Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 12, NUMBER 3 (2009) ISSN 1311-0454

STABILITY OF FRACTIONAL-ORDER SYSTEMS WITH RATIONAL ORDERS: A SURVEY

Ivo Petráš

Abstract

In this survey paper we review the methods for stability investigation of a certain class of fractional order linear and nonlinear systems. The stability is investigated in the time domain and the frequency domain. The general stability conditions and several illustrative examples are presented as well.

Mathematics Subject Classification: 26A33 (main), 93D25, 30F99

Key Words and Phrases: fractional calculus, complex plane, stability, fractional-order system, time domain, frequency domain

1. Introduction

Fractional Calculus (FC) is more than 300 years old topic. A number of applications where FC has been used rapidly grows, especially during last two decades. This mathematical phenomena allow to describe a real object more accurate that the classical "integer" methods. The real objects are generally fractional [38, 40, 48, 66, 68], however, for many of them the fractionality is very low. The main reason for using the integer-order models was the absence of solution methods for fractional differential equations.

Recently, the fractional order linear time invariant (FOLTI) systems have attracted lots of attention in control systems society (e.g. [17, 32, 40, 50, 52]) even though fractional-order control problems were investigated as early as 1960s [33]. In the fractional order controller, the fractional order integration or derivative of the output error is used for the current control force calculation.

The fractional order calculus plays an important role in physics [43, 58, 64], thermodynamics [29, 55], electrical circuits theory and fractances [6, 12, 15, 20, 38, 67], mechatronics systems [53], signal processing [54, 65], chemical mixing [39], chaos theory [60, 63], etc. It is recommended to refer to (e.g. [7,37,41,57,69]) for the further engineering applications of fractional order systems. The question of stability is very important especially in control theory. In the field of fractional-order control systems, there are many challenging and unsolved problems related to stability theory such as robust stability, bounded input - bounded output stability, internal stability, root-locus, robust controllability, robust observability, etc.

For distributed parameter systems with a distributed delay [42], provided an stability analysis method which may be used to test the stability of fractional order differential equations. In [13], the co-prime factorization method is used for stability analysis of fractional differential systems. In [34], the stability conditions for commensurate FOLTI system have been provided. However, the general robust stability test procedure and proof of the validity for the general type of the FOLTI system is still open and discussed in [46]. Stability has also been investigated for fractional order nonlinear system (chaotic system) with commensurate and incomensurate order as well [2, 60, 61].

This paper is organized as follows. Section 2 is a brief introduction to the fractional calculus. Section 3 is on fractional order systems. In Section 4 the stability conditions of the fractional order systems are described. Investigation methods for linear systems are described in Section 5, and for nonlinear systems – in Section 6. Section 7 concludes the paper with some remarks.

2. Fractional calculus fundamentals

The idea of fractional calculus has been known since the development of the classical calculus, with the first reference probably being associated with Leibnitz and l'Hospital in 1695 where half-order derivative was mentioned.

Fractional calculus is a generalization of the integration and differentiation to non-integer order fundamental operator ${}_aD_t^\alpha$, where a and t are the limits of the operation and $\alpha \in \mathrm{R}$.

The three definitions often used for the general fractional differintegral are the Grunwald-Letnikov definition, the Riemann-Liouville and the

Caputo definition, see e.g. [39, 48]. For our purpose we use the Caputo's definition (originated from his 1967's paper [14]), which can be written as

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{1} $$

for $(n-1 < \alpha < n)$. The initial conditions for the fractional order differential equations with the Caputo's derivatives are in the same form as for the integer-order differential equations.

The formula for the Laplace transform of the Caputo's fractional derivative (1) for zero initial conditions has the form [48]:

$$ L\{{}_0D_t^{\alpha} f(t)\} = s^{\alpha} F(s), \tag{2} $$

where $s \equiv j\omega$ denotes the Laplace operator.

Geometric and physical interpretation of fractional integration and fractional differentiation was given in Podlubny's work [49].

3. Fractional-order systems

A general fractional-order linear system can be described by a fractional differential equation of the form

$$ \begin{aligned} &a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \ldots + a_0 D^{\alpha_0} y(t) \\ &\qquad = b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \ldots + b_0 D^{\beta_0} u(t), \end{aligned} \tag{3} $$

where $D^{\gamma} \equiv {}_0D_t^{\gamma}$ denotes the Riemann-Liouville or Caputo's fractional derivative [48], or by the corresponding transfer function of *incommensurate* real orders of the following form [48]:

$$ G(s) = \frac{b_m s^{\beta_m} + \ldots + b_1 s^{\beta_1} + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + \ldots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}} = \frac{Q(s^{\beta_k})}{P(s^{\alpha_k})}, \tag{4} $$

where a_k $(k = 0, \ldots, n)$, b_k $(k = 0, \ldots, m)$ are constant, and α_k $(k = 0, \ldots, n)$, β_k $(k = 0, \ldots, m)$ are arbitrary real or rational numbers and without loss of generality they can be arranged as $\alpha_n > \alpha_{n-1} > \ldots > \alpha_0$, and $\beta_m > \beta_{m-1} > \ldots > \beta_0$.

The incommensurate order system (4) can also be expressed in commensurate form by the multivalued transfer function [9]

$$ H(s) = \frac{b_m s^{m/v} + \ldots + b_1 s^{1/v} + b_0}{a_n s^{n/v} + \ldots + a_1 s^{1/v} + a_0}, \quad (v > 1). \tag{5} $$

Note that every fractional order system can be expressed in the form (5) and domain of the $H(s)$ definition is a Riemann surface with v Riemann sheets, [30].

In the particular case of *commensurate* order systems, it holds that $\alpha_k = \alpha k, \beta_k = \alpha k, (0 < \alpha < 1), \forall k \in \mathrm{Z}$, and the transfer function has the following form:

$$G(s) = K_0 \frac{\sum_{k=0}^{M} b_k (s^\alpha)^k}{\sum_{k=0}^{N} a_k (s^\alpha)^k} = K_0 \frac{Q(s^\alpha)}{P(s^\alpha)}. \tag{6}$$

With $N > M$, the function $G(s)$ becomes a proper rational function in the complex variable s^α which can be expanded in partial fractions of the following form:

$$G(s) = K_0 \left[\sum_{i=1}^{N} \frac{A_i}{s^\alpha + \lambda_i} \right], \tag{7}$$

where $\lambda_i \, (i = 1, 2, ..., N)$ are the roots of the pseudo-polynomial $P(s^\alpha)$ or the system poles which are assumed to be simple without loss of generality.

Consider a control function which acts on the FODE system as follows:

$$a_n \, D_t^{\alpha_n} y(t) + \cdots + a_1 \, D_t^{\alpha_1} y(t) + a_0 \, D_t^{\alpha_0} y(t) = u(t). \tag{8}$$

By Laplace transform, we can get a fractional transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{a_n s^{\alpha_n} + \cdots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0}}. \tag{9}$$

The analytical solution of the FODE (8) for $u(t) = 0$ is given by general formula in the form [48]:

$$\begin{aligned} y(t) &= \frac{1}{a_n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{\substack{k_0+k_1+\ldots+k_{n-2}=m \\ k_0 \geq 0; \ldots, k_{n-2} \geq 0}} (m; k_0, k_1, \ldots, k_{n-2}) \\ &\times \prod_{i=0}^{n-2} \left(\frac{a_i}{a_n} \right)^{k_i} \mathcal{E}_m(t, -\frac{a_{n-1}}{a_n}; \alpha_n - \alpha_{n-1}, \alpha_n \\ &+ \sum_{j=0}^{n-2} (\alpha_{n-1} - \alpha_j) k_j + 1), \end{aligned} \tag{10}$$

where $(m; k_0, k_1, \ldots, k_{n-2})$ are the multinomial coefficients and $\mathcal{E}_k(t, y; \mu, \nu)$ is the function of Mittag-Leffler type introduced by Podlubny [48]. The function is defined by

$$\mathcal{E}_k(t, y; \mu, \nu) = t^{\mu k + \nu - 1} E_{\mu,\nu}^{(k)}(y t^\mu), \quad (k = 0, 1, 2, \ldots), \tag{11}$$

where $E_{\mu,\nu}(z)$ is the Mittag-Leffler function of two parameters [25]:

$$E_{\mu,\nu}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\mu i + \nu)}, \qquad (\mu > 0, \quad \nu > 0), \tag{12}$$

where e.g. $E_{1,1}(z) = e^z$, and where its k-th derivative is given by

$$E_{\mu,\nu}^{(k)}(z) = \sum_{i=0}^{\infty} \frac{(i+k)!\ z^i}{i!\ \Gamma(\mu i + \mu k + \nu)}, \qquad (k = 0, 1, 2, ...). \tag{13}$$

