A quasisymmetric homeomorphism $f : S^1 \to S^1$ is called *G-invariant* if

$$f \circ g \circ f^{-1} \in \text{Möb}(S^1) \text{ for any } g \in G \iff fGf^{-1} \subset \text{Möb}(S^1).$$

Denote by $\text{QS}(S^1)^G$ the subgroup of G-invariant quasisymmetric homeomorphisms in $\text{QS}(S^1)$. Then by definition

$$T(G) := \text{QS}(S^1)^G/\text{Möb}(S^1).$$

The universal Teichmüller space $\mathcal{T}$ itself corresponds to the Fuchsian group $G = \{1\}$.

Remark 3.1. According to definition of $T(G)$, due to Teichmüller, the space $T(G)$ parameterizes different complex structures on the Riemann surface X/Δ which can be obtained from the original complex structure by a quasiconformal deformation.

4. Grassmann realization

4.1. Sobolev space of half-differentiable functions

Definition 4.1. The *Sobolev space of half-differentiable functions* on S^1 is a Hilbert space

$$V := H_0^{1/2}(S^1, \mathbb{R}),$$

consisting of functions $f \in L_0^2(S^1, \mathbb{R})$ with zero average over the circle, which have Fourier decompositions

$$f(z) = \sum_{k \neq 0} f_k z^k \ , \quad f_k = \bar{f}_{-k} \ , \quad z = e^{i\theta} \ ,$$

and a finite Sobolev norm

$$\|f\|_{1/2}^2 = \sum_{k\neq 0} |k||f_k|^2 = 2\sum_{k>0} k|f_k|^2 < \infty \ . \tag{3}$$

Properties of $V = H_0^{1/2}(S^1,\mathbb{R})$:

(1) *Symplectic structure*: define a 2-form ω on V by the formula

$$\omega(\xi,\eta) = 2\,\mathrm{Im}\sum_{k>0} k\xi_k\bar{\eta}_k$$

for vectors $\xi,\eta\in V$ with Fourier series

$$\xi(z) = \sum_{k\neq 0}\xi_k z^k, \quad \eta(z) = \sum_{k\neq 0}\eta_k z^k.$$

This form, which is correctly defined on V due to condition (3), determines a symplectic form on V. Moreover, $H_0^{1/2}(S^1,\mathbb{R})$ is the largest Hilbert space in the scale of Sobolev spaces $H_0^s(S^1,\mathbb{R})$, $s\in\mathbb{R}$, on which this form is correctly defined.

(2) *Complex structure*: the Sobolev space V has a complex structure J^0, defined by the formula

$$\xi(z) = \sum_{k\neq 0}\xi_k z^k \longmapsto (J^0\xi)(z) = -i\sum_{k>0}\xi_k z^k + i\sum_{k<0}\xi_k z^k$$

for a vector $\xi(z) = \sum_{k\neq 0}\xi_k z^k \in V$.

(3) *Riemannian metric*: the introduced symplectic and complex structures on V are compatible with each other in the sense that they generate together a Riemannian metric, defined by

$$g^0(\xi,\eta) = \omega(\xi,J^0\eta) = 2\,\mathrm{Re}\sum_{k>0} k\xi_k\bar{\eta}_k.$$

In other words, V has the structure of a Kähler Hilbert space.

(4) *Complexification* of V, equal to

$$V^{\mathbb{C}} = H_0^{1/2}(S^1,\mathbb{C}),$$

is a complex Hilbert space with a Kähler metric, given by the Hermitian extension of the Riemannian metric g^0 on V to $V^{\mathbb{C}}$. The space $V^{\mathbb{C}}$ is decomposed into the direct sum

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

of ($\mp i$)-eigenspaces of the complex structure operator $J^0 \in \operatorname{End} V^{\mathbb{C}}$. More explicitly,

$$W_+ = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k>0} f_k z^k\}, \quad W_- = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k<0} f_k z^k\}.$$

This splitting is orthogonal with respect to Hermitian inner product on $V^{\mathbb{C}}$.

4.2. QS-action on the Sobolev space V

With any homeomorphism $h : S^1 \to S^1$, preserving the orientation, we can associate a "change-of-variable" operator

$$T_h : L^2_0(S^1, \mathbb{R}) \to L^2_0(S^1, \mathbb{R}),$$

defined by

$$T_h(\xi) := \xi \circ h - \frac{1}{2\pi} \int_0^{2\pi} \xi\left(h(\theta)\right) d\theta.$$

This operator has the following remarkable property.

Theorem 4.1 (Nag–Sullivan [7]). *(i) The operator T_h acts on V, i.e. $T_h : V \to V$, if and only if $h \in QS(S^1)$.*

(ii) The operator T_h with $h \in QS(S^1)$ acts symplectically on V, i.e. it preserves symplectic form ω. Moreover, its complex-linear extension to $V^{\mathbb{C}}$ preserves the subspace W_+ if and only if $h \in Möb(S^1)$. In the latter case, T_h acts as a unitary operator on W_+.

Remark 4.1. We have pointed out in previous Subsection that the Sobolev space V is the largest Hilbert space in the scale of Sobolev spaces, on which the form ω is correctly defined. In other words, this space is "chosen" by symplectic form ω itself. According to Theorem **4.1**, the space V also "chooses" the reparametrization group $QS(S^1)$ in the sense that it is the largest reparametrization group, leaving V invariant. So we get a natural phase space (V, ω) together with a natural group $QS(S^1)$ of its canonical transformations.

According to Theorem **4.1**, we have an embedding

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1) \longrightarrow \mathrm{Sp}(V)/\mathrm{U}(W_+). \tag{4}$$

Here, $\mathrm{Sp}(V)$ is the symplectic group of V, consisting of bounded linear symplectic operators on V, and $\mathrm{U}(W_+)$ is its subgroup, consisting of unitary operators (i.e. the operators, whose complex-linear extensions to $V^{\mathbb{C}}$ preserve the subspace W_+).

Digression 4.1. Recall the definition of symplectic group $\mathrm{Sp}(V)$. In terms of decomposition

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

any linear operator $A : V^{\mathbb{C}} \to V^{\mathbb{C}}$ can be written in the block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Such an operator belongs to symplectic group $\mathrm{Sp}(V)$ if it has the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

with components, satisfying the relations

$$\bar{a}^t a - b^t \bar{b} = 1, \quad \bar{a}^t b = b^t \bar{a}$$

where a^t, b^t denote the transposed operators $a^t : W_- \to W_-$, $b^t : W_- \to W_+$. The unitary group $\mathrm{U}(W_+)$ is embedded into $\mathrm{Sp}(V)$ as a subgroup, consisting of diagonal block matrices of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} .$$

The space

$$\mathcal{J}(V) := \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

in the right hand side of (4), can be identified with the space of complex structures on V, compatible with ω. Indeed, any such structure, given by a linear operator J on V with $J^2 = -I$, determines a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W} \tag{5}$$

of $V^{\mathbb{C}}$ into the direct sum of $(\pm i)$-eigenspaces, isotropic with respect to ω. Conversely, any decomposition (5) of the space $V^{\mathbb{C}}$ into the direct sum of isotropic subspaces determines a complex structure J on $V^{\mathbb{C}}$, equal to iI on W and $-iI$ on $\overline{W}$, which is compatible with ω. Moreover, a complex structure J, obtained from a reference complex structure J^0 by the action of an element A of $\mathrm{Sp}(V)$, is equivalent to J^0 if and only if $A \in \mathrm{U}(W_+)$. Hence,

$$\mathrm{Sp}(V)/\mathrm{U}(W_+) = \mathcal{J}(V) .$$

The space on the right can be, in its turn, identified with the *Siegel disc* $\mathcal{D}$, defined as the set

$$\mathcal{D} = \{Z : W_+ \to W_- \text{ is a symmetric bounded linear operator with } \bar{Z}Z < I\}.$$

The symmetricity of Z means that $Z^t = Z$ and the condition $\bar{Z}Z < I$ means that symmetric operator $I - \bar{Z}Z$ is positive definite. In order to identify $\mathcal{J}(V)$

with $\mathcal{D}$, consider the action of the group $\mathrm{Sp}(V)$ on $\mathcal{D}$, given by fractional-linear transformations $A : \mathcal{D} \to \mathcal{D}$ of the form

$$Z \longmapsto (\bar{a}Z + \bar{b})(bZ + a)^{-1}$$

where $A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(V)$. The isotropy subgroup at $Z = 0$ coincides with the set of operators $A \in \mathrm{Sp}(V)$ such that $b = 0$, i.e. with $\mathrm{U}(W_+)$. So the space

$$\mathcal{J}(V) = \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

can be identified with the Siegel disc $\mathcal{D}$.

It can be proved (cf. [7]) that the constructed embedding of universal Teichmüller space $\mathcal{T}$ into the Siegel disc $\mathcal{D} = \mathrm{Sp}(V)/\mathrm{U}(W_+)$ is an equivariant holomorphic map of Banach manifolds.

The restriction of this map to the regular part $\mathcal{S}$ of universal Teichmüller space yields an embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+), \tag{6}$$

where the *Hilbert–Schmidt subgroup* $\mathrm{Sp}_{\mathrm{HS}}(V)$ of $\mathrm{Sp}(V)$ consists of bounded linear operators $A \in \mathrm{Sp}(V)$, having block representations of the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

where b is a Hilbert–Schmidt operator.

Digression 4.2. Recall that a linear bounded operator $T : H_1 \to H_2$ from a Hilbert space H_1 to a Hilbert space H_2 is called Hilbert–Schmidt if there exists an orthonormal basis $\{e_i\}$ in H_1 such that the Hilbert–Schmidt norm

$$\|T\|_2 := \left(\sum_{i=0}^{\infty} \|Te_i\|_{H_2}^2 \right)^{1/2}$$

is finite. If this is true for some orthonormal basis $\{e_i\}$ in H_1 then it is true for any orthonormal basis in H_1 and the value of the norm $\|T\|_2$ does not depend on the choice of this basis.

We identify, as above, the right hand side of (6) with a subspace $\mathcal{J}_{\mathrm{HS}}(V)$ of the space $\mathcal{J}(V)$ of compatible complex structures on V. As before, the space $\mathcal{J}_{\mathrm{HS}}(V)$ of Hilbert–Schmidt complex structures on V can be realized as a *Hilbert–Schmidt Siegel disc*

$$\mathcal{D}_{\mathrm{HS}} = \{Z : W_+ \to W_- \text{ is a symmetric Hilbert–Schmidt operator with } \bar{Z}Z < I\}.$$

The embedding of $\mathcal{S}$ into the Hilbert–Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$ is an equivariant holomorphic map of Banach manifolds.

II. QUANTIZATION OF UNIVERSAL TEICHMÜLLER SPACE

5. Dirac quantization

5.1. Definition

We start by recalling a general definition of quantization of finite-dimensional classical systems, due to Dirac. A *classical system* is given by a pair $(M, \mathcal{A})$ where M is the phase space of the system and $\mathcal{A}$ is its algebra of observables.

The *phase space* M is a smooth symplectic manifold of even dimension $2n$, provided with symplectic 2-form ω. Locally, it is equivalent to the standard model, given by symplectic vector space $M_0 := \mathbb{R}^{2n}$ together with standard symplectic form ω_0, given in canonical coordinates (p_i, q_i), $i = 1, \dots, n$, on $\mathbb{R}^{2n}$ by

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

The *algebra of observables* $\mathcal{A}$ is a Lie subalgebra of the Lie algebra $C^\infty(M, \mathbb{R})$ of smooth real-valued functions on the phase space M, provided with the Poisson bracket, determined by symplectic 2-form ω. In particular, in the case of standard model $M_0 = (\mathbb{R}^{2n}, \omega_0)$ one can take for $\mathcal{A}$ the Heisenberg algebra, generated by coordinate functions p_i, q_i, $i = 1, \dots, n$, and 1, satisfying the commutation relations

$$\{p_i, p_j\} = \{q_i, q_j\} = 0,$$
$$\{p_i, q_j\} = \delta_{ij} \quad \text{for } i, j = 1, \dots, n.$$

Remark 5.1. One of usual ways to produce algebras of observables is to consider a Lie group Γ of symplectomorphisms of a symplectic manifold (M, ω) and take for $\mathcal{A}$ its Lie algebra $\mathrm{Lie}(\Gamma)$, consisting of Hamiltonian vector fields X_f on M. If M is simply connected then $\mathcal{A}$ can be identified with the dual algebra of functions f, generating Hamiltonian vector fields from $\mathrm{Lie}(\Gamma)$.

Definition 5.1. The *Dirac quantization* of a classical system $(M, \mathcal{A})$ is an irreducible linear representation

$$r : \mathcal{A} \longrightarrow \mathrm{End}^* H$$

of the algebra of observables $\mathcal{A}$ in the space of linear self-adjoint operators, acting on a complex Hilbert space H, called the *quantization space.* The map r should satisfy the condition

$$r(\{f, g\}) = \frac{1}{i}\left(r(f)r(g) - r(g)r(f)\right)$$

for any $f, g \in \mathcal{A}$. We also impose on r the following normalization condition: $r(1) = I$.

Remark 5.2. For complexified algebras of observables $\mathcal{A}^{\mathbb{C}}$ or, more generally, complex involutive Lie algebras of observables (i.e. Lie algebras with conjugation) their Dirac quantization is given by an irreducible Lie-algebra representation

$$r : \mathcal{A}^{\mathbb{C}} \longrightarrow \operatorname{End} H \ ,$$

satisfying the normalization condition and the conjugation law: $r(\bar{f}) = r(f)^*$ for any $f \in \mathcal{A}$.

Remark 5.3. We are going to apply this definition of quantization to infinite-dimensional classical systems, in which both the phase space and algebra of observables are infinite-dimensional. For infinite-dimensional algebras of observables it is more natural to look for their projective representations. Using such a representation for an original algebra $\mathcal{A}$, we can construct the quantization of the extended system $(M, \tilde{\mathcal{A}})$ with $\tilde{\mathcal{A}}$ being a suitable central extension of $\mathcal{A}$.

5.2. Statement of the problem

We shall explain first how to quantize the regular part of universal Teichmüller space $\mathcal{T}$, represented by the classical system

$$(\mathcal{S}, \operatorname{Vect}(S^1))$$

where $\mathcal{S} = \operatorname{Diff}_+(S^1)/\text{Möb}(S^1)$ and $\operatorname{Vect}(S^1)$ is the Lie algebra of $\operatorname{Diff}_+(S^1)$, consisting of smooth vector fields on S^1.

To quantize this system, we first enlarge it to an extended system, using the embedding $\mathcal{S} \hookrightarrow \mathcal{J}_{\text{HS}}(V)$ from Subsection **4.2**. This extended system is given by

$$(\mathcal{J}_{\text{HS}}(V), \text{sp}_{\text{HS}}(V))$$

where $\text{sp}_{\text{HS}}(V)$ is the Lie algebra of $\text{Sp}_{\text{HS}}(V)$.

6. Quantization of $\mathcal{S}$

6.1. Fock space

Fix a compatible complex structure $J \in \mathcal{J}(V)$, generating a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W} \tag{7}$$

of $V^{\mathbb{C}}$ into the direct sum of $\pm i$-eigenspaces of J and provide $V^{\mathbb{C}}$ with a Hermitian inner product

$$< z, w >_J := \omega(z, Jw),$$

determined by J and symplectic form ω.

