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THE COMMUTANT OF THE RIEMANN-LIOUVILLE OPERATOR OF FRACTIONAL INTEGRATION

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Abstract

Characterization results for the continuous linear operators $M : C[a,b] \to C[a,b]$ and $N : L^1[a,b] \to L^1[a,b]$ commuting with a fixed Riemann-Liouville operator for integration of fractional order

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \ , \ \alpha > 0,$$

in the corresponding space are found.

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1. Introduction

It is well known that every two Riemann-Liouville operators I_{a+}^{β} and I_{a+}^{γ} with $\beta, \gamma > 0$ commute (cf. Samko, Kilbas, Marichev [6], pp. 33-34, (2.17) and (2.21)), i.e. $I_{a+}^{\beta} I_{a+}^{\gamma} = I_{a+}^{\gamma} I_{a+}^{\beta}$.

Our aim here is to characterize all continuous linear operators $M : C[a,b] \to C[a,b]$ and $N : L^1[a,b] \to L^1[a,b]$ which commute with a fixed Riemann-Liouville operator

$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-t)^{\alpha-1} f(t)dt \ , \ \alpha > 0, \tag{1}$$

in $C[a,b]$ or $L^1[a,b]$.

Further, we will consider elaborately only the commutation in the space $C = C[a,b]$, since the considerations for the space $L^1[a,b]$ are analogous.

As a basic tools we will use some properties of the Duhamel convolution algebra $(C,*)$ with the multiplication operation

$$(f*g)(x) = \int\limits_a^x f(x+a-t)g(t)dt. \tag{2}$$

As it is well known, the bilinear operation (2) is commutative and associative one, and the Riemann-Liouville operator (1) can be represented as a convolution operator $\left\{\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\right\}*$, i.e.:

$$I^{\alpha}_{a+} f(x) = \left\{\frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)}\right\} * f(x). \tag{3}$$

The operator I^{α}_{a+} is a multiplier of the Duhamel convolution algebra in the sense of the following definition.

Definition 1. (cf. Larsen [4], p.13). A mapping $M : C \to C$ is said to be a multiplier of the algebra $(C,*)$, if the identity

$$(Mf) * g = f * Mg \tag{4}$$

holds for all $f, g \in C$.

As it is shown in [4], pp. 13-14, the set of all multipliers of the convolution algebra $(C,*)$ form a commutative subalgebra of the algebra of all continuous linear operators $A : C \to C$. It contains the identity operator (which is not a convolution operator).

Corollary. *Let $M : C \to C$ be a multiplier of the convolution algebra $(C,*)$. Then, for all $f, g \in C$ it holds the identity*

$$M(f*g) = (Mf)*g.$$

The assertion follows from the commutativity of the algebra of multipliers of $(C,*)$. Indeed, if $f*$ denotes the multiplier operator $(f*)g = f*g$, then it commutes with M, i.e. $M(f*) = (f*)M$. The identity $[M(f*)]g = [(f*)M]g$ holding for arbitrary $g \in C$ can be written as

$$M(f * g) = f * Mg.$$

THEOREM 1. *Let $M : C \to C$ be an arbitrary continuous linear operator, commuting with I^{α}_{a+}, i.e. such that $MI^{\alpha}_{a+} = I^{\alpha}_{a+}M$. Then M is a multiplier of the convolution algebra $(C, *)$.*

P r o o f. We use the continuity of operation (2) with respect to both operands, i.e. the fact that $f_n \to f$ and $g_n \to g$ imply $f_n * g_n \to f * g$ for arbitrary f and g, along with the assumed continuity of the operator M. We start with the evident identity

$$M\{1\} * \{1\} = \{1\} * M\{1\}.$$

We take arbitrary non-negative integers m and n and apply the operator

$$(I^{\alpha}_{a+})^{m+n} = I^{m\alpha}_{a+}\, I^{n\alpha}_{a+}$$

to both sides of the above identity. Thus we get

$$(I^{m\alpha}_{a+}M\{1\}) * I^{n\alpha}_{a+}\{1\} = I^{m\alpha}_{a+}\{1\} * (I^{n\alpha}_{a+}M\{1\})$$

which can be written as

$$M(I^{m\alpha}_{a+}\{1\}) * I^{n\alpha}_{a+}\{1\} = I^{m\alpha}_{a+}\{1\} * M(I^{n\alpha}_{a+}M\{1\}),$$

using the commutation relation $MI^{\alpha}_{a+} = I^{\alpha}_{a+}M$. But

$$I^{\beta}_{a+}\{1\} = \frac{(x-a)^{\beta}}{\Gamma(\beta+1)} \quad \text{for arbitrary} \quad \beta > 0.$$