The fractional-order linear time invariant (LTI) system can also be represented by the following state-space model

$$\begin{aligned} {}_0D_t^{\mathbf{q}} x(t) &= \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) &= \mathbf{C}x(t), \end{aligned} \tag{14}$$

where $x \in \mathrm{R}^n$, $u \in \mathrm{R}^r$ and $y \in \mathrm{R}^p$ are the state, input and output vectors of the system and $\mathbf{A} \in \mathrm{R}^{n\times n}$, $\mathbf{B} \in \mathrm{R}^{n\times r}$, $\mathbf{C} \in \mathrm{R}^{p\times n}$, and $\mathbf{q} = [q_1, q_2, \dots, q_n]^T$ are the fractional orders. If $q_1 = q_2 = \ldots = q_n \equiv \alpha$, system (14) is called a commensurate order system, otherwise it is an incommensurate order system. If Caputo's derivative is considered, the initial conditions are:

$$\begin{aligned} x_1(0) &= x_0^{(1)} = y_0, \ \ x_2(0) = x_0^{(2)} = 0, \ldots \\ x_i(0) &= x_0^{(i)} = \begin{cases} y_0^{(k)}, & \text{if } i = 2k+1, \\ 0, & \text{if } i = 2k, \end{cases} \quad i \le n. \end{aligned} \tag{15}$$

Similar to conventional observability and controllability concept, the controllability is defined as following [36]: System (14) is *controllable* on $[t_0, t_{final}]$ if the controllability matrix

$$C_a = [B|AB|A^2B|\ldots|A^{n-1}B]$$

has rank n.

The observability is defined following [36]: System (14) is *observable* on $[t_0, t_{final}]$ if the observanility matrix

$$O_a = \left[\begin{array}{c} C|CA|CA^2|\ldots|CA^{n-1} \end{array} \right]^T$$

has rank n.

Generally, we consider the following incommensurate fractional-order nonlinear system in the form:

$$\begin{aligned} {}_0D_t^{q_i} x_i(t) &= f_i(x_1(t), x_2(t), \ldots, x_n(t), t) \\ x_i(0) &= c_i, \;\; i = 1, 2, \ldots, n, \end{aligned} \tag{16}$$

where c_i are initial conditions, or in its vector representation:

$$D^{\mathbf{q}}\mathbf{x} = \mathbf{f}(\mathbf{x}), \tag{17}$$

where $\mathbf{q} = [q_1, q_2, \ldots, q_n]^T$ for $0 < q_i < 2$, $(i = 1, 2, \ldots, n)$ and $\mathbf{x} \in \mathrm{R}^n$.

The equilibrium points of system (17) are calculated via solving the following equation

$$\mathbf{f}(\mathbf{x}) = 0 \tag{18}$$

and we suppose that $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is an equilibrium point of system (17).

4. Stability of the fractional-order systems

The stability as an extremely important property of the dynamical systems can be investigated in various domains [18]. Usual concept of bounded input - bounded output (BIBO) or external stability in *time domain* can be defined via the following general stability conditions [35]:

A causal LTI system with impulse response $h(t)$ to be BIBO stable if the necessary and sufficient condition is satisfied

$$\int_0^\infty ||h(\tau)|| d\tau < \infty,$$

where output of the system is defined by convolution

$$y(t) = h(t) * u(t) = \int_0^\infty h(\tau) u(t - \tau) d\tau,$$

where $u, y \in L_\infty$ and $h \in L_1$.

Another very important domain is *frequency domain*. In the case of frequency method for evaluating the stability we transform the s-plane into the complex plane $G_o(j\omega)$ and the transformation is realized according to the transfer function of the open loop system $G_o(j\omega)$. During the transformation, all roots of the characteristic polynomial are mapped from s-plane into the critical point $(-1, j0)$ in the plane $G_o(j\omega)$. The mapping of the

s-plane into $G_o(j\omega)$ plane is conformal, that is, the direction and location of points in the s-plane is preserved in the $G_o(j\omega)$ plane. Frequency investigation method and utilization of the Nyquist frequency characteristics based on argument principle were described in the paper [44].

However, we can not directly use an algebraic tool, as for example Routh-Hurwitz criteria for the fractional order system, because we do not have a characteristic polynomial but pseudo-polynomial with rational power - *multivalued function*. It is possible only in some special cases, cf. [2]. Moreover, modern control methods, as for example LMI (Linear Matrix Inequality) methods [41] or other algorithms [27, 28], have been already developed. The advantage of the LMI methods in control theory is due to their connection with the Lyapunov method (existence of a quadratic Lyapunov function). More generally, the LMI methods are useful to test the matrix eigenvalues belong to a certain region in complex plane. A simple test can be used, [3]. The roots of the polynomial $P(s) = \det(sI - A)$ lie inside in region $-\pi/2 - \delta < \arg(s) < \pi/2 + \delta$ if the eigenvalues of the matrix

$$A_1 = \begin{bmatrix} A\cos\delta & -A\sin\delta \\ A\sin\delta & A\cos\delta \end{bmatrix} \equiv A \otimes \begin{bmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{bmatrix} \tag{19}$$

have negative real part, where $\otimes$ denotes the Kronecker product. This property has been used to stability analysis of ordinary fractional order LTI systems and also for interval fractional order LTI systems [62].

When dealing with incommensurate fractional order systems (or, in general, with fractional order systems) it is important to bear in mind that $P(s^\alpha)$, $\alpha \in \mathrm{R}$ is a multivalued function of s^α, $\alpha = \frac{u}{v}$, the domain of which can be viewed as a Riemann surface with finite number of Riemann sheets v, where the origin is a branch point and the branch cut is assumed at R^- (see Fig. 1). The function s^α becomes holomorphic in the complement of the branch cut line. It is a fact that in multivalued functions only the first Riemann sheet has its physical significance [26]. Note that each Riemann sheet has only one edge at the branch cut and not only poles and singularities originated from the characteristic equation, but branch points and branch cut of given multivalued functions are also important for the stability analysis [10].

In this paper the branch cut is assumed at R^- and the first Riemann sheet is denoted by Ω and defined as (see also Fig. 1)

$$\Omega := \{re^{j\phi} \,|\, r > 0, -\pi < \phi < \pi\}. \tag{20}$$

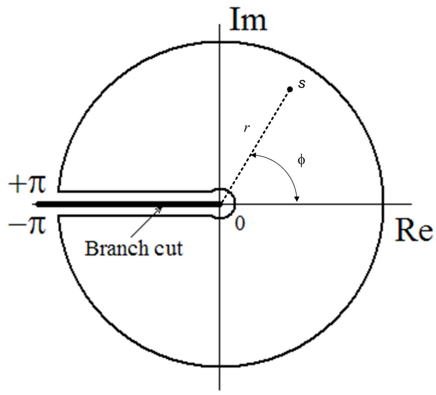


Figure 1: Branch cut $(0, -\infty)$ for branch points in the complex plane.

It is well-known that an integer order LTI system is stable if all the roots of the characteristic polynomial $P(s)$ are negative or have negative real parts if they are complex conjugate (e.g. [18]). This means that they are located on the left of the imaginary axis of the complex s-plane. System $G(s) = Q(s)/P(s)$ is BIBO stable if

$$\exists, \quad ||G(s)|| \leq M < \infty, \quad M > 0, \quad \forall s, \Re(s) \geq 0.$$

A necessary and sufficient condition for the asymptotic stability is [23]:

$$\lim_{t\to\infty} ||X(t)|| = 0.$$

According to the final value theorem proposed in [24], for the fractional order case, when there is a branch point at $s = 0$, we assume that $G(s)$ is multivalued function of s, then

$$x(\infty) = \lim_{s\to 0}[sG(s)].$$

Example 1. Let us investigate the simplest multivalued function defined as follows:

$$w = s^{\frac{1}{2}}, \tag{21}$$

and there will be two s-planes which map onto a single w-plane. The interpretation of the two sheets of the Riemann surface and the branch cut is depicted in Fig. 2.

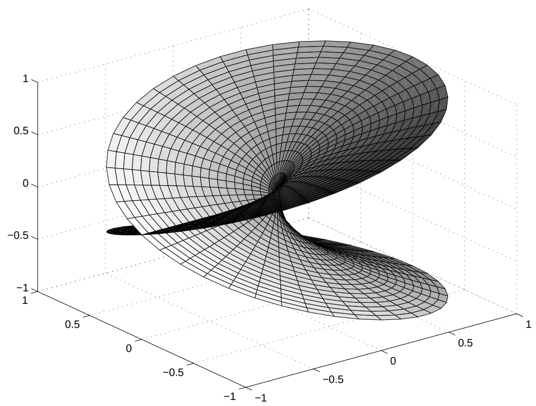


Figure 2: Riemann surface interpretation of the function $w = s^{\frac{1}{2}}$.

