Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 11, NUMBER 1 (2008) ISSN 1311-0454

# ALGORITHMS FOR EVALUATION OF THE WRIGHT FUNCTION FOR THE REAL ARGUMENTS' VALUES

**Yury Luchko**

**Abstract**

The paper deals with analysis of several techniques and methods for the numerical evaluation of the Wright function. Even if the focus is mainly on the real arguments' values, the methods introduced here can be used in the complex plane, too. The approaches presented in the paper include integral representations of the Wright function, its asymptotic expansions and summation of series. Because the Wright function depends on two parameters and on one (in general case, complex) argument, different numerical techniques are employed for different parameters' values. In every case, estimates for accuracy of the computations are provided. The ideas and techniques employed in the paper can be used for numerical evaluation of other functions of the hypergeometric type.

*2000 Math. Subject Classification*: 33E12, 65D20, 33F05, 30E15

*Key Words and Phrases*: Wright function, special functions, integral representations, numerical evaluation of special functions, asymptotic representations

## 1. Introduction

The Wright function was introduced for the first time in [14] in connection with a problem in the number theory regarding the asymptotic of the number of some special partitions of the natural numbers. It was defined trough the convergent series

$$\phi(\rho,\beta;z) := \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\beta+\rho k)},\ \rho > -1,\ \beta \in \mathbb{R},\ z \in \mathbb{C}. \quad (1)$$

Later on, the Wright function (1) found some other applications, e.g. in the Mikusiński operational calculus and in the theory of integral transforms of Hankel type.

But the "golden times" for the Wright function begun just with the start of the active research in Fractional Calculus and its applications. Unexpected enough, the Wright function appeared as the Green function while solving some initial- and boundary-value problems for the fractional diffusion-wave equation, i.e., for the linear partial integro-differential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order $\alpha$ with $0 < \alpha \le 2$ (see e.g. [10]-[12] or [5] for details). In these problems, the Wright function plays the same role as the exponential function for the partial differential equations does. It is worth mentioning that whereas the exponential function is equally important both for the ordinary and partial differential equations, there are two functions - the Mittag-Leffler function and the Wright function - that adopt its properties and value for the ordinary and partial differential equations of fractional order, respectively. Both the Mittag-Leffler function and the Wright function are entire functions of completely regular growth (see [2] for the proof for the Mittag-Leffler function, and [8] for the proof and for many other interesting properties of the Wright function). Moreover, it was shown in [8], that the indicator function $h_\rho = h_\rho(\theta),\ (|\theta| \le \pi)$, that characterizes the growth of the entire function $\phi(\rho, \beta; z)$ along the ray $z = re^{i\theta},\ 0 < r < \infty$, is given by the formulae

$$h_\rho(\theta) = \sigma \cos p\theta \qquad (|\theta| \le \pi)$$

when $\rho \ge 0$ and

$$h_\rho(\theta) = \begin{cases} -\sigma \cos p(\pi + \theta) & \text{for } -\pi \le \theta \le 0 \\ -\sigma \cos p(\theta - \pi) & \text{for } 0 \le \theta \le \pi \end{cases}$$

when $-\frac{1}{3} \le \rho < 0$; $p = \frac{1}{1+\rho}$ being the order and $\sigma = (1+\rho)|\rho|^{-\frac{\rho}{1+\rho}}$ the type of the entire function $\phi(\rho, \beta; z)$.

In particular, it means that the indicator function $h_\rho(\theta)$ of the Wright function $\phi(\rho, \beta; z)$ is reduced to the function $\cos\theta$ (the indicator function of the exponential function $e^z$) if $\rho \to 0$. This property is not valid for another generalization of the exponential function – the Mittag-Leffler function – defined by

$$E_\alpha(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (\alpha > 0, z \in \mathbb{C}).$$

Even though
$$E_1(z) = e^z,$$
the indicator function of the Mittag-Leffler function given for $0 < \alpha < 2, \alpha \neq 1$ by (see e.g. [2])
$$h(\theta) = \begin{cases} \cos\frac{\theta}{\alpha} & \text{for } |\theta| \leq \frac{\pi\alpha}{2} \\ 0 & \text{for } \frac{\pi\alpha}{2} \leq |\theta| \leq \pi \end{cases}$$
does not coincide with the indicator function of $e^z$ if $\alpha \to 1$. In this sense, i.e. from the viewpoint of the theory of the entire functions, the Wright function is a more "natural" generalization of the exponential function comparing with the Mittag-Leffler function. Let us finally mention another connection between the Wright function and Fractional Calculus: as it was shown in [1], [6], [9], the scale-invariant solutions of some partial differential equations of fractional order can be expressed in terms of the Wright and the generalized Wright functions.

In this part of the section, the place of the Wright function in the world of the special functions of the mathematical physics is shortly discussed. Due to the relation
$$\phi(1, \nu + 1; -\frac{1}{4}z^2) = \left(\frac{z}{2}\right)^{-\nu} J_\nu(z)\,,$$
Wright himself considered the function $\phi$ as a generalization of the Bessel function $J_\nu$. If $\rho$ is a positive rational number, the Wright function can be represented in terms of the familiar generalized hypergeometric functions ([5], [8]):
$$\phi(\frac{n}{m}, \beta; z) = \sum_{p=0}^{m-1} \frac{z^p}{p!\Gamma(\beta + \frac{n}{m}p)}$$
$$\times\, {}_0F_{n+m-1}\left(-; \Delta(n, \frac{\beta}{n} + \frac{p}{m}), \Delta^*(m, \frac{p+1}{m}); \frac{z^m}{m^m n^n}\right),$$
where ${}_pF_q((a)_p; (b)_q; z)$ is the generalized hypergeometric function,
$$\Delta(k, a) = \{a, a + \frac{1}{k}, \ldots, a + \frac{k-1}{k}\}, \quad \Delta^*(k, a) = \Delta(k, a) \setminus \{1\}.$$
We note that the set $\Delta^*(k, a)$ is correctly defined since the number 1 is always an element of the set $\Delta(m, \frac{p+1}{m}),\ 0 \leq p \leq m - 1$.