The Fock space $F(V^{\mathbb{C}}, J)$ is the completion of the algebra of symmetric polynomials on W with respect to a natural norm, generated by $< \cdot, \cdot >_J$. In more detail, denote by $S(W)$ the algebra of symmetric polynomials in variables $z \in W$. This algebra is provided with an inner product, generated by $< \cdot, \cdot >_J$. By definition, this inner product on monomials of the same degree is equal to

$$< z_1 \cdot \ldots \cdot z_n, z'_1 \cdot \ldots \cdot z'_n >_J = \sum_{\{i_1,\ldots,i_n\}} < z_1, z'_{i_1} >_J \cdot \ldots \cdot < z_n, z'_{i_n} >_J$$

where the summation is taken over all permutations $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$. The inner product of monomials of different degrees is set to zero. The constructed inner product is extended to the whole algebra $S(W)$ by linearity. The completion $\widehat{S(W)}$ of $S(W)$ with respect to the introduced norm is called the *Fock space* of $V^{\mathbb{C}}$ with respect to complex structure J:

$$F_J = F(V^{\mathbb{C}}, J) := \widehat{S(W)}.$$

If $\{w_n\}$, $n = 1, 2, \ldots$, is an orthonormal basis of W one can take for an orthonormal basis of F_J a family of homogeneous polynomials of the form

$$P_K(z) = \frac{1}{\sqrt{k!}} < z, w_1 >_J^{k_1} \cdot \ldots \cdot < z, w_n >_J^{k_n}, \quad z \in W, \tag{8}$$

where $K = (k_1, \ldots, k_n, 0, \ldots)$, $k_i \in \mathbb{N} \cup 0$, and $k! = k_1! \cdot \cdots \cdot k_n!$.

6.2. Symplectic group action on Fock spaces

We unify different Fock spaces F_J with $J \in \mathcal{J}_{\mathrm{HS}}(V)$ into a single *Fock bundle*

$$\mathcal{F} := \bigcup_{J \in \mathcal{J}_{\mathrm{HS}}(V)} F_J \longrightarrow \mathcal{J}_{\mathrm{HS}}(V) = \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+).$$

Theorem 6.1 (Shale–Berezin). *The Fock bundle*

$$\mathcal{F} \longrightarrow \mathcal{J}_{HS}(V)$$

is a holomorphic Hermitian Hilbert-space bundle. The group $Sp_{HS}(V)$ acts projectively on $\mathcal{F}$ by unitary transformations and this action covers the natural action of $Sp_{HS}(V)$ on $\mathcal{J}_{HS}(V)$ by left translations.

The infinitesimal version of this action yields a projective representation of symplectic Hilbert–Schmidt algebra $\mathrm{sp_{HS}}(V)$ in the Fock space $F_0 = F(V^{\mathbb{C}}, J^0)$, i.e. a quantization of the system

$$\left(\mathcal{J}_{\mathrm{HS}}, \widetilde{\mathrm{sp_{HS}}(V)}\right)$$

where $\widetilde{\mathrm{sp_{HS}}(V)}$ is a central extension of Lie algebra $\mathrm{sp_{HS}}(V)$.

The restriction of the constructed Fock bundle $\mathcal{F}$ to the submanifold $\mathcal{S} \subset \mathcal{J}_{\mathrm{HS}}$ is a holomorphic Hermitian Hilbert-space bundle

$$\mathcal{F}_{\mathcal{S}} := \bigcup_{J\in\mathcal{S}} F_J \longrightarrow \mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$$

together with a projective unitary action of $\mathrm{Diff}_+(S^1)$, covering its action on $\mathcal{S}$ by left translations. The infinitesimal version of this action generates a projective unitary representation of the Lie algebra $\mathrm{Vect}(S^1)$ in the Fock space F_0, i.e. a quantization of the system

$$(\mathcal{S}, \mathrm{vir})$$

where vir is the *Virasoro algebra*, being a central extension of Lie algebra $\mathrm{Vect}(S^1)$.

7. Quantization of $\mathcal{T}$

7.1. Dirac versus Connes quantization

To quantize $\mathcal{S}$, we have used the fact that the symplectic group $\mathrm{Sp_{HS}}(V)$ acts on the Fock bundle $\mathcal{F} \to \mathcal{J}_{\mathrm{HS}}(V)$. For the whole Teichmüller space $\mathcal{T}$ we still have the embedding

$$\mathcal{T} \longrightarrow \mathcal{J}(V) = \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

but we cannot construct an $\mathrm{Sp}(V)$-action on $\mathcal{F}$, covering its action on $\mathcal{J}(V)$. This is forbidden by Shale–Berezin theorem. So we employ another approach for the quantization of $\mathcal{T}$, using Connes' definition of quantization.

Recall that in Dirac's approach we quantize a classical system $(M, \mathcal{A})$, consisting of the phase space M and the algebra of observables $\mathcal{A}$ which is a Lie algebra, consisting of smooth functions on M. The quantization of this system is given by a representation r of $\mathcal{A}$ in a Hilbert space H, sending the Poisson bracket $\{f, g\}$ of functions $f, g \in \mathcal{A}$ into the commutator $\frac{1}{i}[r(f), r(g)]$ of the corresponding operators. In Connes' approach the algebra of observables $\mathfrak{A}$ is an associative involutive algebra, provided with an exterior differential d. Its quantization is, by definition, a representation π of $\mathfrak{A}$ in a Hilbert space H, sending the differential df of a function $f \in \mathfrak{A}$ into the commutator $[S, \pi(f)]$ of the operator $\pi(f)$ with a self-adjoint symmetry operator S with $S^2 = I$.

In the following table we compare Connes and Dirac approaches to quantization:

	Dirac approach	Connes approach
Classical system	$(M, \mathcal{A})$ where: M – phase space $\mathcal{A}$ – involutive Lie algebra of observables	$(M, \mathfrak{A})$ where: M – phase space $\mathfrak{A}$ – involutive associative algebra of observables with differential d
Quantization	representation $r\colon \mathcal{A} \to \operatorname{End} H$, sending $\{f, g\} \mapsto \frac{1}{i}[r(f), r(g)]$	representation $\pi\colon \mathfrak{A} \to \operatorname{End} H$, sending $df \mapsto [S, \pi(f)]$, where $S = S^*$, $S^2 = I$

Remark 7.1. We can reformulate the Connes definition in terms of Lie algebras by switching to the algebra of derivations of associative algebra of observables $\mathfrak{A}$. Recall that the Lie algebra $\operatorname{Der}(\mathfrak{A})$ of derivations of $\mathfrak{A}$ consists of linear maps $\mathfrak{A} \to \mathfrak{A}$, satisfying the Leibnitz rule. The Connes quantization means in these terms the construction of an irreducible representation of $\operatorname{Der}(\mathfrak{A})$ in the space $\operatorname{End} H$, considered as a Lie algebra with a Lie bracket, given by commutator.

Remark 7.2. If all observables are smooth functions on M, both approaches are equivalent to each other. Indeed, the differential df of a smooth observable f is symplectically dual to the Hamiltonian vector field X_f which establishes a relation between the associative algebra $\mathfrak{A} \ni f$ of functions f on M and the Lie algebra $\mathcal{A} \ni X_f$ of Hamiltonian vector fields X_f. A symmetry operator S is determined by a polarization $H = H_+ \oplus H_-$ of the quantization space H and related to the complex structure J (determined by the same polarization) by a simple formula $S = iJ$.

In the case when the algebra of observables $\mathcal{A}$ contains non-smooth functions, the Dirac approach formally cannot be applied. In Connes approach the differential *df* of a non-smooth observable $f \in \mathfrak{A}$ is also not defined but its quantum analogue

$$d^q f := [S, \pi(f)]$$

may still have sense, as it is demonstrated by the example in the next Subsection.

7.2. Example

Suppose that $\mathfrak{A}$ is the algebra $L^\infty(S^1, \mathbb{C})$ of bounded functions on the circle S^1. Any function $f \in \mathfrak{A}$ determines a bounded multiplication operator in the Hilbert space $H = L^2(S^1, \mathbb{C})$ by the formula

$$M_f : v \in H \longmapsto fv \in H.$$

A symmetry operator S in H is given by the *Hilbert transform* $S : L^2(S^1, \mathbb{C}) \to L^2(S^1, \mathbb{C})$:

$$(Sf)(e^{i\varphi}) = \frac{1}{2\pi} P.V. \int_0^{2\pi} K(\varphi, \psi) f(e^{i\psi}) d\psi$$

where the integral is taken in the principal value sense and the kernel is given by

$$K(\varphi, \psi) = 1 + i \cot \frac{\varphi - \psi}{2} \ . \tag{9}$$

Note that for φ, close to ψ, this kernel behaves asymptotically like $2/(\varphi - \psi)$.

The differential *df* of a general observable $f \in \mathfrak{A}$ is not defined in the classical sense but its quantum analogue

$$d^q f := [S, M_f]$$

is a bounded operator in H. Moreover, $d^q f$ for $f \in H$ is a Hilbert–Schmidt operator, given by

$$d^q f(v)(e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} k(\varphi, \psi) v(e^{i\psi}) d\psi \tag{10}$$

with kernel

$$k(\varphi, \psi) := K(\varphi, \psi)(f(\varphi) - f(\psi)) \ ,$$

where $K(\varphi, \psi)$ is defined by (9). The kernel $k(\varphi, \psi)$ for φ, close to ψ, behaves asymptotically like

$$\frac{f(e^{i\varphi}) - f(e^{i\psi})}{\varphi - \psi}.$$

Using this fact, it can be checked that the quasiclassical limit of $d^q f$, arranged by taking the limit $\varphi \to \psi$, coincides (up to a constant) with the multiplication operator $v \mapsto f'v$. So the quantization means in this case simply the replacement of the derivative by its finite-difference analogue.

7.3. Quantization of the universal Teichmüller space

We apply these ideas to the universal Teichmüller space $\mathcal{T}$. In Subsection **4.2** we have defined a natural action of the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of S^1 on the Sobolev space V. As we have remarked, this action does not admit the differentiation, so classically there is no Lie algebra, associated with $\mathrm{QS}(S^1)$. In other words, there is no classical algebra of observables, associated to $\mathcal{T}$. (The situation is similar to that in the example above.) However, we shall construct a *quantum algebra of observables*, associated to $\mathcal{T}$.

For that we define a quantum infinitesimal version of $\mathrm{QS}(S^1)$-action on V, given by the integral operator $d^q f$, defined by formula (10). Then we extend this operator $d^q f$ to the Fock space F_0 by defining it first on elements of the basis (8) of F_0 with the help of Leibnitz rule, and then extending to the whole symmetric algebra $S(W_+)$ by linearity. The completion of the obtained operator yields an operator $d^q f$ on F_0. The operators $d^q f$ with $f \in \mathrm{QS}(S^1)$, constructed in this way, generate a *quantum Lie algebra* $\mathrm{Der}^q(\mathrm{QS})$, associated with $\mathcal{T}$. We consider it as a quantum Lie algebra of observables, associated with $\mathcal{T}$. We can also consider the constructed Lie algebra $\mathrm{Der}^q(\mathrm{QS})$ as a replacement of the (non-existing) classical Lie algebra of the group $\mathrm{QS}(S^1)$.

Compare now the main steps of Connes quantization of $\mathcal{T}$ with the analogous steps in Dirac quantization of $\mathcal{J}_{\mathrm{HS}}$.

In the case of $\mathcal{J}_{\mathrm{HS}}$:

(1) we start with the $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on $\mathcal{J}_{\mathrm{HS}}$;
(2) then, using Shale theorem, extend this action to a projective unitary action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on Fock spaces $F(V,J)$;
(3) an infinitesimal version of this action yields a projective unitary representation of symplectic Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space F_0.

In the case of $\mathcal{T}$:

(1) we have an action of $\mathrm{QS}(S^1)$ on the space V; however, in contrast with Dirac quantization of $\mathcal{J}_{\mathrm{HS}}$, the step (2) in case of $\mathcal{T}$ is impossible, since by Shale theorem we cannot extend the action of $\mathrm{QS}(S^1)$ to Fock spaces $F(V,S)$;
(2) we define instead a quantized infinitesimal action of $\mathrm{QS}(S^1)$ on V, given by quantum differentials $d^q f$;
(3) extending operators $d^q f$ to the Fock space F_0, we obtain a quantum Lie algebra $\mathrm{Der}^q(\mathrm{QS})$, generated by extended operators $d^q f$ on F_0.

Conclusion. The Connes quantization of the universal Teichmüller space $\mathcal{T}$ consists of two steps:

(1) The first step ("the first quantization") is the construction of quantized infinitesimal $\mathrm{QS}(S^1)$-action on V, given by quantum differentials $d^q f$ with $f \in \mathrm{QS}(S^1)$.

(2) The second step ("the second quantization") is the extension of quantum differentials $d^q f$ to the Fock space F_0. The extended operators $d^q f$ with $f \in \mathrm{QS}(S^1)$ generate the quantum algebra of observables $\mathrm{Der}^q(\mathrm{QS})$, associated with $\mathcal{T}$.

Note that the correspondence principle for the constructed Connes quantization of $\mathcal{T}$ means that this quantization, being restricted to $\mathcal{S}$, coincides with Dirac quantization of $\mathcal{S}$.

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FROM HILBERT FRAMES TO GENERAL FRÉCHET FRAMES

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Abstract

Frames extend orthonormal bases and play an important role in both pure and applied mathematics. Frames were first introduced in Hilbert spaces, after that in Banach spaces, and recently in Fréchet spaces. Although Hilbert frames might not be Schauder bases, they still allow series expansions of all the elements of a Hilbert space. However, Banach (resp. Fréchet) frames do not necessarily lead to series expansions in Banach (resp. Fréchet) spaces. In this paper we give a review of results on series expansions via frames in Hilbert, Banach, and Fréchet spaces.

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Key Words and Phrases: frames; Banach frames; Fréchet frames; General Fréchet frames; series representations

1. Introduction and Notation

Frames were introduced by Duffin and Schaeffer [12] in 1952. Around 40 years later, the wavelet era began [11, 9, 10] and the importance of frames was realized. Frames became a topic of a large investigation and started to be involved in many applications (for example, real life applications connected to signal and image processing).

In 1991, Banach frames were introduced [13] and subsequently many papers on the topic appeared, see for example [7, 5, 1, 2, 8, 14, 4, 22, 23, 24].

Recently, the frame-concept was extended to Fréchet spaces [18, 19, 20, 21].

In this paper we review some results on the frame-concepts in Hilbert, Banach, and Fréchet spaces. We concentrate on results connected to series expansions in the corresponding spaces.

In Section **2** we consider frames for Hilbert spaces. First we recall basic properties of orthonormal bases and Schauder bases, and then we consider frames in comparison with orthonormal bases and Schauder bases. In Sections **3** and **4** we consider frame-concepts in Banach and Fréchet spaces, respectively.

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Throughout the paper, $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ denotes a Hilbert space; $(X, \|\cdot\|_X)$ and $(X_s, \|\cdot\|_s)$, $s \in \mathbb{N}_0$, denote Banach spaces; $(\Theta, \|\cdot\|_\Theta)$ and $(\Theta_s, |||\cdot|||_s)$, $s \in \mathbb{N}_0$, denote Banach sequence spaces. The space Θ is called a *BK-space*, if the coordinate functionals are continuous. The n-th canonical vector $(0,\ldots,0,1,0,\ldots)$, having 1 at the n-th position, is denoted by δ_n. The space Θ is called a CB-space if the canonical vectors $(\delta_n)_{n=1}^\infty$ form a Schauder basis for Θ. Recall that when Θ is a CB-space, then $\Theta^\circledast := \{(G\delta_n)_{n=1}^\infty : G \in \Theta^*\}$ with the norm $\|(G\delta_n)_{n=1}^\infty\|_{\Theta^\circledast} := \|G\|_{\Theta^*}$ is a BK-space isometrically isomorphic to Θ^* [17]. Throughout the paper, when Θ is a CB-space, Θ^* is identified with $\Theta^\circledast$.