Define the principal square root function as

$$f_1(s) = |s|^{\frac{1}{2}} e^{\frac{j\phi}{2}} = re^{\frac{j\phi}{2}},$$

where $r > 0$ and $-\pi < \phi < +\pi$. The function $f_1(s)$ is a branch of w. Using the same notation, we can find other branches of the square root function. For example, if we let

$$f_2(s) = |s|^{\frac{1}{2}} e^{\frac{j\phi+2\pi}{2}} = re^{\frac{j\phi+2\pi}{2}},$$

then $f_2(s) = -f_1(s)$ and it can be thought of as "plus" and "minus" square root functions. The negative real axis is called a branch cut for the functions $f_1(s)$ and $f_2(s)$. Each point on the branch cut is a point of discontinuity for both functions $f_1(s)$ and $f_2(s)$. As has been shown in [30], the function described by (21) has a branch point of order 1 at $s = 0$ and at infinity. They are located at the ends of the branch cut (see also Fig. 1).

Example 2. Let us investigate the transfer function of fractional-order system (multivalued function) defined as

$$G(s) = \frac{1}{s^\alpha + b}, \tag{22}$$

where $\alpha \in \mathrm{R}$ $(0 < \alpha \leq 2)$ and $b \in \mathrm{R}$ $(b > 0)$.

The analytical solution of the fractional order system (22) obtained according to relation (10) has the following form:

$$g(t) = \mathcal{E}_0(t, -b; \alpha, \alpha). \tag{23}$$

The Riemann surface of the function (22) contains an infinite number of sheets and infinitely many poles in positions

$$s = b^{\frac{1}{\alpha}} e^{\frac{j(\pi+2\pi n)}{\alpha}}, \quad n = 0, \pm 1, \pm 2, \dots \quad \text{, for } (\alpha > 0) \text{ and } (b > 0).$$

The sheets of the Riemann surface are all different if α is irrational.

For $1 < \alpha < 2$ we have two poles corresponding to $n = 0$ and $n = -1$, and the poles are

$$s = b^{\frac{1}{\alpha}} e^{\pm \frac{i\pi}{\alpha}}.$$

However, for $0 < \alpha < 1$ in (22) the denominator is a multivalued function and singularity of system can not be defined unless it is made singlevalued. Therefore we will use the Riemann surface. Let us investigate transfer function (22) for $\alpha = 0.5$ (half-order system), then we get

$$G(s) = \frac{1}{s^{\frac{1}{2}} + b}, \tag{24}$$

and by equating the denominator to zero we have

$$s^{\frac{1}{2}} + b = 0.$$

Rewriting the complex operator $s^{\frac{1}{2}}$ in exponential form and using the well known relation $e^{j\pi} + 1 = 0$ (or $e^{j(\pm\pi+2k\pi)} + 1 = 0$) we get the following formula:

$$r^{\frac{1}{2}} e^{j(\phi/2+k\pi)} = a e^{j(\pm\pi+2k\pi)}. \tag{25}$$

From relationship (25), it can be deduced that the modulus and phase (arg) of the pole are:

$$r = b^2 \quad \text{and} \quad \phi = \pm 2\pi(1+k) \quad \text{for } k = 0, 1, 2, \dots$$

However the first sheet of the Riemann surface is defined for range of $-\pi < \phi < +\pi$, the pole with the angle $\phi = \pm 2\pi$ does not fall within this range but pole with the angle $\phi = 2\pi$ falls to the range of the second sheet defined for $\pi < \phi < 2\pi$. Therefore this half-order pole with magnitude b^2 is located on the second sheet of the Riemann surface that consequently maps to the left side of the w-plane (see Fig. 3). On this plane the magnitude and phase of the singlevalued pole are b^2 and π, respectively [30].

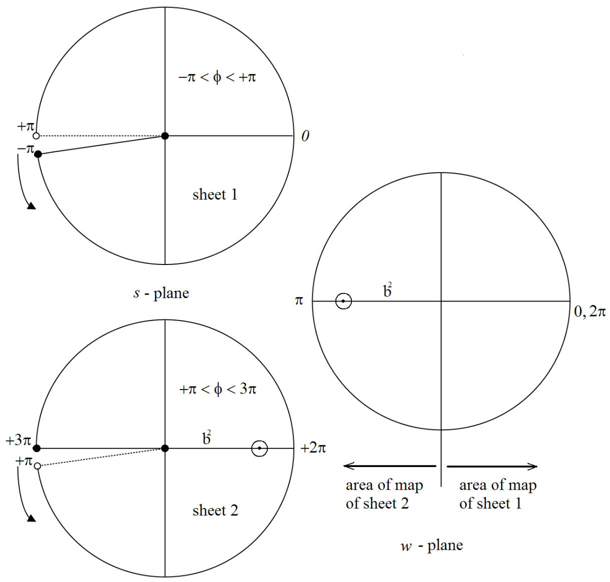
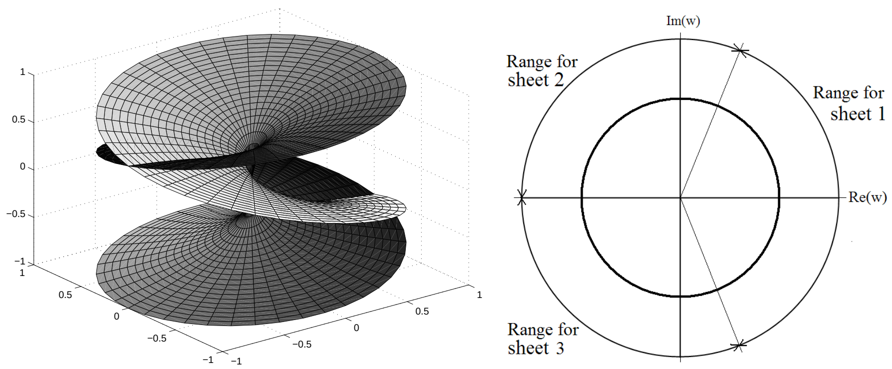


Figure 3: Correspondence between the s-plane and the w-plane.



(a) Riemann surface (b) Complex w-plane

Figure 4: Correspondence between the 3-sheets Riemann surface and w-plane for equation (26).

Example 3. Analogously to the previous examples, we can also investigate function

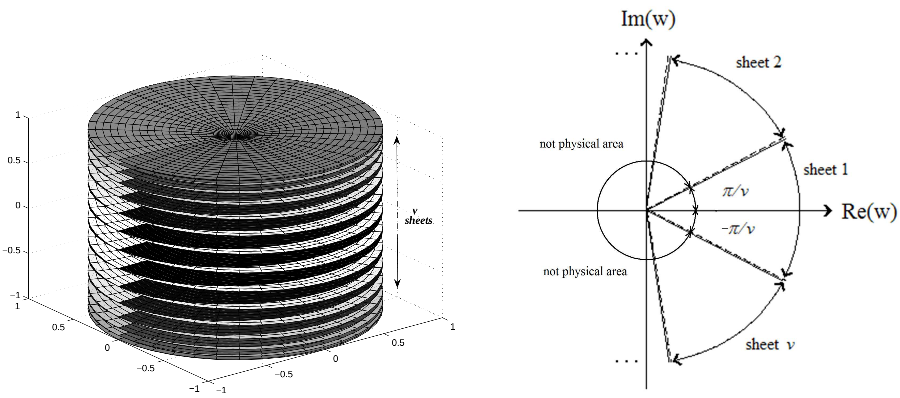
$$w = s^{\frac{1}{3}}, \tag{26}$$

where in this case the Riemann surface has three sheets and each maps onto one-third of the w-plane (see Fig. 4).

Definition 1. Generally, for the multivalued function defined as

$$w = s^{\frac{1}{v}}, \tag{27}$$

where $v \in \mathrm{N}$ $(v = 1, 2, 3, \ldots)$ we get the v sheets in the Riemann surface. In Fig. 5 it is shown the relationship between the w-plane and the v sheets of the Riemann surface, where the sector $-\pi/v < \arg(w) \leq \pi/v$ corresponds to Ω (the first Riemann sheet).