Similar formulae exist in the case of negative rational $\rho$ but under the additional condition that the parameter $\beta$ is also a rational number (see e.g. [5], [8]).

In particular,

$$\phi(-\frac{1}{2}, -n; z) = e^{-\frac{1}{4}z^2} z P_n(z^2) \qquad (n \in I\!N_0),$$

$$\phi(-\frac{1}{2}, \frac{1}{2} - n; z) = e^{-\frac{1}{4}z^2} Q_n(z^2) \qquad (n \in I\!N_0),$$

where $P_n, Q_n$ are polynomials of degree $n$ defined as

$$P_n(z) = \frac{(-1)^{n+1}}{\pi}\Gamma(\frac{3}{2}+n)\ {}_1F_1(-n; \frac{3}{2}; \frac{z}{4}),$$

$$Q_n(z) = \frac{(-1)^{n}}{\pi}\Gamma(\frac{1}{2}+n)\ {}_1F_1(-n; \frac{1}{2}; \frac{z}{4}).$$

In the general case of arbitrary real $\rho > -1$, the Wright function (1) is a particular case of the Fox $H$-function (see [6], [7, App. E], [13, Ch.1]):

$$\phi(\rho, \beta; z) = H^{1,0}_{0,2}\left[ -z \,\middle|\, \begin{array}{c} - \\ (0,1), (1-\mu, \rho) \end{array} \right]. \qquad (2)$$

Along with the Mittag-Leffler function, the Wright function is one of the most prominent and well studied particular cases of the Fox $H$-function that are not reduced in the general case to some particular cases of the Meijer $G$-function. From the viewpoint of properties of the Wright function, the representation (2) is not especially informative since the Fox $H$-function is a very general and still not very well studied object. In particular, there are still no algorithms or programs for numerical evaluation of the Fox $H$-function known. Because the Wright function plays an important role in the theory of the initial- and boundary-value problems for the fractional diffusion-wave equation and in other areas of mathematics (see e.g. [5]), the development of approaches for its numerical evaluation has became an urgent task that this paper tries to contribute to.

The Wright function depends on three variables: on the complex argument $z$ and on the real parameters $\rho$ and $\beta$. Experience in the computation of special functions of mathematical physics teaches us that in distinct parts of the complex plane different numerical techniques should be used. The aim of this paper is the development of some methods for computing the

Wright function (1) as well as the assessment of the range of their applicability and accuracy. In particular, we employ its Taylor series for small $|z|$, asymptotic representations for $|z|$ of large magnitude, and special integral representations for intermediate values of the argument $|z|$. The remainder of the paper is organized as follows: The 2nd section deals with the integral representations of the Wright function, both known and new ones. The 3rd section presents the most important asymptotic formulae for the Wright function that follow from its integral representations. Finally, in the 4th section we formulate several algorithms for the numerical evaluation of the Wright function. Even if the focus of the paper is on the evaluation of the Wright function for the real arguments' values, the presented algorithms can be easy extended to the complex plane. This case will be considered elsewhere.

## 2. Integral representations of the Wright function

We start with the known integral representation ([15])

$$\phi(\rho,\beta;z) = \frac{1}{2\pi i}\int_{\gamma(\epsilon)} e^{\zeta+z\zeta^{-\rho}}\zeta^{-\beta}\,d\zeta,\ \ \rho>-1,\ \ \beta\in \mathbb{R}, \tag{3}$$

where $\gamma(\epsilon)$ denotes the Hankel path in the $\zeta$-plane with a cut along the negative real semi-axis $\arg\zeta=\pi$. The formula (3) can be obtained by substituting the Hankel representation for the reciprocal of the gamma function

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi i}\int_{\gamma(\epsilon)} e^{\zeta}\zeta^{-s}\,d\zeta,\ \ s\in\mathbb{C} \tag{4}$$

for $s=\rho k+\beta$ into the right-hand side of the formula (1) and changing the order of integration and summation.

The primary aim of this section is to transform the integral representation (3) to a form more suitable for numerical evaluations.

We first remark that the contour $\gamma(\epsilon)$ consists of two rays $S_{-\pi}$ and $S_\pi$ ($\arg\zeta=-\pi,\ |\zeta|\geq\epsilon$ and $\arg\zeta=\pi,\ |\zeta|\geq\epsilon$) and a circular arc $C_\delta(0;\epsilon)$ ($|\zeta|=\epsilon,\ -\pi<\arg\zeta<\pi$). With

$$\phi(\zeta,z) = e^{\zeta+z\zeta^{-\rho}}\zeta^{-\beta},$$

we then have

$$\phi(\rho,\beta;z) = \frac{1}{2\pi i}\int_{\gamma(\epsilon)} \phi(\zeta,z)\,d\zeta = \frac{1}{2\pi i}\int_{S_{-\pi}} \phi(\zeta,z)\,d\zeta \tag{5}$$

$$+\frac{1}{2\pi i}\int_{C_\delta(0;\epsilon)}\phi(\zeta,z)\,d\zeta+\frac{1}{2\pi i}\int_{S_\pi}\phi(\zeta,z)\,d\zeta=I_1+I_2+I_3.$$

Now we transform the integrals $I_1,\ I_2$ and $I_3$. For $I_1$ we take $\zeta=re^{-i\pi},\ \epsilon\le r<\infty$ and get

$$I_1=\frac{1}{2\pi i}\int_{S_{-\pi}}\phi(\zeta,z)\,d\zeta=\frac{1}{2\pi i}\int_\epsilon^{+\infty}e^{-r+zr^{-\rho}e^{\pi\rho i}}r^{-\beta}e^{\pi\beta i}\,dr.$$

Analogously for the integral $I_3$ (with $\zeta=re^{i\pi},\ \epsilon\le r<\infty$)

$$I_3=\frac{1}{2\pi i}\int_{S_\pi}\phi(\zeta,z)\,d\zeta=-\frac{1}{2\pi i}\int_\epsilon^{+\infty}e^{-r+zr^{-\rho}e^{-\pi\rho i}}r^{-\beta}e^{-\pi\beta i}\,dr.$$

In what follows, we restrict our attention to the case $z=x,\ x\in I\!R$, that is very important in applications, especially while dealing with the initial- and boundary-value problems for the fractional diffusion-wave equation. Of course, similar but slightly more complicated expressions can be obtained for arbitrary $z=x+iy$; this general case will be considered elsewhere.