2. Frames in Hilbert spaces

First recall some basic facts about Schauder bases and orthonormal bases, which are related to the topic of the paper. A Schauder basis $(g_n)_{n=1}^\infty$ for a Banach space X allows every element $f \in X$ to be represented via the basis elements as

$$f = \sum_{n=1}^\infty c_n g_n \tag{2.1}$$

with unique coefficients $c_n, n \in \mathbb{N}$. For an orthonormal basis $(e_n)_{n=1}^\infty$ of a Hilbert space $\mathcal{H}$ and element $f \in \mathcal{H}$, the unique coefficients c_n in (2.1) have the form $\langle f, e_n\rangle$, i.e. every element $f \in \mathcal{H}$ can be written as

$$f = \sum_{n=1}^\infty \langle f, e_n\rangle e_n,$$

and furthermore, the *Parseval equality* holds, namely,

$$\sum_{n=1}^\infty |\langle f, e_n\rangle|^2 = \|f\|^2.$$

The frame definition extends the Parseval equality:

Definition 2.1. [12] The sequence $(g_n)_{n=1}^\infty$ $(g_n \in \mathcal{H}, n \in \mathbb{N})$ is called a *(Hilbert) frame for* $\mathcal{H}$, if there exist constants $B, A \in (0,\infty)$ so that

$$A\,\|f\|_{\mathcal{H}}^2 \le \sum_{n=1}^\infty |\langle f, g_n\rangle|^2 \le B\,\|f\|_{\mathcal{H}}^2,\ \forall f \in \mathcal{H}. \tag{2.2}$$

We list several known examples of frames and non-frames.

Example 2.1. Let $(e_n)_{n=1}^\infty$ be an orthonormal basis for $\mathcal{H}$. Then the sequence

- $(e_1, e_1, e_2, e_2, e_3, e_3, \ldots)$ is a frame for $\mathcal{H}$ $(A = B = 2)$;

- $(e_1, e_1, e_2, e_3, e_4, ...)$ is a frame for $\mathcal{H}$ $(A = 1, B = 2)$;
- $(e_1, \frac{e_2}{\sqrt{2}}, \frac{e_2}{\sqrt{2}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, \frac{e_3}{\sqrt{3}}, ...)$ is a frame for $\mathcal{H}$ $(A = B = 1)$;
- $(e_1, \frac{e_2}{2}, \frac{e_3}{3}, ...)$ is not a frame for $\mathcal{H}$ (it satisfies the upper frame condition, but not the lower one);
- $(e_1, 2e_2, 3e_3, ...)$ is not a frame for $\mathcal{H}$ (it satisfies the lower frame condition, but not the upper one).

The above examples may lead to a wrong impression that all the frames are obtained using a Schauder basis and adding some more elements. Note that there exists a frame for a Hilbert space $\mathcal{H}$ for which no subset is a Schauder basis for $\mathcal{H}$ [3].

Example 2.2. Let $g(x) = \pi^{-1/4}e^{-x^2/2}$. For positive and small enough a,b, the system $\left(e^{2\pi imbx}g(x - na)\right)_{m,n\in\mathbb{Z}}$ is a frame (Gabor frame) for $L^2(\mathbb{R})$.

Example 2.3. Let $\psi(x) = (1-x^2)e^{-x^2/2}$. For a positive and small enough b, the system $(2^{\frac{j}{2}}\psi(2^j x - bk))_{j,k\in\mathbb{Z}}$ is a frame (wavelet-frame) for $L^2(\mathbb{R})$.

As one can see in Example 2.1, frames do not need to be Schauder bases. Still, they allow series expansions of every element in the space:

Theorem 2.1. [12] *Let $(g_n)_{n=1}^\infty$ be a frame for $\mathcal{H}$. For every $f \in \mathcal{H}$, there exist coefficients c_n, $n \in \mathbb{N}$, so that (2.1) holds. Furthermore, there exists a frame $(f_n)_{n=1}^\infty$ for $\mathcal{H}$ (called a dual frame of $(g_n)_{n=1}^\infty$) so that*

$$f = \sum_{n=1}^{\infty}\langle f, f_n\rangle g_n = \sum_{n=1}^{\infty}\langle f, g_n\rangle f_n, \quad \forall f \in \mathcal{H}. \tag{2.3}$$

When $(g_n)_{n=1}^\infty$ is a frame for $\mathcal{H}$ and also a Schauder basis for $\mathcal{H}$ (i.e., a *Riesz basis for* $\mathcal{H}$), then the coefficients c_n satisfying (2.1) and the frame $(f_n)_{n=1}^\infty$ satisfying (2.3) are unique. When $(g_n)_{n=1}^\infty$ is a frame which is not a Schauder basis for $\mathcal{H}$ (i.e., an *overcomplete frame* for $\mathcal{H}$), then the coefficients c_n satisfying (2.1) and the frame $(f_n)_{n=1}^\infty$ satisfying (2.3) are not unique. This property of overcomplete frames is useful for some applications, it allows to choose appropriate coefficients for the representations according to some additional requirements.

As one can see in Theorem **2.1**, every frame is a complete sequence. While in finite dimensional Hilbert spaces every finite complete sequence is a frame (see [6, Sec. 1.1]), in infinite dimensional Hilbert spaces not every complete sequence is a frame (see the last two sequences in Example **2.1**). For more on frame theory we refer to the books [6, 15, 16].

3. Frame-concepts in Banach spaces

Banach frames [13], p-frames [1], and Θ-frames [4] can be considered as natural extensions of Hilbert frames to Banach spaces. Observe that the inequality (2.2) in the Hilbert frame definition can be written as

$$\sqrt{A}\,\|f\|_{\mathcal{H}} \leq \|(\langle f, g_n\rangle)_{n=1}^{\infty}\|_{\ell^2} \leq \sqrt{B}\,\|f\|_{\mathcal{H}},\ \forall f \in \mathcal{H}. \tag{3.4}$$

Thus, a natural way to extend the frame definition to Banach spaces is to consider more general sequence spaces then ℓ^2.

Definition 3.1. [13, 1, 4] Let X be a Banach space, Θ be a BK-space, and $(g_n)_{n=1}^{\infty} \in (X^*)^{\mathbb{N}}$. The sequence $(g_n)_{n=1}^{\infty}$ is called

- a Θ-*frame for* X if
  (i) $(g_n(f))_{n=1}^{\infty} \in \Theta$, $\forall f \in X$;
  (ii) there exist constants $B, A \in (0, \infty)$ so that
  $$A\|f\|_X \leq \|(g_n(f))_{n=1}^{\infty}\|_{\Theta} \leq B\|f\|_X,\ \forall f \in X;$$
- a p-*frame for* X if it is an ℓ^p-frame for X;
- a *Banach frame for* X *with respect to* Θ if (i) and (ii) hold, and
  (iii) $\exists$ bounded operator $Q : \Theta \to X$ so that $Q(g_n(f))_{n=1}^{\infty} = f$, $\forall f \in X$;
- a Θ-*Bessel sequence in* X if it satisfies (i) and at least the upper condition in (ii).

For a given Θ-Bessel sequence $(g_n)_{n=1}^{\infty}$ in X, its *analysis operator* $U : X \to \Theta$ is defined by $Uf = (g_n(f))_{n=1}^{\infty}$ and $R(U)$ denotes the range of U.

While a Hilbert frame for $\mathcal{H}$ automatically implies the validity of the property (iii) in the above definition and thus, it is automatically a Banach frame for $\mathcal{H}$ with respect to ℓ^2, in general Banach spaces, Θ-frames for X are not automatically Banach frames for X with respect to Θ. Furthermore, a Hilbert frame $(g_n)_{n=1}^{\infty}$ automatically implies the existence of series expansions in the form (2.3), but a Θ-frame $(g_n)_{n=1}^{\infty}$ for X does not automatically lead to series expansions in the form

$$g = \sum_{n=1}^{\infty} g(f_n) g_n,\ \forall g \in X^*, \tag{3.5}$$

$$f = \sum_{n=1}^{\infty} g_n(f) f_n,\ \forall f \in X, \tag{3.6}$$

via some sequence $(f_n)_{n=1}^{\infty} \in X^{\mathbb{N}}$. The next statement gives a necessary and sufficient condition for a Θ-frame $(g_n)_{n=1}^{\infty}$ to be a Banach frame and to imply series expansions in the form (3.5) and (3.6).

Theorem 3.1. [4] *Suppose that Θ is a BK-space, $(g_n)_{n=1}^{\infty} \in (X^*)^{\mathbb{N}}$ is a Θ-frame for X, and U denotes the analysis operator of $(g_n)_{n=1}^{\infty}$. Then the following conditions are equivalent:*

(i) *$(g_n)_{n=1}^{\infty}$ is a Banach frame for X with respect to Θ.*
(ii) *There exists a continuous projection from Θ onto $R(U)$.*
(iii) *The operator $U^{-1} : R(U) \to X$ can be extended to a bounded linear operator $V : \Theta \to X$.*

If Θ is a CB-space, then (i) is equivalent to
(iv) *There exists a Θ^*-Bessel sequence $(f_n)_{n=1}^{\infty} \in X^{\mathbb{N}} \subseteq (X^{**})^{\mathbb{N}}$ for X^* such that (3.6) holds.*

If both Θ and Θ^* are CB-spaces, then (i) is equivalent to
(v) *There exists a Θ^*-frame $(f_n)_{n=1}^{\infty} \in X^{\mathbb{N}} \subseteq (X^{**})^{\mathbb{N}}$ for X^* such that (3.5) holds.*

There exist cases, when p-frames of special structure are automatically Banach frames with respect to ℓ^p and imply series expansions, see [1].

4. Frame-concepts in Fréchet spaces

In this section we consider projective and inductive limits of Banach spaces (for example, the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions and its dual, the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$) and frame-concepts for such spaces.

Let $(Y_s, |\cdot|_s)_{s\in\mathbb{N}_0}$ be a sequence of separable Banach spaces such that

$$\{\mathbf{0}\} \neq \cap_{s\in\mathbb{N}_0} Y_s \subseteq \ldots \subseteq Y_2 \subseteq Y_1 \subseteq Y_0 \tag{4.7}$$

$$|\cdot|_0 \leq |\cdot|_1 \leq |\cdot|_2 \leq \ldots \tag{4.8}$$

$$Y_F := \cap_{s\in\mathbb{N}_0} Y_s \text{ is dense in } Y_s, \ \ s \in \mathbb{N}_0. \tag{4.9}$$

Then Y_F is a Fréchet space with the sequence of norms $|\cdot|_s$, $s \in \mathbb{N}_0$. We use such sequences in two cases:
1. $Y_s = X_s$ with norm $\|\cdot\|_s, s \in \mathbb{N}_0$;
2. $Y_s = \Theta_s$ with norm $|||\cdot|||_s, s \in \mathbb{N}_0$.

Throughout the rest of the section we always assume that $(X_s, \|\cdot\|_s)_{s\in\mathbb{N}_0}$ is a sequence of Banach spaces, which satisfies (4.7)-(4.9), and $(\Theta_s, |||\cdot|||_s)_{s\in\mathbb{N}_0}$ is a sequence of BK-spaces, which satisfies (4.7)-(4.9).

Recall [19] that an operator $G : \Theta_F \to X_F$ is called *F-bounded* if for every $s \in \mathbb{N}_0$ there exists a constant $K_s > 0$ such that $\|G(c_n)_{n=1}^\infty\|_s \leq K_s |||(c_n)_{n=1}^\infty|||_s$ for all $(c_n)_{n=1}^\infty \in \Theta_F$. If $G : \Theta_F \to X_F$ is F-bounded, then G is continuous. The converse does not hold in general.

4.1. Fréchet frames

The Θ-frame and Banach frame concepts were first extended to Fréchet spaces [18, 19] as follows:

Definition 4.1. A sequence $(g_n)_{n=1}^\infty \in (X_F^*)^{\mathbb{N}}$ is called

• a *pre-Fréchet-frame* (in short, *pre-F-frame*) *for* X_F *with respect to* Θ_F if for every $s \in \mathbb{N}_0$ there exist constants $B_s, A_s \in (0, \infty)$ such that

$$(g_n(f))_{n=1}^\infty \in \Theta_F, \ f \in X_F, \tag{4.10}$$

$$A_s\|f\|_s \leq |||(g_n(f))_{n=1}^\infty|||_s \leq B_s\|f\|_s, \ f \in X_F; \tag{4.11}$$

• a *Fréchet frame* (in short, *F-frame*) *for* X_F *with respect to* Θ_F, if it is a pre-F-frame for X_F with respect to Θ_F and there exists an F-bounded operator $V : \Theta_F \to X_F$ so that $V(g_n(f))_{n=1}^\infty = f$ for all $f \in X_F$.

• an *F-Bessel sequence for* X_F *with respect to* Θ_F if (4.10) and at least the upper inequality in (4.11) hold.

When $(g_n)_{n=1}^\infty \in (X_F^*)^{\mathbb{N}}$ is an F-Bessel sequence for X_F with respect to Θ_F, for every $n \in \mathbb{N}$ and every $s \in \mathbb{N}_0$, g_n has a unique continuous extension on X_s which will be denoted by g_n^s.

The next statements give a connection between pre-F-frames and F-frames, as well as results concerning series expansions in the corresponding spaces.