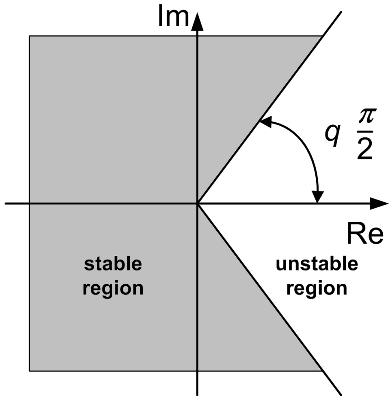


(a) Riemann surface (b) Complex w-plane

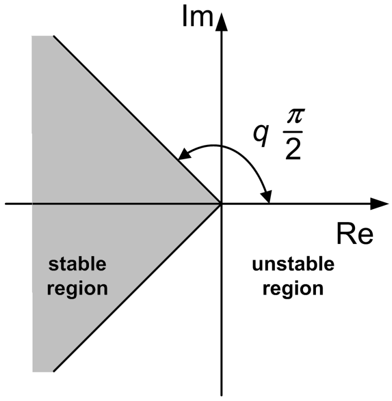
Figure 5: Correspondence between the w-plane and the Riemann sheets.

Definition 2. Mapping the poles from the s^q-plane into the w-plane, where $q \in \mathrm{Q}$ such as $q = \frac{k}{m}$ for $k, m \in \mathrm{N}$ and $|\arg(w)| = |\phi|$, can be done by the following rule: If we assume $k = 1$, then the mapping from s-plane to w-plane is independent of k. Unstable region from s-plane transforms to sector $|\phi| < \frac{\pi}{2m}$ and stable region transforms to sector $\frac{\pi}{2m} < |\phi| < \frac{\pi}{m}$. The region where $|\phi| > \frac{\pi}{m}$ is not physical. Therefore, the system will be stable if all roots in the w-plane lie in the region $|\phi| > \frac{\pi}{2m}$. The stability regions depicted in Fig. 6 correspond to the following propositions:

1. For $k < m$ $(q < 1)$ the stability region is depicted in Fig. 6(a).
2. For $k = m$ $(q = 1)$ the stability region corresponds to the s-plane.
3. For $k > m$ $(q > 1)$ the stability region is depicted in Fig. 6(b).



(a) $0 < q < 1$



(b) $1 < q < 2$

Figure 6: Stability regions of the fractional order system.

5. Stability investigation of fractional LTI systems

As we can see in the previous subsection, in the fractional case, the stability is different from the integer one. Interesting notion is that a stable fractional system may have roots in right half of complex w-plane (see Fig. 6). Since the principal sheet of the Riemann surface is defined by $-\pi < \arg(s) < \pi$, then by using the mapping $w = s^q$, the corresponding w domain is defined by $-q\pi < \arg(w) < q\pi$, and the w plane region corresponding to the right half plane of this sheet is defined by $-q\pi/2 < \arg(w) < q\pi/2$.

Consider the fractional order pseudo-polynomial

$$Q(s) = a_1 s^{q_1} + a_2 s^{q_2} + \ldots + a_n s^{q_n} = a_1 s^{c_1/d_1} + a_2 s^{c_2/d_2} + \ldots + a_n s^{c_n/d_n},$$

where q_i are rational numbers expressed as c_i/d_i, and a_i are real numbers for $i = 1, 2, \ldots, n$. If for some i, $c_i = 0$ then $d_i = 1$. Let v be the least common multiple (LCM) of $d_1, d_2, \ldots, d_n$, denoted as $v = \text{LCM}\{d_1, d_2, \ldots, d_n\}$, then ([24]):

$$Q(s) = a_1 s^{\frac{v1}{v}} + a_2 s^{\frac{v2}{v}} + \ldots + a_n s^{\frac{vn}{v}} = a_1 (s^{\frac{1}{v}})^{v_1} + a_2 (s^{\frac{1}{v}})^{v_2} + \ldots + a_n (s^{\frac{1}{v}})^{v_n}. \tag{28}$$

The fractional degree (FDEG) of the polynomial $Q(s)$ is defined as ([24]):

$$\text{FDEG}\{Q(s)\} = \max\{v_1, v_2, \ldots, v_n\}.$$

The domain of definition for (28) is the Riemann surface with v Riemann sheets where origin is a branch point of order $v - 1$ and the branch cut is assumed at R^-. The number of roots for the fractional algebraic equation (28) is given by the following proposition, [8]: *Let $Q(s)$ be a fractional order polynomial with FDEG$\{Q(s)\} = n$. Then the equation Q(s)=0 has exactly n roots on the Riemann surface.*

Definition 3. The fractional order polynomial

$$Q(s) = a_1 s^{\frac{n}{v}} + a_2 s^{\frac{n-1}{v}} + \ldots + a_n s^{\frac{1}{v}} + a_{n+1}$$

is *minimal* if $\text{FDEG}\{Q(s)\} = n$. We will assume that all fractional order polynomials are minimal. This ensures that there is no redundancy in the number of the Riemann sheets [24].

On the other hand, it has been shown, by several authors and by using several methods, that for the case of FOLTI system of commensurate order, a geometrical method of complex analysis based on the argument principle of the roots of the characteristic equation (a polynomial in this particular case) can be used for the stability check in the BIBO sense (see e.g. [35,44]). The stability condition can then be stated as follows [34, 35, 56]:

Theorem 1. *A commensurate order system described by a rational transfer function (6) is stable if only if*

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2}, \text{ for all } i$$

with λ_i the i-th root of $P(s^\alpha)$.

P r o o f. For proof see [34].

For the FOLTI system with commensurate order where the system poles are in general complex conjugate, the stability condition can also be expressed as follows [34, 35]:

THEOREM 2. *A commensurate order system described by a rational transfer function*

$$G(w) = \frac{Q(w)}{P(w)},$$

where $w = s^q$, $q \in R^+, (0 < q < 2)$, *is stable if only if*

$$|\arg(w_i)| > q\frac{\pi}{2},$$

with $\forall w_i \in C$ *the* i*-th root of* $P(w) = 0$.

P r o o f. For proof see [35].

When $w = 0$ is a single root (singularity at the origin) of P, the system cannot be stable. For $q = 1$, this is the classical theorem of pole location in the complex plane: have no pole in the closed right half plane of the first Riemann sheet. The stability region suggested by this theorem tends to the whole s-plane when q tends to 0, corresponds to the Routh-Hurwitz stability when $q = 1$, and tends to the negative real axis when q tends to 2.

THEOREM 3. *It has been shown that commensurate system (14) is stable if the following condition is satisfied (also if the triplet* $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ *is minimal) [4, 35, 59–61]:*

$$|\arg(\mathrm{eig}(\boldsymbol{A}))| > q\frac{\pi}{2}, \tag{29}$$

where $0 < q < 2$ *and eig(*$\boldsymbol{A}$*) represents the eigenvalues of matrix* $\boldsymbol{A}$.

P r o o f. For proof see [35].

DEFINITION 4. We assume, that some incommensurate order systems described by the FODE (8) or (14), can be decomposed to the following modal form of the fractional transfer function (the so called Laguerre functions, [5]):

$$F(s) = \sum_{i=1}^{N}\sum_{k=1}^{n_k} \frac{A_{i,k}}{(s^{q_i} + \lambda_i)^k} \tag{30}$$

for some complex numbers $A_{i,k}$, λ_i, and positive integer n_k.

A system (30) is BIBO stable if and only if q_i and the argument of λ_i denoted by $\arg(\lambda_i)$ in (30) satisfy the inequalities

$$0 < q_i < 2 \quad \text{and} \quad |\arg(\lambda_i)| < \pi\left(1 - \frac{q_i}{2}\right) \quad \text{for all } i. \tag{31}$$

Henceforth, we will restrict the parameters q_i to the interval $q_i \in (0, 2)$. For the case $q_i = 1$ for all i we obtain a classical stability condition for integer order system (no pole is in right half plane). The inequalities (31) were obtained by applying the stability results given in [1, 35].

THEOREM 4. *Consider the following autonomous system for internal stability definition:*

$$ {}_0D_t^{\boldsymbol{q}} x(t) = \boldsymbol{A}x(t), \quad x(0) = x_0, \tag{32} $$

with $\boldsymbol{q} = [q_1, q_2, \ldots, q_n]^T$ *and its* n*-dimensional representation:*

$$ \begin{aligned} {}_0D_t^{q_1} x_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t) \\ {}_0D_t^{q_2} x_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t) \\ \ldots & \\ {}_0D_t^{q_n} x_n(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t) \end{aligned} \tag{33} $$

where all q_i*'s are rational numbers between* 0 *and* 2*. Assume* m *be the LCM of the denominators* u_i*'s of* q_i*'s , where* $q_i = v_i/u_i$*,* $v_i, u_i \in Z^+$ *for* $i = 1, 2, \ldots, n$ *and we set* $\gamma = 1/m$*. Define [16]:*

$$ det \begin{pmatrix} \lambda^{mq_1} - a_{11} & -a_{12} & \ldots & -a_{1a} \\ -a_{21} & \lambda^{mq_2} - a_{22} & \ldots & -a_{2n} \\ \ldots & & & \\ -a_{n1} & -a_{n2} & \ldots & \lambda^{mq_n} - a_{nn} \end{pmatrix} = 0. \tag{34} $$

The characteristic equation (34) can be transformed to integer order polynomial equation if all q_i*'s are rational number. Then the zero solution of system (33) is globally asymptotically stable if all roots* λ_i*'s of the characteristic (polynomial) equation (34) satisfy*

$$ |\arg(\lambda_i)| > \gamma \frac{\pi}{2} \text{ for all } i. $$

Denoting λ *by* s^γ *in equation (34), we get the characteristic equation in the form* $det(s^\gamma I - A) = 0$.