For $z=x,\ x\in I\!R$ we rewrite the sum $I_1+I_3$ as

$$I_1+I_3=\frac{1}{2\pi i}\int_\epsilon^{+\infty}e^{-r}r^{-\beta}\left(e^{xr^{-\rho}e^{\pi\rho i}}e^{\pi\beta i}-e^{xr^{-\rho}e^{-\pi\rho i}}e^{-\pi\beta i}\right)dr$$

$$=\frac{1}{\pi}\int_\epsilon^{+\infty}K(\rho,\beta,x,r)\,dr, \tag{6}$$

$$K(\rho,\beta,x,r):=e^{-r+xr^{-\rho}\cos(\pi\rho)}r^{-\beta}\sin(xr^{-\rho}\sin(\pi\rho)+\pi\beta). \tag{7}$$

For the integral $I_2$ we have $\zeta=\epsilon e^{i\varphi},\ -\pi<\varphi<\pi$ and

$$I_2=\frac{1}{2\pi i}\int_{C_\delta(0;\epsilon)}\phi(\zeta,z)\,d\zeta=\frac{\epsilon^{1-\beta}}{\pi}\int_0^\pi P(\rho,\beta,x,\epsilon,\varphi)\,d\varphi, \tag{8}$$

$$P(\rho,\beta,x,\epsilon,\varphi):=e^{\epsilon\cos(\varphi)+x\epsilon^{-\rho}\cos(\rho\varphi)+\cos(\varphi(\beta-1))} \tag{9}$$
$$\times\ \cos(\epsilon\sin(\varphi)-x\epsilon^{-\rho}\sin(\rho\varphi)-\sin(\varphi(\beta-1))).$$

To determine the behavior of the integrals (6) and (8) when $\epsilon\to0$ we need to differ between two cases (the case $x=0$ is trivial):

A) $x\in I\!R,\ x<0$,

B) $x\in I\!R,\ x>0$.

The result for the case A) is given in the following

THEOREM 2.1. *Let $z = -x,\ x > 0$. Then the Wright function $\phi$ has the following integral representations depending on its parameters $\rho$ and $\beta$:*

$$\phi(\rho, \beta; -x) = \frac{1}{\pi} \int_0^{+\infty} K(\rho, \beta, -x, r)\, dr, \tag{10}$$

*if* $(-1 < \rho < 0$ *and* $\beta < 1)$ *or* $0 < \rho < \frac{1}{2}$ *or* $(\rho = \frac{1}{2}$ *and* $\beta < 1 + \rho)$,

$$\phi(\rho, \beta; -x) = e + \frac{1}{\pi} \int_0^{+\infty} K(\rho, \beta, -x, r)\, dr, \tag{11}$$

*if* $-1 < \rho < 0$ *and* $\beta = 1$,

$$\phi(\rho, \beta; -x) = \frac{1}{\pi} \int_1^{+\infty} K(\rho, \beta, -x, r)\, dr \ + \ \frac{1}{\pi} \int_0^{\pi} \tilde{P}(\rho, \beta, -x, \varphi)\, d\varphi, \tag{12}$$

*in all other cases, with*

$$K(\rho, \beta, x, r) = e^{-r + x r^{-\rho} \cos(\pi\rho)} r^{-\beta} \sin(x r^{-\rho} \sin(\pi\rho) + \pi\beta), \tag{13}$$

$$\tilde{P}(\rho, \beta, x, \varphi) := e^{\cos(\varphi) + x\cos(\rho\varphi) + \cos(\varphi(\beta-1))} \times \cos(\sin(\varphi) - x\sin(\rho\varphi) - \sin(\varphi(\beta - 1))). \tag{14}$$

To prove the theorem, the behavior of the integrals (6) and (8) for $\epsilon \to 0$ will be investigated. We begin with the integral (6) and consider several cases:

1) $-1 < \rho < 0,\ \beta \notin Z$.

In this case, the integrand of the integral (6) - the function $K(\rho, \beta, -x, r)$ - behaves like the function $r^{-\beta} \sin(\pi\beta)$ when $r \to 0$ and thus the integral (6) converges to the improper integral

$$\int_0^{+\infty} K(\rho, \beta, -x, r)\, dr \tag{15}$$

for $\epsilon \to 0$ under the condition $\beta < 1$.

2) $-1 < \rho < 0,\ \beta \in Z$.

The function $K(\rho, \beta, -x, r)$ from the integral (6) behaves like the function $r^{-\beta} \sin(c r^{-\rho})$ when $r \to 0$, $c$ being a constant not depending on the variable $r$. Then the integral (6) converges to the improper integral (15) for $\epsilon \to 0$ under the condition $\beta < 1 - \rho$. Because $1 - \rho < 2$ and $\beta \in Z$, we can transform the condition $\beta < 1 - \rho$ to the form $\beta \leq 1$.

Combining the cases 1) and 2), the condition for convergence of the integral (6) to the improper integral (15) for $\epsilon \to 0$ in the case $-1 < \rho < 0$ reads $\beta \le 1$.

3) $\rho > 0$, $\cos(\pi\rho) < 0$.

In this case, the main term of the function $K(\rho, \beta, -x, r)$ from the integral (6) behaves like the exponential function $e^{ar^{-\rho}}$, $a > 0$ when $r \to 0$, i.e. the integral (6) diverges when $\epsilon \to 0$ and we cannot transform it to the form (15).

4) $\rho > 0$, $\cos(\pi\rho) > 0$.

In this case, $K(\rho, \beta, -x, r)$ behaves like the exponential function $e^{ar^{-\rho}}$, $a < 0$ when $r \to 0$ and the integral (6) converges to the integral (15) when $\epsilon \to 0$.