Theorem 4.1. [19] *Let* $(g_n)_{n=1}^\infty$ *be a pre-F-frame for* X_F *with respect to* Θ_F*. Then the following holds.*

(a) $(g_n)_{n=1}^\infty$ *is an F-frame for* X_F *with respect to* Θ_F *if and only if there exists an F-bounded projection from* Θ_F *onto* $R(U)$*.*

(b) If $(g_n)_{n=1}^\infty$ *is an F-frame for* X_F *with respect to* Θ_F*, the following statements hold.*

(b1) For every $s \in \mathbb{N}_0$*, the sequence* $(g_n^s)_{n=1}^\infty$ *is a Banach frame for* X_s *with respect to* Θ_s*.*

(b2) If Θ_s, $s \in \mathbb{N}_0$, are CB-spaces, then there exists $(f_n)_{n=1}^{\infty} \in (X_F)^{\mathbb{N}}$ which is a Θ_s^*-Bessel sequence for X_s^* for every $s \in \mathbb{N}_0$, and

$$f = \sum_{n=1}^{\infty} g_n(f) f_n, \; f \in X_F, \; (in\ X_F), \tag{4.12}$$

$$g = \sum_{n=1}^{\infty} g(f_n) g_n, \; g \in X_F^*, \; (in\ X_F^*), \tag{4.13}$$

$$f = \sum_{n=1}^{\infty} g_n^s(f) f_n, \; f \in X_s, \; s \in \mathbb{N}_0. \tag{4.14}$$

(b3) If Θ_s and Θ_s^*, $s \in \mathbb{N}_0$, are CB-spaces, then there exists $(f_n)_{n=1}^{\infty} \in (X_F)^{\mathbb{N}}$, which is a Θ_s^*-frame for X_s^* for every $s \in \mathbb{N}_0$, (4.12)-(4.14) hold and moreover,

$$g = \sum_{n=1}^{\infty} g(f_n) g_n^s, \; g \in X_s^*, \, s \in \mathbb{N}_0.$$

4.2. General Fréchet frames

In [21], Definition **4.1** was extended to a more general case allowing different norms in the upper and the lower inequality:

Definition 4.2. [21] A sequence $(g_n)_{n=1}^{\infty} \in (X_F^*)^{\mathbb{N}}$ is called

• a *General pre-Fréchet frame* (in short, *General pre-F-frame*) *for X_F with respect to* Θ_F if there exist sequences $(\widetilde{s}_k)_{k\in\mathbb{N}_0}$ and $(s_k)_{k\in\mathbb{N}_0}$ (with elements from $\mathbb{N}_0$) which increase to ∞ with the property $s_k \leq \widetilde{s}_k$, $k \in \mathbb{N}_0$, and there exist constants $B_k, A_k \in (0,\infty)$, $k \in \mathbb{N}_0$, satisfying

$$(g_n(f))_{n=1}^{\infty} \in \Theta_F \text{ and } A_k \|f\|_{s_k} \leq |||(g_n(f))_{n=1}^{\infty}|||_k \leq B_k \|f\|_{\widetilde{s}_k}, \; f \in X_F, k \in \mathbb{N}_0;$$

• a *General Fréchet frame* (in short, *General F-frame*) *for X_F with respect to* Θ_F if $(g_n)_{n=1}^{\infty}$ is a General pre-F-frame for X_F with respect to Θ_F and there exists a continuous operator $V : \Theta_F \to X_F$ so that $V(g_n(f))_{n=1}^{\infty} = f$ for all $f \in X_F$.

The next statement shows that some of the results for pre-F-frames are still valid in this more general setting and specifies the changes for the rest of the results.

Theorem 4.2. [21] *Let $(g_n)_{n=1}^{\infty}$ be a General pre-F-frame for X_F with respect to Θ_F. The following statements hold.*

(*a*) $(g_n)_{n=1}^{\infty}$ *is a General F-frame for* X_F *with respect to* Θ_F *if and only if there exists a continuous projection from* Θ_F *onto* $R(U)$.

(*b*) *Let* Θ_s, $s \in \mathbb{N}_0$, *be CB-spaces. If* $(g_n)_{n=1}^{\infty}$ *is a General F-frame for* X_F *with respect to* Θ_F, *then there exists a sequence* $(f_n)_{n=1}^{\infty} \in (X_F)^{\mathbb{N}}$ *such that*

$$f = \sum_{n=1}^{\infty} g_n(f) f_n, \; f \in X_F, \; \text{(in } X_F\text{)}, \tag{4.15}$$

$$g = \sum_{n=1}^{\infty} g(f_n) g_n, \; g \in X_F^*, \; \text{(in } X_F^*\text{)}. \tag{4.16}$$

(*c*) *Let* Θ_s, $s \in \mathbb{N}_0$, *be CB-spaces. The following three statements are equivalent.*

$\mathcal{A}_1$: *There exists an operator* $V : \Theta_F \to X_F$ *so that* $V(g_n(f))_{n=1}^{\infty} = f$, $\forall f \in X_F$, *and for every* $k \in \mathbb{N}_0$ *there is a constant* $C_k > 0$ *satisfying* $\|Vd\|_{s_k} \leq C_k |||d|||_k$ *for all* $d \in \Theta_F$.

$\mathcal{A}_2$: *There exists* $(f_n)_{n=1}^{\infty} \in (X_F)^{\mathbb{N}}$ *such that for every* $k \in \mathbb{N}_0$, $(f_n)_{n=1}^{\infty}$ *is a* Θ_k^**-Bessel sequence for* $X_{s_k}^*$, *and* (4.15) *holds.*

$\mathcal{A}_3$: *There exists* $(f_n)_{n=1}^{\infty} \in (X_F)^{\mathbb{N}}$ *such that for every* $k \in \mathbb{N}_0$, $(f_n)_{n=1}^{\infty}$ *is a* Θ_k^**-Bessel sequence for* $X_{s_k}^*$ *and*

$$f = \sum_{n=1}^{\infty} g_n^{\widetilde{s}_k}(f) f_n \; \text{in } \|.\|_{s_k}\text{-norm}, \; f \in X_{\widetilde{s}_k}.$$

(*d*) *Let* Θ_s *and* Θ_s^*, $s \in \mathbb{N}_0$, *be CB-spaces. Then* $\mathcal{A}_1$ *is equivalent to*

$\mathcal{A}_4$: *There exists* $(f_n)_{n=1}^{\infty} \in (X_F)^{\mathbb{N}}$, *such that for every* $k \in \mathbb{N}_0$, $(f_n)_{n=1}^{\infty}$ *is a* Θ_k^**-Bessel sequence for* $X_{s_k}^*$ *and*

$$g|_{X_{\widetilde{s}_k}} = \sum_{n=1}^{\infty} g(f_n) g_n^{\widetilde{s}_k} \; \text{in } \|.\|_{X_{\widetilde{s}_k}^*}\text{-norm}, \; g \in X_{s_k}^*.$$

At the end of the paper we present an example of a General Fréchet frame which is not a Fréchet frame. It shows that the extension of the F-frame concept to the General F-frame concept is essential.

Example 4.1. [21] Let A be a self-adjoint differential operator, for example, one dimensional normalized harmonic oscillator $(-d^2/dx^2 + 1)/2$) with eigenvalues $\lambda_j = j, j \in \mathbb{N}$, and eigenfunctions $\psi_j, j \in \mathbb{N}$ (Hermite functions) which make an orthonormal basis of $X_0 = L^2(\mathbb{R})$. For $s \in \mathbb{N}$, let X_s be the Hilbert space consisting of L^2-functions $\phi = \sum_{j=1}^{\infty} a_j \psi_j, a_j \in \mathbb{C}, j \in \mathbb{N}$, with the property $\sum_{j=1}^{\infty} |a_j|^2 j^{2s} < \infty$ and with the inner product

$$\langle \phi_1, \phi_2 \rangle_s = \sum_{j=1}^{\infty} a_{1,j} \overline{a_{2,j}} j^{2s}.$$

Then $X_F = \cap_{s \in \mathbb{N}_0} X_s = \mathcal{S}(\mathbb{R})$ (the Schwartz class of rapidly decreasing functions) and $X_F^* = \mathcal{S}'(\mathbb{R})$ (the space of tempered distributions).

For the sequence spaces Θ_s, $s \in \mathbb{N}_0$, take

$$(d_j)_{j=1}^{\infty} \in \Theta_s \text{ if and only if } \sum_{j=1}^{\infty} |d_j|^2 j^{2s} < \infty,$$

with the usual inner product; Θ_F is the space of rapidly decreasing sequences.

Let $r \in \mathbb{N}$ be given and let $(b_j)_{j=1}^{\infty}$ be a sequence of complex numbers such that

$$|b_j| = \begin{cases} 1, & j = 1, 3, 5, \dots ; \\ j^r, & j = 2, 4, 6, \dots . \end{cases}$$

Let $g_j = b_j \psi_j, j \in \mathbb{N}$. Then $(g_j)_{j=1}^{\infty}$ is a General Fréchet frame for X_F wrt Θ_F, but not a Fréchet frame for X_F wrt Θ_F.

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ON THE FUZZY MIKUSINSKI CALCULUS

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We dedicate this paper to the 100th anniversary of the birth of Professor Jan Mikusiński

Abstract

In this paper the fuzzy Mikusiński calculus was introduced and applied. The exact and the approximate solutions of the fuzzy fractional operational equations are constructed and their characters are analyzed. In that manner, the solution of a fuzzy fractional differential equations with fuzzy conditions corresponding to operational equations are considered.

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Key Words and Phrases: fractional calculus, fuzzy calculus, operational calculus, Mikusiński operators

1. Introduction

In recent literature, there are many authors analyzing fuzzy fractional differential equations. In the papers [18] and [10], the solutions of fuzzy fractional differential equations in the sense of Riemann-Liouville H-differentiability are constructed by using fuzzy Laplace transforms and Euler methods respectively. In Section 3, the fuzzy Mikusiński calculus is introduced. The fuzzy Mikusiński operators and their connection with fuzzy numbers and fuzzy functions is analyzed. The well known convergence in the field of Mikusiński operators is used for the definition of the convergence of fuzzy Mikusiński operators. In particular, fuzzy algebraic equations are analyzed and solved. The solutions of fractional fuzzy differential equations, where the fractional derivative are considered in the of Caputo Hukuhara differentiability, for $0 < \beta < 1$, in the frame of Mikusiński fuzzy calculus are constructed.

The techniques used in this paper were presented in our previous papers [19], [20], [21] and [22], on the problems without any fractional derivatives.

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2. Notions and notations

2.1. Some elements of the Mikusiński calculus

The set of continuous functions C_+ with supports in $[0,\infty)$, with the usual addition and the multiplication given by the convolution

$$(f*g)(t) \quad = \quad \int_0^t f(\tau)g(t-\tau)\,d\tau, \qquad t\geq 0,$$

is a commutative ring without unit element. By the Titchmarsh theorem, C_+ has no divisors of zero. The quotients of the form

$$\frac{f}{g},\quad f\in C_+,\ 0\not\equiv g\in C_+,$$

where the last division is considered in the sense of convolution (see [11]), are elements of the Mikusiński operator field, denoted by F, are called *operators*.

Also, among the most important operators are the inverse operator to ℓ, the differential operator s, and I, the identity operator, i.e., $\ell s=I$. Neither s nor I are operators from F_c, representing continuous function.

For the theory of differential equations, the following relation, connecting the operator representing the n-th derivative of an n times derivable function $a=a(t)$ with the operator a is essential:

$$\{a^{(n)}(t)\} \quad = \quad s^n a - a(0)s^{n-1} - \cdots - a^{(n-1)}(0)I. \tag{2.1}$$

In the field F we have the integral operator ℓ representing the constant function 1 on $[0,\infty)$, and the operator ℓ^β :

$$\ell=\{1\},\quad \ell^\beta=\left\{\frac{t^{\beta-1}}{\Gamma(\beta)}\right\},\ \ \beta\geq 0. \tag{2.2}$$

2.2. The fractional calculus

The research on fractional calculus can be found in many resent papers, as [6], [7], [23] and others. In this paper the Riemann-Liouville fractional integral operator J^β, of order $\beta>0$, is used. It is defined as the convolution

$$J^\beta f(t)=\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau)d\tau. \tag{2.3}$$

Since Caputo derivative (originated from [4]) is more suitable for applications to problems with initial and boundary conditions (see [15]) we shall use it for $0<\beta<1$, and $t>0$:

$$D^\beta f(t)=\frac{1}{\Gamma(1-\beta)}\int_0^t (t-\tau)^{-\beta}f'(\tau)d\tau. \tag{2.4}$$

2.3. The Mikusiński and the fractional calculus

From relations (2.3) and (2.4) it can be seen that in the field F the operator ℓ^β corresponds to the Riemann-Liouville fractional integral operator of order β, J^β. In fact, for every continuous function f it holds:

$$\ell^\beta f = \left\{ J^\beta f(t) \right\}. \tag{2.5}$$

The Caputo fractional derivative $D^\beta f(t)$, of order β, $0 < \beta < 1$, applied to a differentiable function f, corresponds to the operator $s^\beta f - f'(0)I$, since

$$\ell^{1-\beta}(sf - f(0)I) = \left\{ D^\beta f(t) \right\}. \tag{2.6}$$

2.4. Fuzzy calculus

Fuzzy set theory was introduced by L. Zadeh in [24]. In the following the fuzzy numbers are considered ([3], [1]) in parametric form as $u^\lambda = (u_1(\lambda), u_2(\lambda))$, $0 \le \lambda \le 1$ where the functions u_1, and u_2, satisfy the following:

(1) u_1 is a bounded left continuous nondecreasing function on $[0, 1]$.
(2) u_2 is a bounded left continuous nonincreasing function over [0, 1].
(3) $u_1 \le u_2$, for $0 \le \lambda \le 1$.

The arithmetic operations on fuzzy numbers $u, v \in E$ are defined as:

$$\begin{aligned}
&(u+v)(\lambda) = (u_1(\lambda) + v_1(\lambda)), u_2(\lambda) + v_2(\lambda)),\\
&(u-v)(\lambda) = (u_1(\lambda) - v_2(\lambda), u_2(\lambda) - v_1(\lambda))\\
&Ku(\lambda) = \begin{cases} (Ku_1(\lambda)), Ku_2(\lambda)) & K \ge 0 \\ (Ku_2(\lambda), Ku_1(\lambda)) & K < 0, \end{cases} \quad K \in R,
\end{aligned} \tag{2.7}$$

- Fuzzy function $f : [0, a] \to E$, where E is the set of fuzzy numbers, has parametric representation:

$$f(t, \lambda) = (f_1(t, \lambda), f_2(t, \lambda)), \quad t \in [0, a], \ \ 0 \le \lambda \le 1. \tag{2.8}$$

- Fuzzy integral of fuzzy function f can be defined as:

$$\int f(t, \lambda)dt = \left(\int f_1(t, \lambda)dt, \int f_2(t, \lambda)dt \right), \quad t \in [0, a], \ \ 0 \le \lambda \le 1. \tag{2.9}$$

- Fuzzy derivative of fuzzy function f can be defined as:

$$(f(t, \lambda))' = \left(f_1'(t, \lambda), f_2'(t, \lambda) \right), \quad t \in [0, a]. \ \ 0 \le \lambda \le 1, \tag{2.10}$$

In this paper we shall use the definition of fuzzy derivative as in relation (2.10), because the other definitions given in papers [18], [1], and others, do not make any essential difference in our considerations.

3. The Mikusiński fuzzy calculus

3.1. Introduction

In this section we shall introduce the fuzzy Mikusiński operators in the sense of definition of fuzzy numbers and Mikusiński calculus.

Every continuous function $a = a(t)$ with support in $[0, \infty)$ can be observed as a (unique) Mikusiński operator denoted by a. Every real valued continuous function $\{a(t, \lambda)\}$ of two variables $t \geq 0$, and $x \in [c, d]$, is observed as the Mikusiński operational function (denoted with $a(\lambda)$).

Next we introduce the following definition of fuzzy Mikusiński operator:

Definition 3.1. The operational function a^λ is a fuzzy Mikusiński operator, if there exists a Mikusiński operator q, satisfying the equality:

$$qa^\lambda = \{a(t, \lambda)\},$$

where a is a fuzzy function with a crisp variable $t \geq 0$, and $0 \leq \lambda \leq 1$.

The fuzzy Mikusiński operator a^λ can be considered in the parametric form as:

$$a^\lambda = (a_1(\lambda), a_2(\lambda)), \quad 0 \leq \lambda \leq 1, \tag{3.11}$$

where $a_1(\lambda)$, and $a_2(\lambda)$ are operational functions. Let us remark that in this paper we consider the fuzzy Mikusiński operators of the form:

$$\ell^\beta \cdot u^\lambda, \tag{3.12}$$

where β is a real number, u is a fuzzy number, for $0 \leq \lambda \leq 1$.