P r o o f. This assumption was proved in [16].

COROLLARY 1. *Suppose* $q_1 = q_2 = \ldots, q_n \equiv q$*,* $q \in (0, 2)$*, all eigenvalues* λ *of matrix* A *in (14) satisfy* $|\arg(\lambda)| > q\pi/2$*, the characteristic equation becomes* $det(s^q I - A) = 0$ *and all characteristic roots of the system (14) have negative real parts [16]. This result is Theorem 1 of paper [34].*

REMARK 1. Generally, when we assume $s = |r|e^{i\phi}$, where $|r|$ is modulus and ϕ is argument of complex number in s-plane, respectively, transformation $w = s^{\frac{1}{m}}$ to complex w-plane can be viewed as $s = |r|^{\frac{1}{m}} e^{\frac{i\phi}{m}}$ and thus $|\arg(s)| = m|\arg(w)|$ and $|s| = |w|^m$. The proof of this statement is obvious.

The stability analysis criteria for a general FOLTI system can be summarized as follows:

The characteristic equation of a general LTI fractional order system of the form:

$$a_n s^{\alpha_n} + \ldots + a_1 s^{\alpha_1} + a_0 s^{\alpha_0} \equiv \sum_{i=0}^{n} a_i s^{\alpha_i} = 0 \tag{35}$$

may be rewritten as

$$\sum_{i=0}^{n} a_i s^{\frac{u_i}{v_i}} = 0$$

and transformed into the w-plane

$$\sum_{i=0}^{n} a_i w^i = 0, \tag{36}$$

with $w = s^{\frac{k}{m}}$, where m is the LCM of v_i. The procedure of stability analysis is (see e.g. [51]):

1. For given a_i calculate the roots of equation (36) and find the absolute phase of all roots $|\phi_w|$.

2. Roots in the primary sheet of the w-plane which have corresponding roots in the s-plane can be obtained by finding all roots which lie in the region $|\phi_w| < \frac{\pi}{m}$ then applying the inverse transformation $s = w^m$ (see Remark 1.). The region where $|\phi_w| > \frac{\pi}{m}$ is not physical. For testing the roots in desired region the matrix approach (19) can be used.

3. The condition for stability is $\frac{\pi}{2m} < |\phi_w| < \frac{\pi}{m}$. Condition for oscillation is $|\phi_w| = \frac{\pi}{2m}$ otherwise the system is unstable (see Fig. 5(b)). If there is not root in the physical s-plane, the system will always be stable.

EXAMPLE 4. Let us consider the linear fractional order LTI system described by the transfer function [17, 48]:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{0.8s^{2.2} + 0.5s^{0.9} + 1}, \tag{37}$$

and let the corresponding FODE have the following form:

$$0.8\,{}_0D_t^{2.2}y(t) + 0.5\,{}_0D_t^{0.9}y(t) + y(t) = u(t) \qquad (38)$$

with zero initial conditions.

The system (38) can be rewritten to its state space representation ($x_1(t) \equiv y(t)$):

$$\begin{aligned} \begin{bmatrix} {}_0D^{\frac{9}{10}}x_1(t) \\ {}_0D^{\frac{13}{10}}x_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1/0.8 & -0.5/0.8 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/0.8 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \qquad (39)$$

The eigenvalues of the matrix $\mathbf{A}$ are $\lambda_{1,2} = -0.3125 \pm 1.0735j$ and then $|\arg(\lambda_{1,2})| = 1.8541$. Because of various derivative orders in (39), Theorem 3 cannot be used directly.

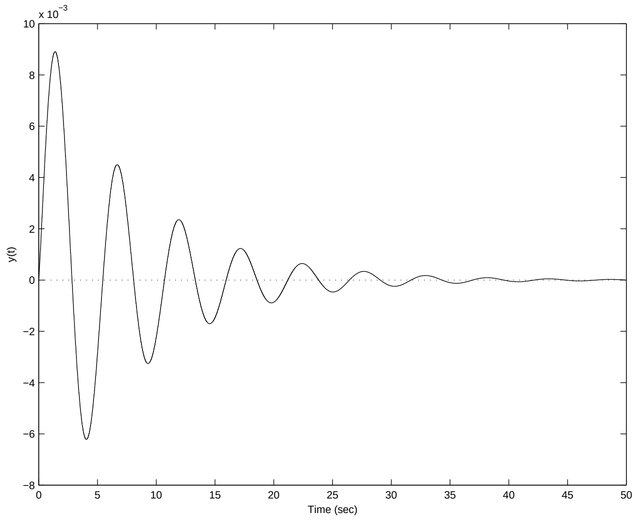


Figure 7: Analytical solution of the FODE (38), where $u(t) = 0$ for $50\,s$ and zeros initial conditions.

The analytical solution of the FODE (38) for $u(t) = 0$ obtained from general solution (10) has the form:

$$y(t) = \frac{1}{0.8}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{1}{0.8}\right)^k \mathcal{E}_k(t, -\frac{0.5}{0.8}; 2.2 - 0.9, 2.2 + 0.9k). \qquad (40)$$

In Fig. 7 is depicted the analytical solution of the FODE (38) where $u(t) = 0$. As we can see in the figure, the solution is stable because $\lim_{t\to\infty} y(t) = 0$. Let us investigate the stability according to the previously described method. The corresponding characteristic equation of the system is:

$$P(s) : 0.8s^{2.2} + 0.5s^{0.9} + 1 = 0 \;\Rightarrow\; 0.8s^{\frac{22}{10}} + 0.5s^{\frac{9}{10}} + 1 = 0, \tag{41}$$

when $m = 10$, $w = s^{\frac{1}{10}}$ then the roots w_i's and their appropriate arguments of polynomial

$$P(w) : 0.8w^{22} + 0.5w^9 + 1 = 0 \tag{42}$$

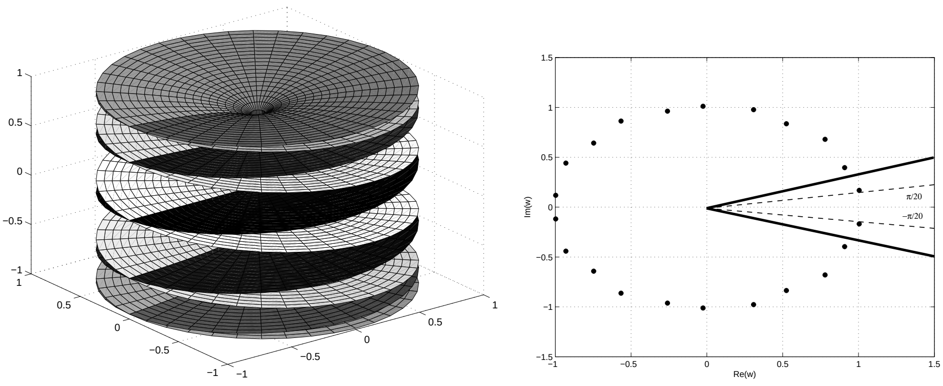
are:
$w_{1,2} = -0.9970 \pm 0.1182j$, $|\arg(w_{1,2})| = 3.023$;
$w_{3,4} = -0.9297 \pm 0.4414j$, $|\arg(w_{3,4})| = 2.698$;
$w_{5,6} = -0.7465 \pm 0.6420j$, $|\arg(w_{5,6})| = 2.431$;
$w_{7,8} = -0.5661 \pm 0.8633j$, $|\arg(w_{7,8})| = 2.151$;
$w_{9,10} = -0.259 \pm 0.9625j$, $|\arg(w_{9,10})| = 1.834$;
$w_{11,12} = -0.0254 \pm 1.0111j$, $|\arg(w_{11,12})| = 1.595$;
$w_{13,14} = 0.3080 \pm 0.9772j$, $|\arg(w_{11,12})| = 1.265$;
$w_{15,16} = 0.5243 \pm 0.8359j$, $|\arg(w_{15,16})| = 1.010$;
$w_{17,18} = 0.7793 \pm 0.6795j$, $|\arg(w_{17,18})| = 0.717$;
$w_{19,20} = 0.9084 \pm 0.3960j$, $|\arg(w_{19,20})| = 0.411$;
$w_{21,22} = 1.0045 \pm 0.1684j$, $|\arg(w_{21,22})| = 0.1661$.