5) $\rho > 0$, $\cos(\pi\rho) = 0$.

Then $K(\rho, \beta, -x, r)$ behaves like the function $r^{-\beta}\sin(cr^{-\rho} + \pi\beta)$ when $r \to 0$, $c$ being a constant not depending on $r$. According to the Abel test for convergence of the improper integrals, the integral (6) converges to the integral (15) for $\epsilon \to 0$ under the condition $\beta < 1 + \rho$.

Now we have to consider the behavior of the integral (8) for $\epsilon \to 0$.

1) $-1 < \rho < 0$, $\beta < 1$.

For $\epsilon \to 0$, the function $P(\rho, \beta, x, \epsilon, \varphi)$ from the integral (8) is a restricted one and, since $\epsilon^{1-\beta} \to 0$ for $\epsilon \to 0$, the right-hand side of the expression (8) tends to zero if $\epsilon \to 0$.

2) $-1 < \rho < 0$, $\beta = 1$.

In this case we can directly evaluate the limit of the function $P(\rho, \beta, x, \epsilon, \varphi)$ when $\epsilon \to 0$ and then the value of the integral (8) using the fact, that this integral is a uniformly convergent one:

$$\lim_{\epsilon \to 0} I_2 = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_0^{\pi} P(\rho, \beta, x, \epsilon, \varphi)\, d\varphi$$

$$= \frac{1}{\pi} \int_0^{\pi} \lim_{\epsilon \to 0} P(\rho, \beta, x, \epsilon, \varphi)\, d\varphi = \frac{1}{\pi} \int_0^{\pi} e\, d\varphi = e.$$

3) $\rho > 0$.

In this case, we have to determine the asymptotic of the integral

$$I(\rho) = \int_0^{\pi} e^{-x\epsilon^{-\rho}\cos(\rho\varphi)}\, d\varphi \tag{16}$$

when $\epsilon \to 0$. This integral belongs to the class of the integrals in the form

$$F(\lambda) = \int_a^b e^{\lambda S(t)}\, dt \tag{17}$$

with a function $S \in C^2[a,b]$ $(S(t) = -\cos(\rho t),\ \lambda = x\epsilon^{-\rho})$. The asymptotic of the integral (17) for $\lambda \to +\infty$ can be easily found using the method of steepest descent (see e.g. [3]). The main term of the asymptotic of the integral (17) is given by

$$F(\lambda) \approx C\lambda^{-1/2}e^{\lambda S(t_0)},\ \lambda \to +\infty \ \text{ if } \ t_0 \in ]a,b[, \tag{18}$$

or

$$F(\lambda) \approx C\lambda^{-1}e^{\lambda S(t_0)},\ \lambda \to +\infty \ \text{ if } \ t_0 = a \ \text{ or } \ t_0 = b, \tag{19}$$

where the point $t_0$ is defined by $S(t_0) = \max_{t\in[a,b]} S(t)$ and the constant $C$ does not depend on $\lambda$. Applying this formula to the integral (16) we get the following results:

3a) If $0 < \rho < 1/2$, then $\max_{t\in[0,\pi]}\{-\cos(\rho t)\} < 0$ and the integral (16) behaves like the exponential function $e^{-C\epsilon^{-\rho}}$, $C > 0$ multiplied with a power function when $\epsilon \to 0$. This means that the right-hand side of the formula (8) tends to zero for any value of the parameter $\beta$ when $\epsilon \to 0$.

3b) If $\rho = 1/2$, then $\max_{t\in[0,\pi]}\{-\cos(\rho t)\} = 0$. The maximum point $t_0 = \pi$ coincides in this case with the endpoint of the integration interval $[0,\pi]$ and we have to use the asymptotic formula (19):

$$I_2 \approx C\epsilon^{1-\beta} \cdot (\epsilon^{-\rho})^{-1} = C\epsilon^{1-\beta+\rho}, \quad \epsilon \to 0.$$

Thus the integral $I_2$ tends to zero for $\epsilon \to 0$ when $\beta < 1 + \rho$.

Putting now the cases 1)-5) for the integral (6) and the cases 1)-3) for the integral (8) together and using the representation (5), (6), and (8) completes the proof of the theorem. ∎

We formulate now a theorem that contains some integral representations of the Wright function in the case $z = x > 0$.

Theorem 2.2. *Let $z = x > 0$. Then the Wright function $\phi$ has the following integral representations depending on its parameters $\rho$ and $\beta$:*

$$\phi(\rho,\beta;x) = \frac{1}{\pi}\int_0^{+\infty} K(\rho,\beta,x,r)\,dr, \tag{20}$$

*if $-1 < \rho < 0$ and $\beta < 1$,*

$$\phi(\rho,\beta;x) = e + \frac{1}{\pi}\int_0^{+\infty} K(\rho,\beta,x,r)\,dr, \tag{21}$$

*if* $-1 < \rho < 0$ *and* $\beta = 1$,

$$\phi(\rho, \beta; x) = \frac{1}{\pi} \int_1^{+\infty} K(\rho, \beta, x, r)\, dr \;+\; \frac{1}{\pi} \int_0^{\pi} \tilde{P}(\rho, \beta, x, \varphi)\, d\varphi, \tag{22}$$

*in all other cases, where the functions* $K(\rho, \beta, x, r)$ *and* $\tilde{P}(\rho, \beta, x, \varphi)$ *are defined as in (13) and (14), respectively.*

The proof of Theorem 2.2 follows the lines of the proof of Theorem 2.1 with some evident changes and we omit it here.

Let us note that the integral representations provided in Theorems 2.1 and 2.2 will be used in the 4th section for numerical evaluation of the Wright function for the "intermediate" values of the argument $z$ $(1 < |z| << \infty)$. Whereas for the small arguments' values we can evaluate the Wright function by means of the series (1), the asymptotic formulae can be used for the large arguments' values. A short overview of the asymptotic of the Wright function is given in the next section.