If $\beta = 0$, then the fuzzy operator given by (3.12), has the form $I \cdot u^\lambda$, does not represent a continuous function, and corresponds to $\delta \cdot u(\lambda)$ where δ is the delta distribution and $0 \leq \lambda \leq 1$. Using relation (2.2), it follows that the parametric form of Mikusiński fuzzy operator is given by

$$\ell^\beta \cdot u^\lambda = \Big(\ell^\beta u_1(\lambda), \ell^\beta u_2(\lambda)\Big) = \left\{ \left(\frac{t^{\beta-1}}{\Gamma(\beta)} u_1(\lambda), \frac{t^{\beta-1}}{\Gamma(\beta)} u_2(\lambda) \right) \right\}, \quad \beta > 0.$$

From (2.1) and (2.10) it follows that the fuzzy derivative Mikusiński operator can be written as:

$$\begin{aligned} sa^\lambda &= (sa_1(\lambda), sa_2(\lambda)) \\ &= (sa_1(\lambda) - a_1(0, \lambda)I, sa_2(\lambda) - a_2(0, \lambda)I), \end{aligned} \tag{3.13}$$

where s is Mikusiński derivative operator, and a is a fuzzy function.

We introduce the arithmetic of fuzzy operators, given by (3.12), in the sense of the arithmetics of fuzzy numbers (see [14]), as follows:

$$\begin{aligned} &(u^\lambda \ell^\beta + v^\lambda \ell^\beta) = (u_1((\lambda) + v_1(\lambda))\ell^\beta), (u_2(\lambda)\ell^\beta + v_2(\lambda))\ell^\beta), \\ &(u^\lambda \ell^\beta - v^\lambda \ell^\beta) = ((u_1(\lambda)\ell^\beta - v_2\ell^\beta(\lambda))\ell^\beta, (u_2(\lambda) - v_1(\lambda))\ell^\beta) \\ &K u^\lambda \ell^\beta = \begin{cases} (Ku_1(\lambda))\ell^\beta, Ku_2(\lambda)\ell^\beta), & K \geq 0 \\ (Ku_2(\lambda)\ell^\beta, Ku_1(\lambda)\ell^\beta), & K < 0, \end{cases}, \end{aligned} \tag{3.14}$$

where u and v are fuzzy numbers, and K is a real number.

Multiplications and divisions of fuzzy operators, given by (3.12), are considered in the sense of multiplications and divisions of fuzzy numbers (see [14]).

3.2. A fuzzy algebraic operational equation

Let us consider the following fuzzy operational equation

$$x^\lambda = A^\lambda x^\lambda + B^\lambda, \quad 0 \leq \lambda \leq 1.$$

In previous relation

$$A^\lambda = \ell^{\beta_1} a^\lambda, \;\; B^\lambda = \ell^{\beta_2} b^\lambda, \tag{3.15}$$

β_1 and β_2 are real numbers, and a, b are fuzzy numbers.

The solution of previous equation can be written in the form:

$$x^\lambda = \frac{B^\lambda}{I - A^\lambda} = B^\lambda \cdot \sum_{i=0}^{\infty} (A^\lambda)^i.$$

In the previous series fuzzy infinite operational sum appeared and thus we have to analyze its convergence.

The sequence of fuzzy numbers is analyzed in the paper [5] and its convergence is given by:

- The sequence of fuzzy numbers $u_n^\lambda = (u_{1,n}(\lambda), u_{2,n}(\lambda))$ converges to fuzzy number $u^\lambda = (u_1(\lambda), u_2(\lambda))$, if it holds:

  $$u_{1,n}(\lambda) \to u_1(\lambda), \quad u_{2,n}(\lambda) \to u_2(\lambda),$$

  uniformly, for $0 \leq \lambda \leq 1$.

Theorem 3.1. *If fuzzy operators A^λ and B^λ, are given by relation (3.15) for $0 \leq \lambda \leq 1$, then*

- *the fuzzy operator*

$$\sum_{i=0}^{\infty} (A^\lambda)^i = \sum_{i=0}^{\infty} (a^\lambda)^i \ell^{\beta_1 i}$$

converges for $\beta_1 > 0$;

- *the fuzzy operator*

$$x^\lambda = B^\lambda \sum_{i=0}^{\infty} (A^\lambda)^i = b^\lambda \ell^{\beta_2} \sum_{i=0}^{\infty} (a^\lambda)^i \ell^{\beta_1 i} \tag{3.16}$$

represents

  - *local integrable function for* $\beta_1 > 0$ *and* $\beta_2 > 0$;
  - *continuous function for* $\beta_1 > 0$, $\beta_2 \geq 1$.

From relations (2.2) and (3.16) it follows that the fuzzy operator x^λ for $\beta_1 > 0$, $\beta_2 \geq 1$, and $0 \leq \lambda \leq 1$, corresponds to the continuous fuzzy function

$$x^\lambda = b^\lambda \sum_{i=0}^{\infty} (a^\lambda)^i \frac{t^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)}. \tag{3.17}$$

3.3. Fuzzy Mittag-Lefler function

Let us remember that b^λ and a^λ, $0 \leq \lambda \leq 1$, are fuzzy numbers expressed in parametric forms as:

$$b^\lambda = (b_1(\lambda), b_2(\lambda)), \quad a^\lambda = (a_1(\lambda), a_2(\lambda)).$$

Then in the sense of powers of fuzzy numbers (see [14]) we have

$$(a^\lambda)^i = (a_1(\lambda)^i, a_2(\lambda)^i), \quad i = 1, \ldots,$$

and we can denote by

$$c_i^\lambda := b^\lambda \cdot (a^\lambda)^i = (c_{1,i}(\lambda), c_{2,i}(\lambda)), \quad i = 1, \ldots.$$

The continuous fuzzy function given by relation (3.17), after the corresponding arithmetic operations on fuzzy numbers a_i^λ, $i = 1, \ldots$, and b^λ, (see [14]) can be expressed in parametric forms as:

$$x^\lambda = \sum_{i=0}^{\infty} c_i^\lambda \frac{t^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)} = \left(\sum_{i=0}^{\infty} c_{1,i}(\lambda) \frac{t^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)}, \sum_{i=0}^{\infty} c_{2,i}(\lambda) \frac{t^{\beta_1+\beta_2-1}}{\Gamma(\beta_1+\beta_2)} \right).$$

The fuzzy function given by previous relations is in fact the fuzzy Mittag-Lefler function.

3.4. Mikusiński fuzzy fractional calculus

Fuzzy fractional integral operator, of order β, $0 < \beta < 1$, applied to fuzzy operator f^λ, in the sense of the Definition **3.1**, is given by:

$$\ell^\beta f^\lambda = (\ell^\beta f_1^\lambda, \ell^\beta f_2^\lambda),$$

and it corresponds in the to fuzzy fractional integral of fuzzy function f, given by relation (2.9).

Analogously, fuzzy fractional differential operator of order β, $0 < \beta < 1$, applied to fuzzy operator f^{λ}, i.e., $s^{\beta} f^{\lambda} - f'(\lambda, 0)s^{\beta-1}$ can be expressed as:

$$(s^{\beta} f_1^{\lambda} - f'(\lambda, 0)s^{\beta-1}, s^{\beta} f_2^{\lambda} - f'(\lambda, 0)s^{\beta-1})$$

and it corresponds to fuzzy fractional derivative, of order $\beta > 0$, of fuzzy function f, given by relation (2.10).

3.5. An application

Next we consider the fuzzy fractional differential equation:

$$x_t^{\beta}(t, \lambda) = x(t, \lambda) + u^{\lambda} t, \quad 0 < \beta < 1, \quad 0 \leq \lambda \leq 1, \tag{3.18}$$

with the fuzzy initial condition

$$x(0, \lambda) = x_0(\lambda), \quad 0 \leq \lambda \leq 1 \tag{3.19}$$

where x is a fuzzy function with crisp variable t, and x_0 and u are fuzzy numbers.

In the field of fuzzy Mikusiński operators the problem (3.18), (3.19), corresponds to the fuzzy operational equation

$$s^{\beta} x^{\lambda} - x^{\lambda} = \ell^{1-\beta} x_0^{\lambda} + \ell^2 u^{\lambda}. \tag{3.20}$$

The solution of previous equation has a form:

$$x^{\lambda} = \frac{\ell^{1-\beta} x_0^{\lambda} + \ell^2 u^{\lambda}}{s^{\beta} - 1} = \frac{\ell x_0^{\lambda} + \ell^{2+\beta} u^{\lambda}}{1 - \ell^{\beta}} = (\ell x_0^{\lambda} + \ell^{2+\beta} u^{\lambda}) \sum_{i=0}^{\infty} (-1)^i \ell^{\beta i}. \tag{3.21}$$

By taking

$$\begin{aligned} u(\lambda) &= (0,1,2) = (1-(1-\lambda), 1+(1-\lambda)), \quad 0 <\leq \lambda \leq 1 \\ x(0,\lambda) &= (1,2,3) = (2-(1-\lambda), 2+(1-\lambda)),, \quad 0 <\leq \lambda \leq 1, \end{aligned}$$

then the operational solution (3.21) has the form:

$$x^{\lambda} = \frac{(0,1,2)\ell + (1,2,3)\ell^{2+\beta} u^{\lambda}}{1 - \ell} = \Big((0,1,2) + (1,2,3)\ell^{1+\beta} u^{\lambda}\Big) \sum_{i=0}^{\infty} \ell^{\beta i + 1}.$$

If we denote by $M(t, \beta)$ and $L(t, \beta)$ the Mitag-Lefler functions corresponding to the operators $\sum_{i=0}^{\infty} \ell^{\beta i+1}$ and $\sum_{i=0}^{\infty} \ell^{(i+1)\beta+2}$, respectivelly, then the fuzzy solution of the problem (3.18), (3.19) is of the form:

$$\begin{aligned} x(t,\lambda) &= (0,1,2)M(t,\beta) + (1,2,3)L(t,\beta) \\ &= ((\lambda, 2-\lambda)M(t,\beta) + (1+\lambda, 3-\lambda)L(t,\beta)) \\ &= (\lambda M(t,\beta) + (1+\lambda)L(t,\beta), (2-\lambda)M(t,\beta) + (3-\lambda)L(t,\beta)). \end{aligned}$$

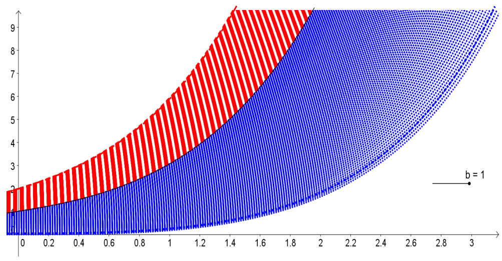

Figure 1.

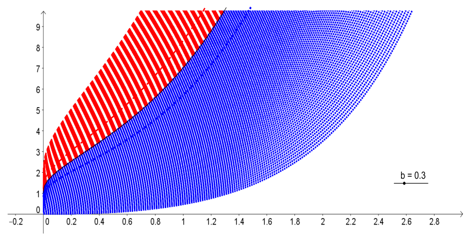

Figure 2.

On Figures 1 and 2 we visualized the approximate solutions $x_n(t,\lambda)$ for $n = 7$, (by taking seven addends in Mitag-Lefler function), and $\beta = 1$ and $\beta = 0.3$, respectively. The blue curves are the graphs of the function $\lambda M_7(t,\beta) + (1+\lambda)L_7(t,\beta)$ and the red curves are the graphs of the function $(2-\lambda)M_7(t,\beta) + (3-\lambda)L_7(t,\beta)$.

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BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS — REAL AND COMPLEX METHODS IN COMPARISON

Wolfgang Tutschke

Abstract

Using fundamental solutions, boundary value problems for elliptic equations and elliptic systems can be reduced to fixed-point problems for a suitably defined operator. In order to solve the related fixed-point problems, we apply both the contraction-mapping principle and the second version of the Schauder Fixed-point Theorem as well.

Since the right-hand sides are supposed to be only locally bounded or only locally Lipschitz-continuous, the fixed-pont theorems are applicable only in balls of the underlying function space, not in the whole function space. We show also how one can determine the best radius of the ball which leads to solvability conditions which are as weak as possible.

Elliptic first order systems in higher dimensions can be reduced to operators whose definition contains a monogenic function. The real-valued components of a monogenic function are solutions of the Laplace equation. The paper constructs also so-called distinguishing parts of the boundary from which the real-valued components can completely be recovered.

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Key Words and Phrases: reduction of boundary value problems for elliptic equations and systems to fixed-point methods using fundamental solutions; complex and Clifford-analytic normal forms of real systems; distinguishing parts of the boundary; optimization of fixed-point methods

1. Reduction of boundary value problems to fixed-point problems

As it is well-known, the initial value problem

$$y' = f(x, y)$$
$$y(x_0) = y_0$$

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can be reduced to the fixed-point problem

$$Y(x) = y_0 + \int_{x_0}^{x} f(\xi, y(\xi))\, d\xi.$$

Similarly, boundary value problems

$$Lu = F(x, u) \text{ in } \Omega \tag{1.1}$$

$$lu = g \text{ on } \partial\Omega \tag{1.2}$$

in domains Ω of $\mathbf{R^n}$ can be reduced to fixed-point problems. The associated operator is defined by a fundamental solution of the (elliptic) operator L.

This reduction of boundary value problems to fixed-point problems is based on a *Green Integral Formula* for differential operators of divergence type.

1.1. Green Integral Formula

A differential operator L of order k is said to be a *differential operator of divergence type* if there exists another differential operator L^* of order k and if there exist n differential operators P_i of order $k-1$ (depending on two functions u and v) such that

$$vLu + (-1)^{k+1} uL^*v = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} \ .$$

Then the Gauss Integral Theorem leads to the *Green Integral Formula*

$$\int_{\Omega} \Big(vLu + (-1)^{k+1} uL^*v\Big) dx = \int_{\partial\Omega} P[u, v] d\mu \tag{1.3}$$

where

$$P[u, v] = \sum_{i=1}^{n} P_i[u, v] N_i \ ,$$

$N = (N_1, ..., N_n)$ is the outer unit normal and $d\mu$ is the measure element of the boundary $\partial\Omega$.

1.2. The Cauchy-Pompeiu Integral Formula

Let $E(x, \xi)$ be a fundamental solution having an isolated singularity at the point ξ. A fundamental solution $E(x, \xi)$ of the equation $Lu = 0$ is a special solution having a special singular behaviour at an isolated point ξ. For the Laplace operator $L = \Delta$ in $\mathbf{R^3}$, for instance, a fundamental solution with singularity at ξ is given by the Newton potential

$$E(x, \xi) = -\frac{1}{4\pi} \cdot \frac{1}{|x - \xi|} \ .$$

Now let v be an arbitrary k times continuously differentiable function in the domain Ω of $\mathbf{R^n}$. Denote the ε-neighbourhood of ξ by $U_\varepsilon(\xi)$. Applying the Green Integral Formula (1.3) to $u = E(x,\xi)$ and v in $\Omega_\varepsilon = \Omega \setminus \overline{U_\varepsilon}$, one obtains

$$(-1)^{k+1} \int\limits_{\Omega_\varepsilon} E(x,\xi) L^* v dx = \int\limits_{\partial\Omega} P[E(x,\xi), v] d\mu + \int\limits_{|x-\xi|=\varepsilon} P[E(x,\xi), v] d\mu \quad (1.4)$$

because we have $Lu = 0$ in Ω_ε and the boundary of Ω_ε consists of the boundary $\partial\Omega$ of Ω and the ε-sphere $|x - \xi| = \varepsilon$. A fundamental solution $E(x,\xi)$ with singularity at ξ is, first, weakly singular at ξ and, second, the integral over the ε-sphere $|x-\xi| = \varepsilon$ tends to $-v(\xi)$ if $\varepsilon \to 0$. Weakly singular means that $E(x,\xi)$ can be estimated by

$$|E(x,\xi)| \leq \frac{\text{const}}{|x-\xi|^\alpha}, \text{ where } \alpha < n.$$

Therefore the limiting process $\varepsilon \to 0$ in (1.4) leads to the *Cauchy-Pompeiu Integral Formula* (or Borel-Pompeiu Integral Formula):

Theorem 1.

$$v(\xi) = \int\limits_{\partial\Omega} P[E(x,\xi), v] d\mu + (-1)^k \int\limits_{\Omega} E(x,\xi) L^* v dx.$$

Two important special cases of this formula are, first, the boundary integral representation

$$v(\xi) = \int\limits_{\partial\Omega} P[E(x,\xi), v] d\mu$$

for solutions v of the differential equation $Lv^* = 0$ and, second, the formula

$$\varphi(\xi) = (-1)^k \int\limits_{\Omega} E(x,\xi) L^* \varphi dx \quad (1.5)$$

which allows to recover a test function $v = \varphi$ from $L^*\varphi$. In the next subsection **1.3** we shall show that the last formula can be used in order to solve inhomogeneous differential equations $Lu = h$ with a given right-hand side h.