The physical significance roots are in the first Riemann sheet, which is expressed by the relation $-\pi/m < \phi < \pi/m$, where $\phi = \arg(w)$. In this case they are complex conjugate roots $w_{21,22} = 1.0045 \pm 0.1684j$ ($|\arg(w_{21,22})| = 0.1661$), which satisfy conditions $|\arg(w_{21,22})| > \pi/2m = \pi/20$. It means that system (38) is stable (see Fig. 8). Other roots of the polynomial equation (42) lie in region $|\phi| > \frac{\pi}{m}$ which is not physical (outside of closed angular sector limited by thick line in Fig. 8(b)).

In Fig. 8(a) is depicted the Riemann surface of the function $w = s^{\frac{1}{10}}$ with the 10-Riemann sheets and in Fig. 8(b) are depicted the roots in complex w-plane with angular sector corresponds to stability region (dashed line) and the first Riemann sheet (thick line).

The interesting notion of Remark 1 should be mentioned here. The characteristic equation (41) has the following poles:

$$s_{1,2} = -0.10841 \pm 1.19699j,$$



(a) 10-sheets Riemann surface

(b) Poles in complex w-plane

Figure 8: Riemann surface of function $w = s^{\frac{1}{10}}$ and roots of equation (42).

in the first Riemann sheet in s-plane, which can be obtained e.g. via the *MATLAB* routine, as for instance:

```
>>s=solve('0.8*s^2.2+0.5*s^0.9+1=0','s')
```

When we compare $|\arg(w_{21,22})| = 0.1661$ and $|\arg(s_{1,2})| = 1.661$, we can see that $|\arg(s_{1,2})| = m|\arg(w_{21,22})|$, where $m = 10$ in transformation $w = s^{\frac{1}{m}}$. The first Riemann sheet is transformed from the s-plane to the w-plane as follows: $-\pi/10 < \arg(w) < \pi/10$ and in order to $-\pi < 10\arg(w) < \pi$. Therefore from this consideration we then obtain $|\arg(s)| = 10\,|\arg(w)|$.

Example 5. Let us examine an interesting example of application, the so called Bessel function of the first kind, which transfer function is ([35]):

$$H(s) = \frac{1}{\sqrt{s^2+1}} \quad \forall s,\ \Re(s) > 0. \tag{43}$$

We have two branch points $s_1 = i$, and $s_2 = -i$ and two cuts. One along the half line $(-\infty+i, i)$ and another one along the half line $(-\infty-i, -i)$. In this doubly cut complex plane, we have the identity $\sqrt{s^2+1} = \sqrt{s-i}\sqrt{s+i}$. The well known asymptotic expansion of equation (43) is:

$$h(t) \approx \sqrt{\frac{2}{\pi t}}\cos(t-\frac{\pi}{4}) = \sqrt{\frac{2}{\pi}}\,t^{-\frac{1}{2}}\,E_{2,1}\left(-(t-\frac{\pi}{4})^2\right).$$

According to the branch points and the above asymptotic expansion, we can state that the system described by the Bessel function (43) is on boundary of stability and has oscillation behaviour.

EXAMPLE 6. Consider the closed loop system with controlled system (electrical heater)

$$G(s) = \frac{1}{39.96s^{1.25} + 0.598} \tag{44}$$

and integer order controller

$$C(s) = 64.47 + 12.46s. \tag{45}$$

The resulting closed loop transfer function $G_c(s)$ becomes ([47]):

$$G_c(s) = \frac{Y(s)}{W(s)} = \frac{12.46s + 64.47}{39.69s^{1.25} + 12.46s + 65.068}. \tag{46}$$

The analytical solution (impulse response) of the fractional order control system (46) is:

$$\begin{aligned} y(t) &= \frac{12.46}{39.69}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{12.46}{39.69}\right)^k \times \mathcal{E}_k(t, -\frac{65.068}{39.69}; 1.25, 0.25-k) \quad (47)\\ &+ \frac{64.47}{39.69}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{65.068}{39.69}\right)^k \times \mathcal{E}_k(t, -\frac{12.46}{39.69}; 1.25-1, 1.25+k) \end{aligned}$$

with zero initial conditions.

The characteristic equation of this system is

$$39.69s^{1.25} + 12.46s + 65.068 = 0 \;\Rightarrow\; 39.69s^{\frac{5}{4}} + 12.46s^{\frac{4}{4}} + 65.068 = 0. \tag{48}$$

Using the notation $w = s^{\frac{1}{m}}$, where LCM is $m = 4$, we obtain a polynomial of complex variable w in the form

$$39.69w^5 + 12.46w^4 + 65.068 = 0. \tag{49}$$

Solving the polynomial (49) we get the following roots and their arguments:

$$w_1 = -1.17474, |\arg(w_1)| = \pi,$$

$$w_{2,3} = -0.40540 \pm 1.0426j, |\arg(w_{2,3})| = 1.9416,$$

$$w_{4,5} = 0.83580 \pm 0.64536j, |\arg(w_{4,5})| = 0.6575.$$

This first Riemann sheet is defined as a sector in w-plane within interval $-\pi/4 < \arg(w) < \pi/4$. The complex conjugate roots $w_{4,5}$ lie in this interval and satisfy the stability condition given as $|\arg(w)| > \frac{\pi}{8}$, therefore the system is stable. The region where $|\arg(w)| > \frac{\pi}{4}$ is not physical.

6. Stability investigation of fractional nonlinear systems

As it has been mentioned in [34], the exponential stability cannot be used to characterize the asymptotic stability of fractional order systems. A new definition was introduced in [41].

DEFINITION 5. The trajectory $x(t) = 0$ of the system (16) is t^{-q} asymptotically stable if there is a positive real q such that:

$$\forall ||x(t)|| \text{ with } t \leq t_0, \exists\, N(x(t)), \text{ such that } \forall t \geq t_0, ||x(t)|| \leq N t^{-q}.$$

The fact that the components of $x(t)$ slowly decay towards 0 following t^{-q} leads to fractional systems sometimes being called long memory systems. The power law stability t^{-q} is a special case of the Mittag-Leffler stability [31].

According to the stability theorem from [63], the equilibrium points are asymptotically stable for $q_1 = q_2 = \ldots = q_n \equiv q$ if all the eigenvalues λ_i, $(i = 1, 2, \ldots, n)$ of the Jacobian matrix $\mathbf{J} = \partial \mathbf{f} / \partial \mathbf{x}$, where $\mathbf{f} = [f_1, f_2, \ldots, f_n]^T$, evaluated at the equilibrium, satisfy the condition ([59, 60]):

$$|\arg(\mathrm{eig}(\mathbf{J}))| = |\arg(\lambda_i)| > q\frac{\pi}{2}, \quad i = 1, 2, \ldots, n. \tag{50}$$

Fig. 6 shows stable and unstable regions of the complex plane for such case.

Now, consider the incommensurate fractional order system $q_1 \neq q_2 \neq \ldots \neq q_n$ and suppose that m is the LCM of the denominators u_i's of q_i's, where $q_i = v_i/u_i$, $v_i, u_i \in Z^+$ for $i = 1, 2, \ldots, n$ and we set $\gamma = 1/m$. System (17) is asymptotically stable if:

$$|\arg(\lambda)| > \gamma\frac{\pi}{2}$$

for all roots λ of the following equation

$$\det(\mathrm{diag}([\lambda^{mq_1}\, \lambda^{mq_2}\, \ldots\, \lambda^{mq_n}]) - \mathbf{J}) = 0. \tag{51}$$

A necessary stability condition for fractional order systems (17) to remain chaotic is keeping at least one eigenvalue λ in the unstable region [60]. The number of saddle points and eigenvalues for one-scroll, double-scroll and multi-scroll attractors was exactly described in the work [61]. Assume that a 3D chaotic system has only three equilibria. Therefore, if the system has double-scroll attractor, it has two saddle points surrounded by scrolls and one additional saddle point. Suppose that the unstable eigenvalues of scroll saddle points are: $\lambda_{1,2} = \alpha_{1,2} \pm j\beta_{1,2}$. The necessary condition to exhibit double-scroll attractor of system (17) is the eigenvalues $\lambda_{1,2}$ remaining in the unstable region [61]. The condition for commensurate derivatives order is

$$q > \frac{2}{\pi}\text{atan}\left(\frac{|\beta_i|}{\alpha_i}\right), \quad i = 1, 2. \tag{52}$$

This condition can be used to determine the minimum order for which a nonlinear system can generate chaos [60]. In other words, when the instability measure $\pi/2m - \min(|\arg(\lambda)|)$ is negative, the system can not be chaotic.