## 3. Asymptotic behavior of the Wright function

The complete picture of the asymptotic behavior of the Wright function for large values of $|z|$ was given by Wright [15] in the case $\rho > 0$ and by Wright [16] in the case $-1 < \rho < 0$. In both cases he used the method of steepest descent and the integral representation (3) to get the asymptotic expansions of his function.

It is well known that one can use the asymptotic formulae for the numerical evaluation of the special functions for $|z|$ of large magnitude. In the paper, we present only the results needed for our further discussions; for other results see [5], [15], [16]. Let us first consider the case $\rho > 0$.

Theorem 3.1. *Let* $\rho > 0$, $\arg z = \theta$, $|\theta| \le \pi - \epsilon$, $\epsilon > 0$. *Then*

$$\phi(\rho, \beta; z) = Z^{\frac{1}{2}-\beta} e^{\frac{1+\rho}{\rho} Z} \left\{ \sum_{m=0}^{M} \frac{(-1)^m a_m}{Z^m} + \mathrm{O}\left( \frac{1}{|Z|^{M+1}} \right) \right\}, \; Z \to \infty, \tag{23}$$

*where* $Z = (\rho|z|)^{1/(\rho+1)} e^{i\theta/(\rho+1)}$ *and the* $a_m$, $m = 0, 1, \dots,$ *are defined as the coefficients of* $v^{2m}$ *in the expansion of*

$$\frac{\Gamma(m+\frac{1}{2})}{2\pi} \left( \frac{2}{\rho+1} \right)^{m+\frac{1}{2}} (1-v)^{-\beta} \left( 1 + \frac{\rho+2}{3} v + \frac{(\rho+2)(\rho+3)}{3 \cdot 4} v^2 + \dots \right)^{-\frac{2m+1}{2}}.$$

If $z = -x,\ x > 0$ (the case not covered by Theorem 3.1), we get the asymptotic expansion of the Wright function $\phi(\rho, \beta; -x)$ for $x \to +\infty$ in the form ([5])

$$\phi(\rho, \beta; -x) = x^{p(\frac{1}{2}-\beta)} e^{\sigma x^p \cos \pi p} \cos\left(\pi p(\frac{1}{2} - \beta) + \sigma x^p \sin \pi p\right) \left\{c_1 + O(x^{-p})\right\},$$

where $p = \frac{1}{1+\rho}$, $\sigma = (1+\rho)\rho^{-\frac{\rho}{1+\rho}}$ and the constant $c_1$ can be exactly evaluated.

In the case $\rho = 0$, the Wright function is reduced to the exponential function with the constant factor $1/\Gamma(\beta)$: $\phi(0, \beta; z) = \exp(z)/\Gamma(\beta)$ which turns out to vanish identically for $\beta = -n,\ n = 0, 1, \ldots$.

The case $\rho < 0$ is considered in the following

Theorem 3.2. *Let* $-1 < \rho < 0,\ y = -z,\ -\pi < \arg z \le \pi,\ -\pi < \arg y \le \pi,\ |\arg y| \le \min\{\frac{3}{2}\pi(1+\rho), \pi\} - \epsilon,\ \epsilon > 0$. *Then*

$$\phi(\rho, \beta; z) = Y^{\frac{1}{2}-\beta} e^{-Y} \left\{ \sum_{m=0}^{M-1} A_m Y^{-m} + \mathrm{O}(Y^{-M}) \right\},\ Y \to \infty, \tag{24}$$

*where* $Y = (1+\rho)\left((-\rho)^{-\rho} y\right)^{1/(1+\rho)}$ *and the coefficients* $A_m,\ m = 0, 1 \ldots$ *are defined by the asymptotic expansion*

$$\frac{\Gamma(1-\beta-\rho t)}{2\pi(-\rho)^{-\rho t}(1+\rho)^{(1+\rho)(t+1)}\Gamma(t+1)} = \sum_{m=0}^{M-1} \frac{(-1)^m A_m}{\Gamma((1+\rho)t + \beta + \frac{1}{2} + m)} + \mathrm{O}\left(\frac{1}{\Gamma((1+\rho)t + \beta + \frac{1}{2} + M)}\right),$$

*valid for* $\arg t$, $\arg(-\rho t)$, *and* $\arg(1 - \beta - \rho t)$ *all lying between* $-\pi$ *and* $\pi$ *and* $t$ *tending to infinity.*

The asymptotic expansion of the Wright function $\phi(\rho, \beta; x)$ for $x \to +\infty$ in the case $-1/3 < \rho < 0$ (the case not covered by Theorem 3.2) can be given in the form ([5])

$$\phi(\rho, \beta; x) = x^{p(\frac{1}{2}-\beta)} e^{-\sigma x^p \cos \pi p} \cos(\pi p(\frac{1}{2} - \beta) - \sigma x^p \sin \pi p) \left\{c_2 + O(x^{-p})\right\},$$

where $p = \frac{1}{1+\rho}$, $\sigma = (1+\rho)(-\rho)^{-\frac{\rho}{1+\rho}}$ and the constant $c_2$ can be exactly evaluated.

For $-1 < \rho < -1/3$, there is a region of the complex plane where the asymptotic expansion of the Wright function is an algebraic one:

Theorem 3.3. *If* $-1 < \rho < -1/3, \ |\arg z| \le \frac{1}{2}\pi(-1-3\rho) - \epsilon, \ \epsilon > 0,$ *then*

$$\phi(\rho,\beta;z) = \sum_{m=0}^{M-1} \frac{z^{(\beta-1-m)/(-\rho)}}{(-\rho)\Gamma(m+1)\Gamma\left(1+\frac{m+1-\beta}{\rho}\right)} + \mathrm{O}(z^{\frac{\beta-1-M}{-\rho}}), \ z \to \infty. \quad (25)$$

## 4. Numerical algorithms

The aim of this section is to suggest several algorithms for computation of the Wright function with the prescribed accuracy $\epsilon > 0$. Some of the algorithms will be formulated for the complex arguments' values, but the main focus of the section is on the evaluation of the Wright function with the real arguments' values.