1.3. Solution of inhomogeneous partial differential equations

Theorem 2. *Suppose h is an integrable function given in Ω. Then*

$$u(x) = \int\limits_{\Omega} E(x,\xi) h(\xi) d\xi$$

is a special distributional solution of the inhomogeneous differential equation

$$Lu = h$$

in Ω.

Indeed, using the Fubini Teorem and applying formula (1.5), we obtain

$$\begin{aligned}
\int\limits_{\Omega_x} uL^*\varphi dx &= \int\limits_{\Omega_x}\Big(\int\limits_{\Omega_\xi} E(x,\xi)h(\xi)d\xi\Big)L^*\varphi dx \\
&= \int\limits_{\Omega_\xi} h(\xi)\Big(\int\limits_{\Omega_x} E(x,\xi)L^*\varphi(x)dx\Big)d\xi \\
&= (-1)^k\int\limits_{\Omega_\xi} h(\xi)\varphi(\xi)d\xi
\end{aligned}$$

where Ω_x and Ω_ξ mean the domain Ω as domains in the x-space and in the ξ-space, respectively.

1.4. The associated fixed-point problem

Consider again the boundary value problem (1.1), (1.2). Suppose $E(x,\xi)$ is a fundamental solution of the (elliptic) equation $Lu = 0$. Define

$$U(x) = u_0(x) + \tilde{u}(x) + \int\limits_{\Omega} E(x,\xi)F\left(\xi, u(\xi)\right)d\xi \tag{1.6}$$

where u_0 is a solution of the (uniquey solvable) boundary value problem

$$\begin{aligned}
Lu_0 &= 0 \text{ in } \Omega \\
lu_0 &= g \text{ on } \partial\Omega
\end{aligned}$$

and $\tilde{u}$ is a solution of the boundary value problem.

$$\begin{aligned}
&L\tilde{u} = 0 \text{ in } \Omega \\
&l\tilde{u} = -\left(\int\limits_{\Omega} E(\cdot,\xi)F\left(\xi, u(\xi)\right)d\xi\right) \text{ on } \partial\Omega.
\end{aligned}$$

Since u_0 has the prescibed boundary values g, and since $\tilde{u}$ compensates the boundary values of the integral term on (1.6), we obtain the following result:

Theorem 3. *Fixed points of the operator (*1.6*) are solutions of the boundary value problem (*1.1*), (*1.2*) and vice versa.*

So it remains to prove the existence of fixed points of the operator (1.6). This can be done either by the contraction-mapping principle or by the Schauder fixed-point theorem.

1.5. Estimates of the operator

In order to apply fixed-point theorems, one has to estimate the operator (1.6). To be short, we consider only the case of second-order differential operators L. Then one has to distinguish two cases:

Case I: The right-hand side of the differental equation (1.1) depends only on the desired solution u, not on its first-order derivatives.

Case II: The right-hand side of the differential equation (1.1) depends on u and also on the first-order derivatives of u.

In order to estimate the operator (1.6) in the case I, one needs only an estimate of $|u|$, and so one can use the supremum norm $\|\cdot\|$, see the Example in Subsection **4.1**.

1.6. Right-sides depending on first order derivatives

There are many papers dealing with differential equations of type (1.1) in the case II. Many papers such as the paper [2] of L. Boccardo, F. Murat and J. P. Puel consider special right hand sides such as $u|\text{grad } u|^2$ (see also the review MR766983). Newer papers such as [4] of D. Giachetti and S. Segura de Len solve boundary value problems of type (1.1), (1.2) in the case that the right-hand side has a quadratic gradient term with a singularity.

However, the above introduced operator (1.6) can also be used in order to solve boundary value problems in the case II. This has be done in the Thesis [11] of Muhammad Sajid Iqbal. Using the operator (1.6), it is not necessary to suppose that the right-hand side has a special structure (such as a quadratic gradient term). In this Thesis [11] the right-hand side can depend arbitrarily on the first order derivatives $\partial_i u$, $i = 1, ..., n$, that is, one has an arbitrary right-hand side $F(x, u, \partial_1 u, ..., \partial_n u)$.

In order to estimate the related operator (1.6), one has to estimate not only $|u|$ but also one needs an estimate of the absolute values $|\partial_i u|$ of the first order derivatives. Therefore it is not possible to use the supremum norm. A suitable function space is the space $\mathcal{C}^{1,\alpha}(\Omega)$ of Hölder-continuously differentiable functions. The integral term of the operator (1.6) can dirctly be estimated in $\mathcal{C}^{1,\alpha}(\Omega)$, while the auxiliary functions u_0 and $\tilde{u}$ can be estimated using the Schauder estimates (see, for instance, the paper [1] of E. A. Baderko).

1.7. Real and complex versions

In Section **2** we shall apply this method to non-linear first order systems in the plane, and in Section **3** we consider first order systems in higher dimensions.

Generally speaking, the fundamental solution $E(x,\xi)$ depends on the differential operator L (fundamental solutions for elliptic second order differential equations are constructed, for instance, in the book [9] of C. Miranda). We shall see, however, that in the case of first order systems the reduction to the operator (1.6) is always possible using the Cauchy kernels of complex and Clifford analysis, respectively. This is possible using special normal forms for first order sytems (see Subsection **2.1** for first order systems in the plane and Subsection **3.3** for first order systems in higher dimensions).

See also the paper [14] concerning the advantages of complex methods compared with real ones. This paper is Chapter 4.7 of the Proceedings [7].

2. First-order systems in the plane

2.1. A complex normal form

Consider the system

$$H_j\left(x, y, u, v, \partial_x u, \partial_y u, \partial_x v, \partial_y v\right) = 0, \quad j = 1, 2, \tag{2.7}$$

of two real equations for two desired real-valued functions u and v. Now introduce the following abbreviations:

$$\begin{aligned} \frac{1}{2}\left(\partial_x u + \partial_y v\right) &= p_1 \\ \frac{1}{2}\left(\partial_x v - \partial_y u\right) &= p_2 \\ \frac{1}{2}\left(\partial_x u - \partial_y v\right) &= q_1 \\ \frac{1}{2}\left(\partial_x v + \partial_y u\right) &= q_2. \end{aligned}$$

Then one has

$$\begin{aligned} \partial_x u &= p_1 + q_1 \\ \partial_y u &= -p_2 + q_2 \\ \partial_x v &= p_2 + q_2 \\ \partial_y v &= p_1 - q_1. \end{aligned}$$

Substituting these expressions into the system (2.7), this system passes into

$$H_j\left(x, y, u, v, p_1 + q_1, -p_2 + q_2, p_2 + q_2, p_1 - q_1\right) = 0, \quad j = 1, 2.$$

Now suppose that this system can be solved for q_1 and q_2. Then one gets real-valued representations

$$q_j = F_j(x, y, u, v, p_1, p_2), \quad j = 1, 2. \tag{2.8}$$

Introduce the partial complex differentiations

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \tag{2.9}$$

and
$$\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{2.10}$$

Since $x+iy = z$, $u+iv = w$ and $p_1+ip_2 = \partial_z w$, the variables on the right-hand sides of these equations can be expressed by z, w and $\partial_z w$ (and their conjugate complex values). Denoting $F_1 + iF_2$ by F, and taking into consideration that $q_1 + iq_2 = \partial_{\overline{z}} w$, the two equations (2.8) can be combined to the one complex equation:

Theorem 4. = Complex normal form of systems of the form (2.7)

$$\partial_{\overline{z}} w = F\,(z, w, \partial_z w)\,. \tag{2.11}$$

Special cases of (2.11):

- $F\,(z, w, \partial_z w) \equiv 0$ (classical Cauchy-Riemann system).
- $F\,(z, w, \partial_z w) \equiv A(z)w + B(z)\overline{w}$ (Vekua equation).
- $F\,(z, w, \partial_z w) \equiv q(z)\partial_z w$ (Beltrami equation).

The Beltrami equation is uniformly elliptic if $|q(z)| \leq q_0 < 1$.

Fundamental solutions of real (elliptic) second order equation depend on the coefficients of the equation (see Subsection **1.7**). However, for systems (2.11) one has:

Theorem 5. *In the case of all systems of the form (*2.11*) one can work with the same kernels:*

- *the Cauchy kernel* $\frac{1}{\pi} \cdot \frac{1}{z-\zeta}$ *and*
- *its derivative* $-\frac{1}{\pi} \cdot \frac{1}{(z-\zeta)^2}$.

In other words, it is not necessary to construct fundamental solutions depending on the right-hand side $F\,(z, w, \partial_z w)$.

2.2. Cauchy-Pompeiu Integral Formula in the complex plane

Let Ω be a domain in the complex plane whose boundary $\partial\Omega$ is sufficiently smooth. Let, further, f be a (complex-valued) function definied and continuously differentiable (with respect to the real variables x and y) in $\overline{\Omega}$. Suppose the domain is positively oriented, that is, the domain is on the left-hand side when transversing the boundary. Using the partial complex differentiations (2.9), (2.10), the Gauss Integral formulae

$$\iint_{\Omega} \partial_x f dxdy = \int_{\partial\Omega} f dy$$

$$\iint_{\Omega} \partial_y f dxdy = -\int_{\partial\Omega} f dx$$

can be written in the form

$$\iint_{\Omega} \partial_{\overline{z}} f dxdy = \frac{1}{2i} \int_{\partial\Omega} f dz \tag{2.12}$$

$$\iint_{\Omega} \partial_z f dxdy = -\frac{1}{2i} \int_{\partial\Omega} f d\overline{z}.$$

Consider

$$g(z) = \frac{f(z)}{z - \zeta}$$

where ζ is a point in Ω. This new function has g an isolated singularity at ζ. In order to apply the complex version (2.12) of the Gauss Integral Formula, the singularity ζ has be be omitted. For this purpose we introduce the domain Ω_ε:

$$\Omega_\varepsilon = \Omega \setminus \overline{U_\varepsilon(\zeta)}$$

where $U_\varepsilon(\zeta)$ s the ε-neighbourhood of ζ. Applying the complex Green-Gauss Integral Formula (2.12) to g in Ω_ε, it follows

$$\iint_{\Omega_\varepsilon} \partial_{\overline{z}} g dxdy = \frac{1}{2i} \int_{\partial\Omega_\varepsilon} g dz. \tag{2.13}$$

Observe that the boundary $\partial\Omega_\varepsilon$ of Ω_ε consists of the boundary $\partial\Omega$ of Ω and the circle $|z - \zeta| = \varepsilon$ with radius ε and centred at ζ. Since the boundary has to be oriented positively, the last circle has to be oriented negatively. Therefore formula (2.13) implies

$$\iint_{\Omega_\varepsilon} \frac{\partial_{\overline{z}} f(z)}{z - \zeta} dxdy = \frac{1}{2i} \int_{\partial\Omega} \frac{f(z)}{z - \zeta} dz - \frac{1}{2i} \int_{|z-\zeta|=\varepsilon} \frac{f(z)}{z - \zeta} dz, \tag{2.14}$$

where now the circle centred at ζ must have positive orientation. Carrying out the limiting process $\varepsilon \to 0$, one obtains the following *Cauchy-Pompeiu Integral Formula* in the complex plane:

Theorem 6. *Suppose f is in $\overline{\Omega}$ continuously differentiable with respect to the real variables x and y. Then*

$$f(\zeta) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(z)}{z-\zeta} dz - \frac{1}{\pi} \iint_{\Omega} \frac{\partial_{\overline{z}} f(z)}{z-\zeta} dxdy. \tag{2.15}$$

2.3. Solution of the inhomogeneous Cauchy-Riemann differential equation

Applying Theorem 2 to the inhomogeneous Cauchy-Riemann differential equation

$$\partial_{\overline{z}} w = h \tag{2.16}$$

in the complex plane leads to the following special solution:

Theorem 7. *Provided h is integrable, the function*

$$w(\zeta) = -\frac{1}{\pi} \iint_{\Omega} \frac{h(z)}{z-\zeta} dxdy$$

is a special (distributional) solution of the inhomogeneous Cauchy-Riemann differential equation (2.16).

2.4. Weyl Lemma for holomorphic functions

Applying formula (2.12) to $f = w\varphi$ (where φ is a continuously differentiable test function), one obtains

$$\iint_{\Omega} (\partial_{\overline{z}} w \varphi + w \partial_{\overline{z}} \varphi) dxdy = 0$$

because $\partial_{\overline{z}} f = \partial_{\overline{z}} w \varphi + w \partial_{\overline{z}} \varphi$. And so a distributional solution of the homogeneous Cauchy-Riemann equation $\partial_{\overline{z}} w = 0$ is an integrable function satisfying the relation

$$\iint_{\Omega} w \partial_{\overline{z}} \varphi dxdy = 0$$

for each test function φ.

The complex version of the Weyl Lemma is the following theorem:

Theorem 8. *An (integrable) weak solution of the homogeneous Cauchy-Riemann system is necessarily a holomorphic function in the classical sense.*

2.5. Boundary value problems for holomorphic functions

One of the simplest boundary value problems for holomorphic functions $w = u + iv$ is the following Dirichlet boundary value problem:

Prescribe the values of the imaginary part v on the whole boundary $\partial\Omega$ (which is supposed to be sufficiently smooth). Then the Cauchy-Riemann system is a (completely integrable) first order system for the real part u. And so u is uniquely determined by its value at one point (in simply conected domains).

A more general boundary value problem for holomorphic functions is the Riemann-Hilber problem. Here one prescribes the values of a liniear combination of u and v on the boundary. For the sake of simplicity we consider only the Dirichlet problem in simply connected domains.

2.6. Boundary value problems for generalized analytic functions

By analogy with Subsection **1.4** we consider now a boundary value problem which is solvable for holomorphic functions. In order to solve the same boundary value problem for the non-linear equation (2.11), we consider the operator

$$W(\zeta) = w_0(\zeta) + \tilde{w}(\zeta) - \frac{1}{\pi}\iint\limits_{\Omega} \frac{F(z, w(z))}{z - \zeta} dxdy \tag{2.17}$$

where w_0 is a holomorphic solution of the boundary value problem under consideration, and $\tilde{w}$ compensates the boundary values of the integral term to zero. Then the Statements 7 and 8 show that also the following statement is true:

Theorem 9. *Fixed points of the operator* (2.17) *are solutions of the considered boundary value problem for the non-linear system* (2.11).