Example 7. Let us investigate the Chen's system with a double scroll attractor. The fractional order form of such system can be described as [63]

$$\begin{aligned} {}_0D_t^{0.8}x_1(t) &= 35[x_2(t) - x_1(t)], \\ {}_0D_t^{1.0}x_2(t) &= -7x_1(t) - x_1(t)x_3(t) + 28x_2(t), \\ {}_0D_t^{0.9}x_3(t) &= x_1(t)x_2(t) - 3x_3(t). \end{aligned} \tag{53}$$

The system has three equilibria at $(0, 0, 0)$, $(7.94, 7.94, 21)$, and $(-7.94, -7.94, 21)$. The Jacobian matrix of the system evaluated at (x_1^*, x_2^*, x_3^*) is:

$$\mathbf{J} = \begin{bmatrix} -35 & 35 & 0 \\ -7 - x_3^* & 28 & -x_1^* \\ x_2^* & x_1^* & -3 \end{bmatrix}. \tag{54}$$

The two last equilibrium points are saddle points and surrounded by a chaotic double scroll attractor. For these two points, equation (51) becomes as follows:

$$\lambda^{27} + 35\lambda^{19} + 3\lambda^{18} - 28\lambda^{17} + 105\lambda^{10} - 21\lambda^8 + 4410 = 0. \tag{55}$$

The characteristic equation (55) has unstable roots $\lambda_{1,2} = 1.2928 \pm 0.2032j$, $|\arg(\lambda_{1,2})| = 0.1560$ and therefore system (53) satisfies the necessary condition for exhibiting a double scroll attractor. The instability measure is 0.0012.

Numerical simulation result of the system (53) for initial conditions $(-9, -5, 14)$, obtained via approximation technique described in [17], for simulation time $30\,s$ and time step $h = 0.0005$, is depicted in Fig. 9.

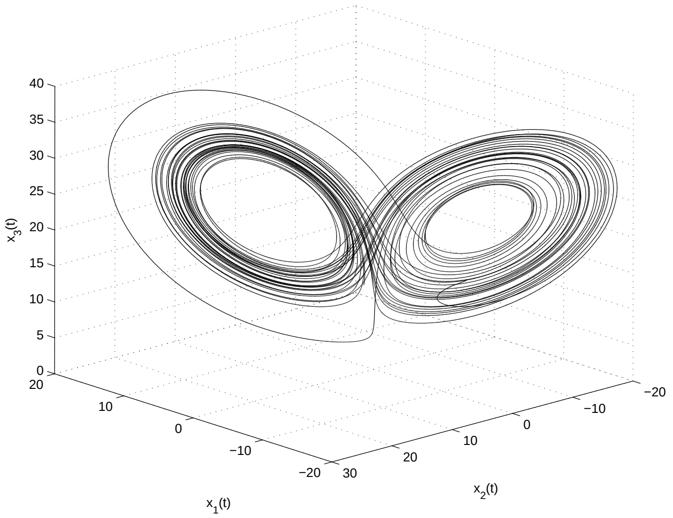


Figure 9: Double scroll attractor of Chen's system (53) in state space.

7. Conclusions

In this paper we have presented the definitions for internal and external stability condition of certain class of the linear and nonlinear fractional order system of finite dimension given in state space, FODE or transfer function representation (polynomial). It is important to note that the stability and asymptotic behavior of fractional order system is not of exponential type [11] but it is in the form of power law $t^{-\alpha}$ ($\alpha \in \mathrm{R}$), the so called long memory behavior [34].

The results presented in this survey are also applicable in the robust stability investigation [22, 45–47], stability of delayed system [16, 21] and stability of discrete fractional order system [19, 35]. Investigation of the fractional incommensurate order systems in state space, where space is deformed by various order of derivatives in various directions is still open problem.

Acknowledgements

This work was partially supported by the Slovak Grant Agency for Science under Grants VEGA: 1/4058/07, 1/0365/08, 1/0404/08, and Project APVV-0040-07.

References

[1] H. Akcay and R. Malti, On the completeness problem for fractional rationals with incommensurable differentiation orders. In: *Proc. of the 17th World Congress IFAC*, Soul, Korea, July 6-11 (2008), 15367–15371.

[2] E. Ahmed, A. M. A. El-Sayed and Hala A. A. El-Saka, On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems. *Physics Letters A* **358** (2006), 1–4.

[3] B. D. O.Anderson, N. I. Bose and E. I.Jury, A simple test for zeros of a complex polynomial in a sector. *IEEE Transactions on Automatic Control*, *Tech. Notes and Corresp.* **AC-19** (1974), 437-438.

[4] M. Aoun, R. Malti, F. Levron and A. Oustaloup, Numerical simulations of fractional systems: An overview of existing methods and improvements. *Nonlinear Dynamics* **38** (2004), 117-131.

[5] M. Aoun, R. Malti, F. Levronc and A. Oustaloup, Synthesis of fractional Laguerre basis for system approximation. *Automatica* **43** (2007), 1640–1648.

[6] P. Arena, R. Caponetto, L. Fortuna and D. Porto, *Nonlinear Noninteger Order Circuits and Systems - An Introduction.* World Scientific, Singapore (2000).

[7] M. Axtell and E. M. Bise, Fractional calculus applications in control systems. In: *Proc. of the IEEE 1990 Nat. Aerospace and Electronics Conf.*, New York (1990), 563–566.

[8] F. M.-Bayat, M. Afshar and M. K.-Ghartemani, Extension of the root-locus method to a certain class of fractional-order systems. *ISA Transactions* **48**, Issue 1 (Jan. 2009), 48-53 (Elsevier, doi:10.1016/j.isatra.2008.08.001).

[9] F. M.-Bayat and M. Afshar, Extending the Root-Locus Method to Fractional-Order Systems. *Journal of Applied Mathematics*, **2008** (2008), Article ID 528934, 13p. (Hindawi Publ. Corp., doi:10.1155/2008/528934).

[10] F. M.-Bayat and M. K.-Ghartemani, On the essential instabilities caused by multi-valued transfer functions. *Math. Problems in Engineering* **2008** (2008), Article ID 419046, 13p. (Hindawi Publ. Corp.).

[11] R. Bellman, *Stability theory of differential equations.* McGraw-Hill Book Company, New York (1953).

[12] H. W. Bode, *Network Analysis and Feedback Amplifier Design.* Tung Hwa Company (1949).

[13] C. Bonnet and J. R. Partington, Coprime factorizations and stability of fractional differential systems. *Systems and Control Letters* **41** (2000), 167-174.

[14] M. Caputo, Linear models of dissipation whose Q is almost frequency independent. *Geophys. J. R. Astr. Soc.* **13** (1967), 529-539 (Reprinted recently in: *Fract. Calc. Appl. Anal.* **11**, No 1 (2008), 3–14).

[15] G. E. Carlson and C. A. Halijak, Approximation of fractional capacitors $(1/s)^{1/n}$ by a regular Newton process. *IEEE Trans. on Circuit Theory* **11** (1964), 210–213.

[16] W. Deng, Ch. Li and J. Lu, Stability analysis of linear fractional differential system with multiple time delays. *Nonlinear Dynamics* **48** (2007), 409-416.

[17] Ľ. Dorčák, *Numerical Models for Simulation the Fractional-Order Control Systems*, UEF-04-94, The Academy of Sciences, Inst. of Experimental Physic, Košice, Slovakia (1994).

[18] R. C. Dorf and R. H. Bishop, *Modern Control Systems.* Addison-Wesley, New York (1990).

[19] A. Dzieliski and D. Sierociuk, Stability of discrete fractional order state-space systems. *Journal of Vibration and Control* **14** (2008), 1543–1556.

[20] A. Charef, Modeling and analog realization of the fundamental linear fractional order differential equation. *Nonlinear Dynamics* **46** (2006), 195-210.

[21] Y. Q. Chen and K. L. Moore, Analytical stability bound for a class of delayed fractional-order dynamic systems. *Nonlinear Dynamics* **29** (2002), 191-200.

[22] Y. Q. Chen H. -S. Ahna and D. Xue, Robust controllability of interval fractional order linear time invariant systems. *Signal Processing* **86** (2006), 2794-2802.

[23] S. A. Abd El-Salam and A. M. A. El-Sayed, On the stability of some fractional-order non-autonomous systems. *Electr. J. of Qualitative Theory of Diff. Equations* **6** (2007), 1–14.

[24] M. K. Ghartemani and F. M. Bayat, Necessary and sufficient conditions for perfect command following and disturbance rejection in fractional order systems. In: *Proc. of the 17th World Congress IFAC*, Soul, Korea, July 6-11 (2008), 364–369.