Following the lines of the paper [4], where similar numerical algorithms were introduced for the Mittag-Leffler function, we distinguish three cases while formulating the algorithms for the numerical evaluation of the Wright function (1):

A) $|z| \le q_1, \ 0 < q_1 < 1$ ($q_1$ is a fixed small number),

B) $q_1 < |z| < q_2$ ($q_1$ is a fixed small number, $q_2$ is a fixed big number),

C) $|z| > q_2$ ($q_2$ is a fixed big number).

The reason for considering these three cases is as follows: Even if the series (1) converges for any value of $z$, we cannot use it practically for evaluation of the Wright function for $|z|$ of large magnitude because the first terms of the series become very large and we need to sum a huge number of these terms. This problem is especially sharp in the case $z = -x, \ x > 0$, $x$ being a large number, because the series (1) is an alternating series having the terms with the large absolute values whereas the sum of the series - the value of the Wright function - is a small number.

The cases A) and C) are considered in this section for the complex arguments' values, but in the case B) (probably, the most important one in the applications) we restrict ourselves to the real arguments' values.

We begin with the case A): $|z| \le q_1, \ 0 < q_1 < 1$.

Theorem 4.1. *If* $|z| \le q_1, \ 0 < q_1 < 1,$ *then the Wright function (1) can be computed with the prescribed accuracy* $\epsilon > 0$ *through the following finite sum:*

$$\phi(\rho,\beta;z) = \sum_{k=0}^{k_0} \frac{z^k}{k!\Gamma(\beta+\rho k)} + \mu(z), \ |\mu(z)| \le \epsilon, \quad (26)$$

*where*

$$k_0 > \max\left\{\frac{1-\beta}{\rho},\ \frac{\ln(\epsilon)}{\ln(|z|)-1}\right\} \quad \textit{if}\ \ \rho > 0,$$

$$k_0 > \max\left\{-\frac{\beta}{\rho},\ \frac{1-\beta}{1+\rho},\ \frac{\ln(\pi(1+\rho)\epsilon|z|^{\frac{\beta-1}{1+\rho}})}{\ln(|z|)-1-\rho} + \frac{1-\beta}{1+\rho}\right\} \quad \textit{if}\ \ -1 < \rho < 0.$$

To prove the theorem, the series

$$S_{k_0}(z) := \phi(\rho,\beta;z) - \sum_{k=0}^{k_0} \frac{z^k}{k!\Gamma(\beta+\rho k)} = \sum_{k=k_0+1}^{\infty} \frac{z^k}{k!\Gamma(\beta+\rho k)} \tag{27}$$

is estimated by using the standard technique and properties of the Γ-function. If $\rho > 0$ and $k \geq \frac{1-\beta}{\rho}$, then $\beta + \rho k \geq 1$ and $\Gamma(\beta+\rho k) \geq 1$. We have then the following simple estimate:

$$|S_{k_0}(z)| \leq \sum_{k=k_0+1}^{\infty} \frac{|z|^k}{k!\Gamma(\beta+\rho k)} \leq \sum_{k=k_0+1}^{\infty} \frac{|z|^k}{k!} < \frac{e|z|^{k_0+1}}{(k_0+1)!} < \frac{|z|^{k_0}}{e^{k_0}}.$$

If $k_0 > \frac{\ln(\epsilon)}{\ln(|z|)-1}$, then

$$|S_{k_0}(z)| < \frac{|z|^{k_0}}{e^{k_0}} < \epsilon,$$

what we wanted to reach. Of course, we cannot use the same method in the case $-1 < \rho < 0$ and another approach is required. Using the complement formula for the Gamma function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}, \quad z \in \mathbb{C},$$

let us rewrite the series (27) as follows

$$S_{k_0}(z) = \sum_{k=k_0+1}^{\infty} \frac{\sin\pi(\beta+\rho k)\Gamma(1-\beta-\rho k)\, z^k}{\pi k!}. \tag{28}$$

The next trick is to represent the coefficient of the series (28) with the help of the relation

$$\frac{\Gamma(1-\beta-\rho k)}{k!} = \frac{\Gamma(1-\beta-\rho k)\Gamma(\beta+(\rho+1)k)}{\Gamma(\beta+(\rho+1)k)\Gamma(1+k)} = \frac{B(1-\beta-\rho k, \beta+(\rho+1)k)}{\Gamma(\beta+(\rho+1)k)},$$

that is valid due to the property

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}$$

of the $B$-function that is defined by

$$B(s,t) = \int_0^1 (1-u)^{s-1} u^{t-1}\, du, \quad \Re(s), \Re(t) > 0. \tag{29}$$

The integral representation (29) of the $B$-function leads to the inequality

$$|B(s,t)| \le \int_0^1 |(1-u)^{s-1}||u^{t-1}|\,|du| \ \le\ 1, \quad \Re(s), \Re(t) \ge 1,$$

that we use to estimate the series (28):

$$|S_{k_0}(z)| \le \sum_{k=k_0+1}^{\infty} \frac{|\sin \pi(\beta+\rho k)\Gamma(1-\beta-\rho k)|\,|z|^k}{\pi k!}$$

$$\le \sum_{k=k_0+1}^{\infty} \frac{B(1-\beta-\rho k, \beta+(\rho+1)k)|z|^k}{\Gamma(\beta+(\rho+1)k)\pi} \le \sum_{k=k_0+1}^{\infty} \frac{|z|^k}{\Gamma(\beta+(\rho+1)k)\pi}$$

under the conditions $1-\beta-\rho(k_0+1) \ge 1,\ \beta+(1+\rho)(k_0+1) \ge 1.$ As to the last series, the following chain of the inequalities is valid:

$$\sum_{k=k_0+1}^{\infty} \frac{|z|^k}{\Gamma(\beta+(\rho+1)k)\pi} \le \int\limits_{k_0}^{\infty} \frac{|z|^x}{\Gamma(\beta+(\rho+1)x)\pi} dx =$$

$$\frac{1}{(1+\rho)\pi} \int\limits_{(1+\rho)k_0+\beta-1}^{\infty} \frac{|z|^{\frac{u+1-\beta}{1+\rho}}}{\Gamma(1+u)} du \le \frac{1}{(1+\rho)\pi} |z|^{\frac{1-\beta}{1+\rho}} \sum_{i=[(1+\rho)k_0+\beta-1]}^{\infty} \frac{|z|^{\frac{i}{1+\rho}}}{\Gamma(1+i)}.$$