3. First-order systems in higher dimensions

3.1. The Clifford algebra $\mathcal{A}_n$ - a generalization of complex numbers

Complex analysis of the plane is based on the fact that a product of vectors in the plane is defined. In order to apply similar methods in higher dimensions, one has to define also a product of vectors of $\mathbf{R^{1+n}}$, $n \geq 2$.

An arbitrary point $(a_0, a_1, ..., a_n)$ of $\mathbf{R^{1+n}}$ can be written as linear combination $\sum_{i=0}^{n} a_i e_i$ where $e_i = (0, 0, ..., 0, 1, 0, ..., 0)$, $i = 0, 1, ..., n$, and the component 1 is located at the $(i+1)$-th digit. Denote the vector e_i, $i = 1, ..., n$, by X_i.

Then the vectors of $\mathbf{R^{1+n}}$ can be interpreted as linear polynomials

$$a_0 + a_1 X_1 + \cdots + a_n X_n. \tag{3.18}$$

Clearly, products of vectors of $\mathbf{R^{1+n}}$ can now be defined as products of the corresponding polynomials (3.18). We agree to distinguish two products

$$X_{\mu_1} \cdots X_{\mu_m}$$

if they differ in the order of the factors.

While vectors of $\mathbf{R^{1+n}}$ are linear polynomials, arbitrary polynomials form a certain extension of the Euclidean space $\mathbf{R^{1+n}}$. In order to obtain a finite-dimensional extension, one has to use suitable equivalence relations for polynomials in X_1, ..., X_n. In order to find such relations, we consider the ring $\mathcal{R}[X_1, ..., X_n]$ of all polynomials with real coefficients in X_1, ..., X_n.

Firstly define the Cauchy-Riemann operator

$$D = \partial_0 + \sum_{k=1}^{n} e_k \partial_k \tag{3.19}$$

where ∂_i means differentiation with respect to x_i, $i = 0, 1, ..., n$. This operator is (up to the factor 1/2) a generalization of the Cauchy-Riemann operator $\partial_{\overline{z}}$ of the complex plane (cf. (2.10)) to $\mathbf{R^{1+n}}$. Together with (3.19) consider the adjoint operator

$$\overline{D} = \partial_0 - \sum_{j=1}^{n} e_j \partial_j.$$

Then it follows

$$\begin{aligned} \overline{D}D &= \Big(\partial_0 - \sum_{j=1}^{n} e_j \partial_j\Big)\Big(\partial_0 + \sum_{k=1}^{n} e_k \partial_k\Big) \\ &= \partial_0^2 - \sum_{j=1}^{n} e_j^2 \partial_j^2 - \sum_{j \neq k} e_j e_k \partial_j \partial_k \\ &= \partial_0^2 - \sum_{j=1}^{n} e_j^2 \partial_j^2 - \sum_{j<k} (e_j e_k + e_k e_j) \partial_j \partial_k. \end{aligned} \tag{3.20}$$

Denote by Δ_{n+1} the Laplace operator in $\mathbf{R^{1+n}}$. Then relation (3.20) implies

$$\overline{D}D = \Delta_{n+1}$$

provided one uses the equivalence relations

$$X_j^2 + 1 = 0 \quad \text{and} \quad X_j X_k + X_k X_j = 0 \tag{3.21}$$

where $j, k = 1, ..., n$ and $j \neq k$. These so-called structure relations in the ring $\mathcal{R}[X_1, ..., X_n]$ define the Clifford algebra $\mathcal{A}_n$ which is an extension of the Euclidean space $\mathbf{R^{1+n}}$ (see also the paper [12] in the Proceedings [10]).

The Clifford algebra $\mathcal{A}_n$ has 2^n basis elements

$$1,\ e_1, ...,\ e_n,\ e_{12}, ...,\ e_{12\cdots n}.$$

3.2. Left- and right-monogenic functions

Holomorphic functions $w = w(z)$ in the complex plane are defined by the Cauchy-Riemann equation $\partial_{\overline{z}} w = 0$. Analogously, left-monogenic (or shortly monogenic) functions u are defined by $Du = 0$ where D is the Cauchy-Riemann operator in $\mathbf{R^{1+n}}$ (see formula (3.19) in Subsection **3.1**).

Similarly, right-monogenic functions are defined by $uD = 0$.

3.3. A Clifford-analytic normal form

Consider a fully non-linear first order system of 2^n equations for 2^n desired real-valued functions $u_0,\ u_1,\ ...,\ u_n,\ u_{12},\ ...,\ u_{12...n}$ depending on $n+1$ real variables $x_0,\ x_1,\ ...,\ x_n$:

$$H_j\left(x_0, ..., x_n, u_0, ..., u_{12...n}, \partial_{x_0} u_0, ..., \partial_{x_n} u_{12...n}\right) = 0,\ j = 1, ..., 2^n. \qquad (3.22)$$

The 2^n desired real-valued functions can be interpreted as the components of a Clifford-algebra-valued function $u(x)$ depending on $x = (x_0, x_1, ..., x_n)$. Clearly, Du has also 2^n real components q_A which can be expressed by the first order derivatives of the real-valued components u_A of u. The definition (3.19) of the Cauchy-Riemann operator D shows that

$$q_A = \partial_{x_0} u_A + \text{ derivatives with respect to } x_1,\ ...,\ x_n.$$

Consequently, all derivatives with respect to x_0 can be expressed by the 2^n real variables q_A and derivatives with respect to x_1, ..., x_n. Carrying out this substitution, the system (3.22) turns out to be a system of 2^n equations depending on the $(n+1)+(n+2)2^n$ variables x_0, ..., x_n, u_A, q_A, $\partial_{x_k} u_A$, $k = 1, ..., n$. Now assume that this system can be solved for the 2^n variables q_A. Then the given system (3.22) can be written in the following form (see [13, 14]):

Theorem 10. = Clifford-analytic normal form

$$Du = F(x, u, \partial_{x_1} u, ..., \partial_{x_n} u). \qquad (3.23)$$

3.4. The Cauchy kernel of Clifford analysis

Next we are going to show how the Cauchy kernel

$$\frac{1}{\pi} \cdot \frac{1}{z-\zeta} = \frac{1}{\pi} \cdot \frac{\overline{z} - \overline{\zeta}}{|z-\zeta|^2}$$

of complex analysis in the plane can be generalized to the case of higher dimensions.

Let $x = x_0 + \sum_{j=1}^{n} x_j e_j$ be a point of $\mathbf{R^{1+n}}$. To this point x we define a conjugate point $\overline{x}$ by

$$\overline{x} = x_0 - \sum_{j=1}^{n} x_j e_j.$$

Note that sometimes x is called a "Clifford number" and $\overline{x}$ the "conjugate" Clifford number.

The definition of $\overline{x}$ implies

$$x\overline{x} = |x|^2.$$

Now let ω_{n+1} the surface measure of the unit sphere in $\mathbf{R^{1+n}}$. Define

$$E(x,\xi) = \frac{1}{\omega_{n+1}} \cdot \frac{\overline{x} - \overline{\xi}}{|x-\xi|^{n+1}} \tag{3.24}$$

where ξ is a fixed point of R^{1+n}. Then a direct calculation shows:

The function $E(x,\xi)$ is both left-monogenic and right-monogenic for each $x \neq \xi$.

3.5. The Green Integral Formula for the Cauchy-Riemann operator in Clifford analysis

Consider two Clifford-algebra-valued functions

$$u = \sum_A u_A e_A \quad \text{and} \quad v = \sum_B v_B e_B$$

with values in $\mathcal{A}_n$ which are continuously differentiable in $\overline{\Omega}$ where u_A and v_B are the real-valued components of u and v, respectively. Applying the (real) Gauss Integral Formula one can prove the following *Green Integral Formula* of Clifford analysis:

$$\int_\Omega \Big(vD \cdot u + v \cdot Du\Big) dx = \int_{\partial\Omega} v \cdot d\sigma \cdot u, \tag{3.25}$$

where $d\sigma$ is the Clifford-algebra-valued surface element

$$d\sigma = \sum_{j=0}^{n} e_j N_j d\mu$$

of Clifford analysis, $N = (N_0, N_1, ..., N_n)$ is the outer unit normal of $\partial\Omega$.

3.6. Cauchy-Pompeiu Integral Formula in Clifford analysis

Again let u be an arbitrary continuously differentiable Clifford-algebra-valued function in $\overline{\Omega}$. In order to apply the Green Integral Formula to this function u and the Cauchy kernel $v(x) = E(x,\xi)$, one has to omit the singularity ξ. Introduce the domain $\Omega_\varepsilon = \Omega \setminus \overline{U_\varepsilon(\xi)}$ where $U_\varepsilon(\xi) = \{x : |x-\xi| < \varepsilon\}$ is the ε-neighbourhood of ξ. Taking into account the properties of fundamental solutions (weak singularity and behaviour of the boundary integral over the ε-sphere centred at ξ), the limiting process $\varepsilon \to 0$ leads the following Cauchy Pompeiu Integral Formula:

$$u(\xi) = \int\limits_{\partial\Omega} E(x,\xi) \cdot d\sigma \cdot u - \int\limits_{\Omega} E(x,\xi) \cdot Du \cdot dx.$$

First special case: This formula is the Cauchy Integral Formula for monogenic functions

$$u(\xi) = \int\limits_{\partial\Omega} E(x,\xi) \cdot d\sigma \cdot u$$

because one has $Du = 0$ for monogenic functions.

Second special case: Test functions $u = \varphi$ can be recovered from $D\varphi$:

$$\varphi(\xi) = -\int\limits_{\Omega} E(x,\xi) \cdot D\varphi \cdot dx.$$

3.7. Another version of the Cauchy-Pompeiu Integral Formula

In the preceding Subsection **3.6** we obtained the Cauchy-Pompeiu Integral Formula by applying the Green Integral Formula to an arbitrary function u and $v = E(x,\xi)$. Now we apply the Green Integral Formula to $u = E(x,\xi)$ and an arbitrary function v. Then one obtains the following version of the Cauchy-Pompeiu Integral Formula:

$$v(\xi) = \int\limits_{\partial\Omega} v \cdot d\sigma \cdot E(x,\xi) - \int\limits_{\Omega} vD \cdot E(x,\xi) \cdot dx.$$

First special case: For a right-monogenic function v (that means $vD = 0$) one obtains the Cauchy Integral Formula

$$v(\xi) = \int\limits_{\partial\Omega} v \cdot d\sigma \cdot E(x,\xi).$$

Second special case: If $v = \varphi$ is a test function, then the Cauchy-Pompeiu Integral Formula leads to the relation

$$\varphi(\xi) = -\int\limits_{\Omega} \varphi D \cdot E(x,\xi) \cdot dx \tag{3.26}$$

showing that test functions φ can also be recovered from φD.

3.8. Solution of inhomogeneous differential equations

Suppose h is an integrable Clifford-algebra-valued function in the (bounded) domain Ω. Denote Ω as domain in the x- and in the ξ-space by Ω_x and Ω_ξ, respectively. Then the following theorem holds:

Theorem 11. *The function*

$$u(x) = \int\limits_{\Omega_\xi} E(x,\xi)h(\xi)d\xi$$

is a distributional solution of the inhomogeneous Cauchy-Riemann equation $Du = f$.

For the proof of this theorem we have to apply the special case (3.26) of the Cauchy-Pompeiu Integral Formula of the Subsection **3.7**.

Proof of Theorem 11. Using the Fubini Theorem for weakly singular integrals, the definition of u and the relation (3.26) implies

$$\begin{aligned}
\int\limits_{\Omega_x} \varphi D \cdot u dx &= \int\limits_{\Omega_x} \varphi D \left(\int\limits_{\Omega_\xi} E(x,\xi)h(\xi)d\xi \right) dx \\
&= \int\limits_{\Omega_\xi} \left(\int\limits_{\Omega_x} \varphi D \cdot E(x,\xi)dx \right) h(\xi)d\xi \\
&= -\int\limits_{\Omega_\xi} \varphi(\xi)h(\xi)d\xi.
\end{aligned}$$

Replacing the integration variable ξ of the last integral by x, we have thus proved that for each test function φ the relation

$$\int\limits_{\Omega_x} \Big(\varphi D \cdot u + \varphi \cdot h\Big) dx = 0$$

holds. ■

3.9. Weyl Lemma for monogenic functions

If u is a continuously differentiable monogenic function (that is, $Du = 0$), then the Green Integral Formula (3.25) of Subsection **3.5** implies

$$\int\limits_{\Omega} \varphi D \cdot u dx = 0 \tag{3.27}$$

for each test function φ. This relation (3.27) leads to the concept of monogenic functions in the distributional sense:

An integrable function u is said to be a *monogenic functions in the distributional sense* if relation (3.27) is satisfied for each test function.

Then the following Weyl Lemma is true:

Theorem 12. *A monogenic functions in the distributional sense is necessarily a monogenic function in the classical sense, that is, u is continuously differentiable and the equation $Du = 0$ is pointwise satisfied.*

3.10. Boundary value problems for monogenic functions

Let Ω be a (bounded) domain in the complex plane whose boundary is sufficiently smooth. Let, further, $w = u_0 + iu_1$ be a holomorphic function. Since the imaginary part u_1 is a solution of the Laplace equation, u_1 is uniquely determined by its boundary values. Knowing u_1, the real part u_0 can be calcutated using the Cauchy-Riemann system

$$\partial_0 u_0 = \partial_1 u_1, \quad \partial_1 u_0 = -\partial_0 u_1.$$

This system is completely integrable because $\Delta u_1 = 0$. Thus u_0 is (in simply connected domains) uniquely determined by its value at one point.

A similar situation occurs for monogenic functions $u = u_0 + u_1 e_1 + u_2 e_2 + u_{12} e_{12}$ in $\mathbf{R^3}$ (see the paper [17]). The four real-valued components satisfy the Cauchy-Riemann system

$$\partial_0 u_0 - \partial_1 u_1 - \partial_2 u_2 = 0 \tag{3.28}$$
$$\partial_0 u_1 + \partial_1 u_0 + \partial_2 u_{12} = 0 \tag{3.29}$$
$$\partial_0 u_2 - \partial_1 u_{12} + \partial_2 u_0 = 0 \tag{3.30}$$
$$\partial_0 u_{12} + \partial_1 u_2 - \partial_2 u_1 = 0. \tag{3.31}$$

Let u be given in $\overline{\Omega}$ where Ω is a cylincrical domain in x_0-direction

$$\Omega = \Big\{ x = (x_0, x_1, x_2) : \psi_1(x_1, x_2) < x_0 < \psi_2(x_1, x_2),\ (x_1, x_2) \in \Omega_0 \Big\}.$$

Here is Ω_0 a simply connected domain in the x_1, x_2-plane. Again in view of the Laplace equation, the two components u_1 and u_2 are already uniquely determined by their values on the whole boundary.

Knowing u_1 and u_2, and knowing the values of u_{12} on the lower covering surface

$$S_0 = \left\{ x = (x_0, x_1, x_2) : x_0 = \psi_1(x_1, x_2),\ (x_1, x_2) \in \overline{\Omega}_0 \right\}$$

of the cylindrical domain Ω, the last equation (3.31) allows to calculate u_{12} in the whole cylindrical domain by a simple integration in x_0-direction.