[25] R. Gorenflo, Yu. Luchko and S. Rogosin. Mittag-Leffler type functions: Notes on growth properties and distribution of zeros, *Preprint No. A-97-04*, Freie Univ. Berlin, Germany (2004).

[26] B.Gross and E.P. Braga, *Singularities of Linear System Functions.* Elsevier Publishing, New York (1961).

[27] S. E. Hamamci, Stabilization using fractional-order PI and PID controllers. *Nonlinear Dynamics* **51** (2008), 329-343.

[28] Ch. Hwang and Y. -Ch. Cheng, A numerical algorithm for stability testing of fractional delay systems. *Automatica* **42** (2006), 825-831.

[29] I. S. Jesus and J. A. T. Machado, Fractional control of heat diffusion systems. *Nonlinear Dynamics* **54**, No 3 (Nov. 2008), 263–282 (doi:10.1007/s11071-007-9322-2).

[30] W. R. LePage, *Complex Variables and the Laplace Transform for Engineers*. McGraw-Hill (1961).

[31] Y. Li, Y.Q.Chen, I. Podlubny and Y. Cao, Mittag-Leffler stability of fractional order nonlinear dynamic system. In: *Proc. of the 3rd IFAC Workshop on Fractional Differentiation and its Applications*, Ankara, Turkey, 05-07 November (2008).

[32] B. J. Lurie, Three-Parameter Tunable Tilt-Integral-Derivative (TID) Controller, *United States Patent, 5 371 670*, USA (1994).

[33] S. Manabe, The non-integer integral and its application to control systems. *ETJ of Japan* **6** (1961), 83–87.

[34] D. Matignon, Stability result on fractional differential equations with applications to control processing. In: *IMACS-SMC Proceedings*, Lille - France (July 1996), 963-968.

[35] D. Matignon, Stability properties for generalized fractional differential systems. In: *Proc. of Fractional Differential Systems: Models, Methods and App.* **5** (1998), 145–158.

[36] D. Matignon and B. D'Andrea-Novel, Some results on controllability and observability of finite-dimensional fractional differential systems. In: *Computational Engineering in Systems Applications*, Vol. 2, Lille - France (1996), 952–956.

[37] C. A. Monje, B. M. Vinagre, V. Feliu and Y. Q. Chen, Tuning and auto-tuning of fractional order controllers for industry applications. *IFAC J. of Control Engineering Practice* **16** (2008), 798–812.

[38] M. Nakagava and K. Sorimachi, Basic characteristics of a fractance device. *IEICE Trans. Fundamentals* **E75 - A** (1992), 1814–1818.

[39] K. B. Oldham and J. Spanier, *The Fractional Calculus.* Academic Press, New York (1974).

[40] A. Oustaloup, *La Derivation Non Entiere: Theorie, Synthese et Applications.* Hermes, Paris (1995).

[41] A. Oustaloup, J. Sabatier, P. Lanusse, R. Malti, P. Melchior, X. Moreau and M. Moze, An overview of the CRONE approach in system analysis, modeling and identification, observation and control. In: *Proc. of the 17th World Congress IFAC*, Soul, Korea, July 6-11 (2008), 14254–14265.

[42] N. Özturk and A. Uraz, An analysis stability test for a certain class of distributed parameter systems with delay. *IEEE Trans. on Circuit and Systems* **CAS-32** (1985), 393–396.

[43] F. J. V. -Parada, J. A. O. -Tapia and J. A. -Ramirez, Effective medium equations for fractional Ficks law in porous media. *Physica A* **373** (2007), 339–353.

[44] I. Petráš and Ľ. Dorčák, The frequency method for stability investigation of fractional control systems, *J. of SACTA*, **2** (1999), 75–85.

[45] I. Petráš, Y. Q. Chen and B. M. Vinagre, A robust stability test procedure for a class of uncertain LTI fractional order systems. In: *Proc. of the Intern. Carpathian Control Conf.*, Malenovice, Czech republic, May 27-30 (2002), 247–252.

[46] I. Petráš, Y. Q. Chen and B. M. Vinagre, Robust stability test for interval fractional order linear systems. Volume 208-210, Chapter 6.5: V. D. Blondel and A. Megretski (Eds.), *Unsolved Problems in the Mathematics of Systems and Control*, Princeton University Press (2004).

[47] I. Petráš, Y. Q. Chen, B. M. Vinagre and I. Podlubny, Stability of linear time invariant systems with interval fractional orders and interval coefficients. In: *Proc. of the Intern. Conf. on Computation Cybernetics*, Vienna, Austria, 8/30-9/1 (2004), 1-4.

[48] I. Podlubny, *Fractional Differential Equations.* Academic Press, San Diego (1999).

[49] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation. *Fract. Calc. Appl. Anal.* **5** (2002), 367-386.

[50] I. Podlubny, Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers. *IEEE Trans. on Automatic Control* **44** (1999), 208-213.

[51] A. G. Radwan, A. M. Soliman, A. S. Elwakil and A. Sedeek, On the stability of linear systems with fractional-order elements. *Chaos, Solitons & Fractals* **40**, Issue 5 (June 2009), 2317-2328 (doi:10.1016/j.chaos.2007.10.033).

[52] H.F. Raynaud and A. Zergaïnoh, State-space representation for fractional order controllers. *Automatica* **36** (2000), 1017–1021.

[53] M. F. Silva, J. A. T. Machado and A. M. Lopes, Fractional order control of a hexapod robot. *Nonlinear Dynamics* **38** (2004), 417-433.

[54] B. M. Vinagre, Y. Q. Chen and I. Petráš, Two direct Tustin discretization methods for fractional-order differentiator/integrator. *J. Franklin Institute* **340** (2003), 349–362.

[55] B. M. Vinagre, I. Petráš, P. Merchan and Ľ. Dorčák, Two digital realizations of fractional controllers: Application to temperature control of a solid. In: *Proc. of the ECC'01*, Porto, Portugal, September 4 - 7 (2001), 1764–1767.

[56] B. M. Vinagre and V. Feliu, Optimal fractional controllers for rational order systems: A special case of the Wiener-Hopf spectral factorization method. *IEEE Trans. on Automatic Control* **52** (2007), 2385-2389.

[57] B. M. Vinagre, C. A. Monje, A. J. Calderon and J. I. Suarez, Fractional PID controllers for industry application. A brief introduction. *J. of Vibration and Control* **13** (2007), 1419–1429.

[58] J. A. O-Tapia, F. J. V. -Parada and J. A. -Ramirez, A fractional-order Darcys law. *Physica A* **374** (2007), 1-14.

[59] M. S. Tavazoei and M. Haeri, Unreliability of frequency-domain approximation in recognising chaos in fractional-order systems. *IET Signal Proc.* **1** (2007), 171–181.

[60] M. S. Tavazoei and M. Haeri, A necessary condition for double scroll attractor existence in fractional-order systems. *Physics Letters A* **367** (2007), 102–113.

[61] M. S. Tavazoei and M. Haeri, Limitations of frequency domain approximation for detecting chaos in fractional order systems. *Nonlinear Analysis* **69** (2008), 1299–1320.

[62] M. S. Tavazoei and M. Haeri, A note on the stability of fractional order systems. *Mathematics and Computers in Simulation* **79**, Issue 5 (Jan. 2009), 1566–1576 (doi: 10.1016/j.matcom.2008.07.003).

[63] M. S. Tavazoei and M. Haeri, Chaotic attractors in incommensurate fractional order systems. *Physica D* **237** (2008), 2628–2637.

[64] P. J. Torvik and R. L. Bagley, On the appearance of the Fractional Derivative in the Behavior of real materials. *Transactions of the ASME* **51** (1984), 294–298.

[65] Ch. -Ch. Tseng, Design of FIR and IIR fractional order Simpson digital integrators. *Signal Processing* **87** (2007), 1045-1057.

[66] J. C. Wang, Realizations of generalized Warburg impedance with RC ladder networks and transmission lines. *J. of Electrochem. Soc.* **34** (1987), 1915–1920.

[67] S. Westerlund and L. Ekstam, Capacitor theory. *IEEE Trans. on Dielectrics and Electrical Insulation* **1** (1994), 826–839.

[68] S. Westerlund, *Dead Matter Has Memory!*. Causal Consulting, Kalmar (2002).

[69] D. Xue and Y. Q. Chen, A comparative introduction of four fractional order controllers, In: *Proc. of the* 4^{th} *World Congress on Intelligent Control and Automation*, Shanghai, China, June 10-14 (2002).

Received: November 25, 2008

Technical University of Košice
BERG Faculty
Institute of Control and Informatization of Production Processes
B. Němcovej 3, 042 00 Košice – SLOVAKIA

e-mail: ivo.petras@tuke.sk