Using the same method as the one we employed in the case $\rho > 0$, we can estimate the last series and thus arrive to the inequality

$$|S_{k_0}(z)| \le \frac{1}{(1+\rho)\pi} |z|^{\frac{1-\beta}{1+\rho}} \sum_{i=[(1+\rho)k_0+\beta-1]}^{\infty} \frac{|z|^{\frac{i}{1+\rho}}}{\Gamma(1+i)} < \epsilon$$

under the condition $k_0 > \frac{\ln(\pi(1+\rho)\epsilon|z|^{\frac{\beta-1}{1+\rho}})}{\ln(|z|)-1-\rho} + \frac{1-\beta}{1+\rho}$ that finishes the proof of the theorem. ∎

We proceed with the case B): $q_1 < |z| < q_2,\ z = x \in I\!R$, that is very important for applications and presents the main result of the paper. In this case, we use the integral representations from Theorems 2.1 and 2.2 to numerically evaluate the Wright function. In fact, the integral representations are valid for all values of $z = x,\ x \in I\!R$, so that the restrictions $q_1 < |z| < q_2$ can be omitted. According to Theorems 2.1 and 2.2, to evaluate the Wright function we have to compute numerically either the improper integral

$$I = \int_a^\infty K(\rho,\beta,x,r)\,dr,\ a \in \{0,\ 1\}, \tag{30}$$

$$K(\rho,\beta,x,r) = e^{-r+xr^{-\rho}\cos(\pi\rho)}r^{-\beta}\sin(xr^{-\rho}\sin(\pi\rho)+\pi\beta), \tag{31}$$

or even the integral

$$J = \int_0^\pi \tilde{P}(\rho,\beta,x,\varphi)\,d\varphi,$$

$\tilde{P}$ being defined by the formula (14). In the integral $J$, the integrand (function $\tilde{P}$) is bounded for all values of the argument and parameters and the limits of integration are finite, so that it can be calculated with prescribed accuracy $\epsilon > 0$ by one of many known product quadrature methods.

For calculating the (improper) integral (30) over the bounded function $K$ with the prescribed accuracy $\epsilon$ we use the following

THEOREM 4.2. *If $x\cos(\pi\rho) \le 0$, then the representation*

$$I = \int_a^{r_0} K(\rho,\beta,x,r)\,dr\ + \mu(x),\ |\mu(x)| \le \epsilon,\ a \in \{0,\ 1\},\ r_0 \ge 1 \tag{32}$$

*is valid with*

$$r_0 \ge -\ln(\epsilon)$$

*under the condition $\beta \ge 0$ and with*

$$r_0 \ge \max\{|\beta|+1,\ -2\ln(\epsilon(-2\beta)^\beta/(|\beta|+2))\}$$

*under the condition $\beta < 0$. If $x\cos(\pi\rho) > 0$, then the representation (32) is valid with*

$$r_0 \ge \max\{-2\ln(\epsilon 2^{\beta-1}),\ (2x\cos(\pi\rho))^{\frac{1}{1+\rho}}\}$$

*under the condition $\beta \ge 0$ and with*

$$r_0 \ge \max\{2(|\beta|+1),\ -4\ln(\epsilon(-4\beta)^\beta/(2(|\beta|+2))),\ (2x\cos(\pi\rho))^{\frac{1}{1+\rho}}\}$$

*under the condition* $\beta < 0$.

The proof of the theorem consists in the estimation of the function (31) and then of the improper integral

$$I_\rho := \int_a^\infty K(\rho,\beta,x,r)\,dr - \int_a^{r_0} K(\rho,\beta,x,r)\,dr = \int_{r_0}^\infty K(\rho,\beta,x,r)\,dr. \qquad (33)$$

We begin with the estimation of the function (31):

$$|K(\rho,\beta,x,r)| \le e^{-r+xr^{-\rho}\cos(\pi\rho)}r^{-\beta} = e^{-r(1-xr^{-1-\rho}\cos(\pi\rho))}r^{-\beta}.$$

For $\rho > -1$ the function $r^{-1-\rho}$ tends to zero as $r \to \infty$ because its exponent $-1-\rho$ is negative and the function $K$ behaves like the function $e^{-Cr}r^{-\beta},\ C > 0$. More precisely, if $x\cos(\pi\rho) \le 0$, then $1 - xr^{-1-\rho}\cos(\pi\rho)$ is always greater than or equal to 1 (remember that $r \ge r_0 \ge 1$) and we have the estimate

$$|K(\rho,\beta,x,r)| \le e^{-r}r^{-\beta}. \qquad (34)$$

If $x\cos(\pi\rho) > 0$, then $1 - xr^{-1-\rho}\cos(\pi\rho)$ is greater than or equal to $1/2$ under the condition $r \ge (2x\cos(\pi\rho))^{\frac{1}{1+\rho}}$ and the estimate reads

$$|K(\rho,\beta,x,r)| \le e^{-r/2}r^{-\beta}. \qquad (35)$$

To estimate the integral (33), we need the following two lemmas (for their proofs see [4]):

LEMMA 4.1. *For the incomplete gamma function* $\Gamma(1-\beta,x)$ *the following estimates hold:*

$$|\Gamma(1-\beta,x)| \le e^{-x},\ x \ge 1,\ \beta \ge 0, \qquad (36)$$

$$|\Gamma(1-\beta),x) \le (|\beta|+2)x^{-\beta}e^{-x},\ x \ge |\beta|+1,\ \beta < 0, \qquad (37)$$