It remains to calculate the real part u_0. The three first equations (3.28), (3.29) and (3.30) form a first order system of the form

$$\partial_j u_0 = p_j,\ j = 0, 1, 2,$$

where

$$p_0 = \partial_1 u_1 + \partial_2 u_2,\ p_1 = -\partial_0 u_1 - \partial_2 u_{12},\ p_2 = -\partial_0 u_2 + \partial_1 u_{12}.$$

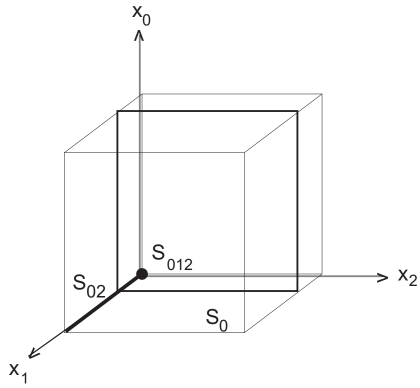
The system (3.28) - (3.30) is completely integrable because using the Laplace equation for u_1 we have

$$\begin{aligned} \partial_1 p_0 - \partial_0 p_1 &= \partial_1^2 u_1 + \partial_1 \partial_2 u_2 + \partial_0^2 u_1 + \partial_0 \partial_2 u_{12} \\ &= -\partial_2^2 u_1 + \partial_1 \partial_2 u_2 + \partial_0 \partial_2 u_{12} = \partial_2(-\partial_2 u_1 + \partial_1 u_2 + \partial_0 u_{12}). \end{aligned}$$

Since the last bracket is equal to zero in view of equation (3.31), the condition $\partial_1 p_0 - \partial_0 p_1 = 0$ is satisfied. Similarly, the Laplace equation for u_2 shows that $\partial_2 p_0 - \partial_0 p_2 = 0$. The compatibility condition $\partial_2 p_1 - \partial_1 p_2 = 0$, finally, is satisfied if we use $\Delta u_{12} = 0$.

To sum up, a monogenic function in a cylindrical domain in $\mathbf{R^3}$ is completely determined if one knows the two components u_1 and u_2 on the whole boundary, the the component u_{12} on the basis S_0 of the cylindrical domain, and one has to know the value of u_0 at one point.

In order to carry out analogous constructions in Euclidean spaces $\mathbf{R^{1+n}}$ of arbitrary dimensions, one has to consider domains which can be decomposed into μ-dimensional fibres (see the paper [15], cylindrical domains can be decomposed into one-dimensional fibres). The fibres are defined by so-called distinguishing $(1 + n - \mu)$-dimensional parts of the boundary. In order to explain this concept, we consider the interval $\Omega = \{x : 0 < x_j < 1,\ j = 0, 1, 2\}$ in $\mathbf{R^3}$:



This interval can be decomposed into one-dimensional fibres x_0-direction where a distinguishing part of the boundary is the lower covering surface $S_0 = \{x \in \overline{\Omega} : x_0 = 0\}$. The domain Ω is, however, can also be decomposed into two-dimensional fibres inx_0, x_2-direction where a distinguishing part of the boundary is the one-dimensional interval $S_{02} = \{x \in \overline{\Omega} : 0 \leq x_1 \leq 1\}$ on the x_1-axis. However, the interval Ω can also be interpreted as a three-dimensional fibre where the corresponding distinguishing part is the (zero-dimensional) point $S_{012} = \{x \in \overline{\Omega} : x = (0, 0, 0)\}$.

In accordance with the the decomposition of domains into μ-dimensional fibres, the Cauchy-Riemann system can be decomposed into μ-dimensional subsystems for particular components. These subsystems turn out to be completely integrable so that the corresponding components can be calculated from their values in the distinguishing part of the boundary.

The components of monogenic functions in $\mathbf{R^4}$ satisfy the system

$$\partial_0 u_0 - \partial_1 u_1 - \partial_2 u_2 - \partial_3 u_3 = 0 \tag{3.32}$$

$$\partial_0 u_1 + \partial_1 u_0 + \partial_2 u_{12} - \partial_3 u_{13} = 0 \tag{3.33}$$

$$\partial_0 u_2 - \partial_1 u_{12} + \partial_2 u_0 + \partial_3 u_{23} = 0 \tag{3.34}$$

$$\partial_0 u_3 + \partial_1 u_{13} - \partial_2 u_{23} + \partial_3 u_0 = 0 \tag{3.35}$$

$$\partial_0 u_{12} + \partial_1 u_2 - \partial_2 u_1 - \partial_3 u_{123} = 0 \tag{3.36}$$

$$\partial_0 u_{23} - \partial_1 u_{123} + \partial_2 u_3 - \partial_3 u_2 = 0 \tag{3.37}$$

$$\partial_0 u_{13} - \partial_1 u_3 - \partial_2 u_{123} + \partial_3 u_1 = 0 \tag{3.38}$$

$$\partial_0 u_{123} + \partial_1 u_{23} + \partial_2 u_{13} + \partial_3 u_{12} = 0. \tag{3.39}$$

Suppose Ω can be decomposed into 1-dimensional fibres in x_0-directions, into 2-dimensional fibres in x_0, x_2-direction and also that Ω is a 4-dimensional fibre in all directions. Suppose, further, the corresponding distinguishing parts of the boundary are S_0, S_{02} and S_{0123}. Suppose, finally, that all fibres are homotopically simply connected. Then the monogenic function can be reconstructed from the following data (see the paper [15]):

The four components u_1, u_2, u_3, u_{123} can be found from their values on the whole boundary. Using (3.36) and (3.37), the two components u_{12} and u_{23} resp. can be found from their values on the three-dimensional distinguishing part S_0 of the boundary, while in view of the two-dimensional system (3.38) the component u_{13} can be calculated from the values on the boundary curve S_{02}. The component u_0, finally, is as solution of (3.32) - (3.35) completely determined by its value at the point S_{0123}.

3.11. Boundary value problems for generalized monogenic functions

Now we consider again the general first order system (3.22) in its Clifford-analytic normal form (3.23) (see Subsection **3.3**). Consider, further, a boundary value problem which is solvable for monogenic functions. Define the operator

$$U(x) = u_0(x) + \tilde{u}(x) + \int\limits_{\Omega} E(x,\xi)F(\xi, u(\xi), \partial_1 u(\xi), ..., \partial_n u(\xi))d\xi, \qquad (3.40)$$

where $E(x, \xi)$ is the Cauchy kernel of Clifford analysis, u_0 is a monogenic function of the boundary value problem and $\tilde{u}$ is a monogenic function compensating the boundary values of the integral to zero.

By analogy with Theorem 3 in Subsection **1.4** and Theorem 9 in Subsection **2.6**, one has

Theorem 13. *Fixed points of the operator (*3.40*) are solutions of the boundary value problem for the system (*3.23*).*

The investigation of boundary value problems for monogenic functions and for systems of the form (3.23) is in progress.

4. Outlook

4.1. Optimization of fixed-point methods

If L is the Laplace operator in $\mathbf{R^3}$, then the operator (1.6) reads

$$U(x) = u_0(x) + \tilde{u}(x) - \frac{1}{4\pi}\int\limits_{\Omega} \frac{F(\xi, u(\xi))}{|\xi - x|}d\xi \qquad (4.41)$$

because

$$-\frac{1}{4\pi} \cdot \frac{1}{|\xi - x|}$$

is a fundamental solution of the Laplace equation in $\mathbf{R^3}$. The functions u_0 and $\tilde{u}$ are solutions of the Laplace equation, $\Delta u_0 = 0$ and $\Delta\tilde{u} = 0$. The right-hand side $F(x,u)$ is supposed to be (locally) bounded by $K(R)$, $|F(x,u)| \le K(R)$ for $x \in \overline{\Omega}$ and $|u| \le R$.

In order to estimate the operator (4.41), we use the Schmidt Inequality for domains Ω in $\mathbf{R^n}$ with bounded measure $m\Omega$. Suppose $0 \le \alpha < n$. Then

$$\int\limits_{\Omega} \frac{1}{|\xi - x|^\alpha} d\xi \le \frac{\omega_n}{n-\alpha}\left(\frac{m\Omega}{\tau_n}\right)^{1-\frac{\alpha}{n}}$$

for each x of $\mathbf{R^n}$ where τ_n and ω_n are the volume and the surface measure resp. of the unit ball. And so the absolute value of the integral term in (4.41) can for domains Ω in $\mathbf{R^3}$ be estimated by

$$\frac{1}{2}K(R)\left(\frac{3m\Omega}{4\pi}\right)^{\frac{2}{3}} \tag{4.42}$$

in $|F(x,u)| \le K(R)$ for $x \in \overline{\Omega}$ and $|u| \le R$. In case the boundary values g of the desired solution u can be estimated by $|g| \le C$, the maximum-minimum principle for the Laplace equation leads to the estimate $|u_0| \le C$. Analogously, $|\tilde{u}|$ can be estimated by the expression (4.42).

Now suppose that Ω is the unit ball in $\mathbf{R^3}$. To sum up, we obtain the estimate $|U| \le C + K(R)$. Thus the operator (4.41) maps the ball $\|u\| \le R$ into itself provided $C + K(R) \le R$. In other words, the maximal possible bound C for the boundary values can be estimated by $C \le \Lambda_1(R)$ where $\Lambda_1(R) = R - K(R)$. Clearly, possible radii R have to satisfy the condition $\Lambda_1(R) \ge 0$.

Since

$$|F(x,u)| \le |F(x,u) - F(x,0)| + |F(x,0)|,$$

we obtain $K(R) \le L(R)\cdot R + M$ if $F(x,u)$ is Lipschitz continuous with the Lipschitz constant $L(R)$ in the ball $\|u\| \le R$, and if $|F(x,0)| \le M$. And so the operator (4.41) maps the ball $\|R\| \le R$ into itself also under the condition $\Lambda_2(R) \ge 0$ where $\Lambda_2(R) = R - (L(R)\cdot R + M)$.

Since the boundary value problem has been reduced to the fixed-point problem for the operator (4.41), one has to apply suitable fixed-point theorems such as the contraction-mapping principle or the second version of the Schauder Fixed-Point Theorem.

In order to compare the solvability conditions, we consider now the special right-hand side

$$\Delta u = \frac{1}{4}(1 + u^2).$$

In this case we have

$$K(R) = \frac{1}{4}(1+R^2), \quad L(R) = \frac{R}{2} \quad \text{and} \quad M = \frac{1}{4}.$$

The above introduced bounds $\Lambda_1(R)$ and $\Lambda_2(R)$ for C have then the form

$$\Lambda_1(R) = R - \frac{1}{4}R^2 - \frac{1}{4} \quad \text{and} \quad \Lambda_2(R) = R - \frac{1}{2}R^2 - \frac{1}{4}.$$

The bound $\Lambda_1(R)$ is positive only in the interval $R_1 < R < R_2$ where

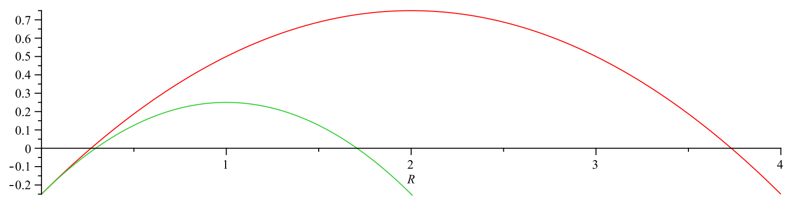
$$R_1 = 2 - \sqrt{3} \text{ and } R_2 = 2 + \sqrt{3}.$$

Further, $\Lambda_1(R)$ takes is maximal at $R_* = 2$, its maximal value is $\Lambda_1(2) = 3/4$.

Similarly, $\Lambda_2(R)$ is positive only in (R_1, R_2) where we have now

$$R_1 = 1 - \frac{1}{2}\sqrt{2} \quad \text{and} \quad R_2 = 1 + \frac{1}{2}\sqrt{2}.$$

The maximal value $\Lambda_2(R_*) = 1/4$ is taken at $R_* = 1$.



And so the contraction-mapping principle is at least applicable to boundary values with $0 \le C \le 1/4$, whereas the Schauder Fixed-point Theorem can be applied for C with $0 \le C \le 3/4$.

The boundary value problem $g \equiv 0$ can be solved in a ball whose radius is equal to the minimal value R_1 of the admissible interval $[R_1, R_2]$. And so the boundary values $g \equiv 0$ lead to the optimal a-priori estimates

$$\|u\| \le 2 - \sqrt{3} \text{ and } \|u\| \le 1 - \frac{1}{2}\sqrt{2}$$

in case one applies the Schauder Fixed-Point Theorem and the contraction-mapping principle, resp. (where the first bound is the better one).

The operator turns out to be contractive if only $L(R) < 1$, and thus the boundary value problems are always (uniquely) solvable by the contraction-mapping principle if only $C < 3/4$, whereas the Schauder Fixed-Point Theorem is also applicable for $C = 3/4$.

4.2. General structure relation

The classical Clifford algebras $\mathcal{A}_n$ are defined by the structure relations (3.21) in Subsection **3.1**. These structure relations imply that the related monogenic functions satisfy the Laplace equation. In order to include more general (elliptic or non-elliptic) differential equations, one can replace the structure relations (3.21) by more general ones, see [12, 16, 18, 20] (the paper [16] is Chapter

14 of the book [3], and [20] is Chapter 8 in the Proceedings [6]). Piecewise constant structure relations are considered in [19].

4.3. Multi-monogenic functions

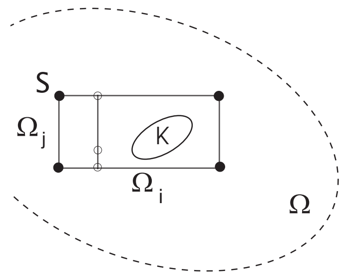
Let u be a function depending on n variables $x^{(i)}$, $i = 1, ..., n$, where $x^{(i)}$ runs in an $(m_i + 1)$-dimensional Euclidean space $\mathbf{R^{m_i+1}}$. The function u is said to be *multi-monogenic* in case u is monogenic with respect to each separate variable $x^{(i)}$.

Suppose Ω is a domain in $\mathbf{R^{m_1+1}} \times \cdots \times \mathbf{R^{m_n+1}}$. Suppose, further, that K is a compact subset of Ω such that $\Omega \setminus K$ is connected. Suppose, additionally, that there exist domains Ω_i (with sufficiently smooth boundaries $\partial\Omega_i$) in $\mathbf{R^{m_i+1}}$ such that the polycylinder

$$\Omega_0 = \Omega_1 \times \cdots \times \Omega_n$$

has the following properties (see the figure below):

- The closure of Ω_0 is completely contained in Ω.
- The compact subset K of Ω is completely contained in Ω_0.



Then the following continuation theorem holds:

Theorem 14. = Hartogs continuation theorem for multi-monogenic functions
Suppose u is multi-monogenic in $\Omega \setminus K$. Then u can be uniquely extended to a multi-monogenic function in the whole domain Ω.

Indeed, applying the Cauchy Integral Formula for separately monogenic functions, the function u can in $\Omega_0 \setminus K$ be represented by an integral over $S = \partial\Omega_1 \times \cdots \times \partial\Omega_n$. Since the Cauchy kernels are monogenic functions of the particular variables $x^{(i)}$, the integral defines the desired continuation of u to the whole domain Ω_0, concerning details see the paper [5] and [8] which is the Chapter 5 of the book [6].

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