*where the incomplete gamma function is defined as*

$$\Gamma(\alpha,x) = \int_x^\infty e^{-t}t^{\alpha-1}\,dt,\ x > 0.$$

LEMMA 4.2. *For arbitrary* $x,y,q > 0$ *there holds the inequality*

$$x^y \le (qy)^y e^{x/q}. \qquad (38)$$

Now let us estimate the integral (33) in the case $x\cos(\pi\rho) \le 0$. Using the inequality (34) we get

$$|I_\rho| \le \int_{r_0}^{\infty} |K(\rho,\beta,x,r)|\,dr \le \int_{r_0}^{\infty} e^{-r} r^{-\beta}\,dr = \Gamma(1-\beta,r_0).$$

Applying Lemma 4.1 lets to continue the last estimate as follows:

$$|I_\rho| \le \Gamma(1-\beta,r_0) \le \begin{cases} e^{-r_0} & \text{for } \beta \ge 0, \\ (|\beta|+2) r_0^{-\beta} e^{-r_0} & \text{for } \beta < 0, \quad r_0 \ge |\beta|+1. \end{cases}$$

From this last estimate we immediately get the desired inequality

$$|I_\rho| \le \epsilon \ \text{ if } \ r_0 \ge -\ln(\epsilon)$$

in the case $\beta \ge 0$. As to the case $\beta < 0$, we first apply the lemma 4.2 to get the estimate

$$r_0^{-\beta} \le (-2\beta)^{-\beta} e^{r_0/2}$$

and then the inequality

$$|I_\rho| \le (|\beta|+2)(-2\beta)^{-\beta} e^{r_0/2} e^{-r_0} = (|\beta|+2)(-2\beta)^{-\beta} e^{-r_0/2} \le \epsilon$$

under the condition $r_0 \ge -2\ln(\epsilon(-2\beta)^{\beta}/(|\beta|+2))$. Collecting all the estimates, we arrive at the statement of Theorem 4.2 in the case $x\cos(\pi\rho) \le 0$. The proof of the theorem in the case $x\cos(\pi\rho) > 0$ uses the estimate (35) instead of (34) and the same technique we used in the previous case, so that we omit it here. ■

Let us remark that Theorem 4.2 builds the base for the numerical evaluation of the Wright function for the real values of its argument. In fact, it reduces the problem under consideration to the numerical evaluation of an integral with a bounded function over a known finite integration interval that can be solved with the prescribed accuracy $\epsilon > 0$ by one of many known product quadrature methods.

Finally, the evaluation of the Wright function in the case C): $|z| > q_2$ ($q_2$ is a fixed big number) can be performed by using the asymptotic formulae from the 3rd section, too. However, in this case we cannot theoretically predict the evaluation accuracy and have to use different experimental techniques for its determination. We remark here that the method of the integral representations we used in the case B) can be used in the case C), too. Of course, the asymptotic formulae from the 3rd section deliver results much faster than the evaluation by means of the integral representations. Still, the method we introduced for the case B) has the advantage of the prescribed accuracy and should be used for precise calculations.

The algorithm described in this section can serve as a basis for creation of a programming package for numerical evaluation of the Wright function; both the package and the numerical results obtained with its help will be presented elsewhere.

## References


[1] E. Buckwar, Yu. Luchko, Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations. *J. Math. Anal. Appl.* **227** (1998), 81-97.

[2] M.A. Evgrafov, *Asymptotic Estimates and Entire Functions.* Gordon and Breach, New York (1961).

[3] M.V. Fedorjuk, *Asymptotic. Integrals and Series.* Nauka, Moscow (1987).

[4] R. Gorenflo, J. Loutchko, Yu. Luchko, Computation of the Mittag-Leffler function $E_{\alpha,\beta}$ and its derivatives. *Fractional Calculus and Applied Analysis* **5**, No 4 (2002), 491-518.

[5] R. Gorenflo, Yu. Luchko, F. Mainardi, Analytical properties and applications of the Wright function. *Fractional Calculus and Applied Analysis* **2**, No 4 (1999), 383-415.

[6] R. Gorenflo, Yu. Luchko, F. Mainardi, Wright functions as scale-invariant solutions of the diffusion-wave equation. *Journal of Computational and Applied Mathematics* **11** (2000), 175-191.

[7] V. Kiryakova, *Generalized Fractional Calculus and Applications.* Longman, Harlow & J. Wiley, N. York (1994).

[8] Yu. Luchko, On the asymptotics of zeros of the Wright function. *Zeitschrift für Analysis und ihre Anwendungen* **19** (2000), 597-622.

[9] Yu. Luchko, R. Gorenflo. Scale-invariant solutions of a partial differential equation of fractional order. *Fractional Calculus and Applied Analysis* **1**, No 1 (1998), 63-78.

[10] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation. In: *Waves and Stability in Continuous Media* (Eds. S. Rionero and T. Ruggeri), World Scientific, Singapore (1994), 246-251.

[11] F. Mainardi, Fractional diffusive waves in viscoelastic solids. In: *IUTAM Symposium - Nonlinear Waves in Solids*(Eds. J.L. Wagner and F.R. Norwood), ASME/AMR, Fairfield NJ (1995), 93-97.

[12] F. Mainardi, The fundamental solutions for the fractional diffusion-wave equation. *Applied Mathematics Letters* **9**, No 6 (1996), 23-28.

[13] H.M. Srivastava, K.C. Gupta, S.P. Goyal, *The H-Functions of One and Two Variables with Applications.* South Asian Publishers, New Delhi - Madras (1982).

[14] E.M. Wright, On the coefficients of power series having exponential singularities. *Journal London Math. Soc.* **8** (1933), 71-79.

[15] E.M. Wright, The asymptotic expansion of the generalized Bessel function. *Proc. London Math. Soc. (Ser. II)* **38** (1935), 257-270.

[16] E.M. Wright, The generalized Bessel function of order greater than one. *Quart. J. Math., Oxford ser.* **11** (1940), 36-48.

*Received: December 8, 2007*

*Dept. of Mathematics II*
*Technical University of Applied Sciences Berlin*
*Luxemburger Str. 10*
*13353 – Berlin, GERMANY*

*e-mail: luchko@tfh-berlin.de*