Hence,

$$M\{(x-a)^{m\alpha}\} * \{(x-a)^{n\alpha}\} = \{(x-a)^{m\alpha} * M\{(x-a)^{n\alpha}\}$$

for arbitrary $m, n \in \mathbb{N}_0$. From Weierstrass' approximation theorem it follows that

$$\overline{\text{span}}\{(x-a)^{\alpha n}\}_{n=0}^{\infty} = C[a,b].$$

It remains to use the continuity of the convolution product, and thus to obtain the identity

$$Mf * g = f * Mg$$

for arbitrary $f, g \in C[a,b]$. Hence, M is a multiplier of the convolution algebra $(C, *)$. ∎

Theorem 1 reduces the problem of characterizing of the continuous linear operators $M : C[a,b] \to C[a,b]$ commuting with the Riemann-Liouvile operator I^{α}_{a+} to the problem of characterizing the multipliers of the convolution algebra $(C,*)$. The solution of the last problem is given by the following theorem.

THEOREM 2. (Dimovski [2], pp. 6-7) *The following assertions are equivalent:*

i) M is a multiplier of the convolution algebra $(C,*)$;

ii) a) $m(t) = M\{1\} \in C$ is a function with bounded variation on $[a,b]$;

b) It holds the integral representation

$$Mf(x) = \frac{d}{dx}\int\limits_a^x m(x+a-t)f(t)dt \tag{6}$$

for arbitrary $f \in C[a,b]$.

REMARK. In [2] it is considered the case $a = 0$, but the result for arbitrary $a \in \mathbb{R}$ could be obtained by a simple translation.

2. Characterization of the commutant of I^{α}_{a+} on $C[a,b]$

Theorems 1 and 2 give a complete characterization of the commutant of a fixed Riemann-Liouville operator on $C[a,b]$, which we state as the following theorem.

THEOREM 3. *A continuous linear operator $M : C[a,b] \to C[a,b]$ commutes with a fixed Riemann-Liouvive operator I^{α}_{a+} in $C[a,b]$ iff:*

a) the function $m(t) = M\{1\}$ is a function with bounded variation on $[a,b]$;

and

b)
$$Mf(x) = \frac{d}{dx}\int_a^x m(x+a-t)f(t)dt\,.$$

This representation can be written in the equivalent form

$$Mf(x) = m(a)f(x) + \int\limits_a^x f(x+a-t)\,dm(t)\,, \tag{7}$$

where the integral is to be understood in the sense of Stieltjes.

The characterization, given by Theorem 3, can be stated in a more transparent form as the following theorem.

Theorem 4. *The commutant of an arbitrary Riemann-Liouville operator is a commutative algebra, isomorphic to the convolution algebra $BV \cap C$ of the functions of $C[a,b]$ with bounded variation, where the multiplication operation is*

$$(f \widetilde{*} g)(x) = \frac{d}{dx} \int_a^x f(x+a-t) g(t) dt. \tag{8}$$

The isomorphism is given by the mapping $M \mapsto M\{1\} = m(t)$ and it is both algebraic and topological isomorphism.

The fact that (8) is an inner operation in $\mathrm{BV} \cap C$ is proved in Mikusiński and Ryll-Nardzewski [5].

3. Characterization of the commutant of I^{α}_{a+} on $L^1[a,b]$

We use almost the same approach as in the case of $C[a,b]$. Instead of Weierstrass' approximation theorem, we rely on the analogous theorem which assert that each function of $L^1[a,b]$ can be approximated by polynomials in the L^1-metrix. An analogue of Theorem 1 holds with the only difference that for the function $n(t) = N\{1\}$, it is asserted that it is a function with essentially bounded variation, i.e. that it is equal almost everywhere to a function with bounded variation on $[a,b]$ (cf. Dixmier [3]). In [3] it is proposed a characterization of the commutant of the integration operator $I = I^1_{0+}$ in $L^1[0,1]$.

Theorem 5. *The following assertions are equivalent:*

i) A continuous linear operator $N : L^1[a,b] \mapsto L^1[a,b]$ commutes with the Riemann-Liouville operator I^{α}_{a+} on $L^1[a,b]$;

ii) a) $n(x) = N\{1\} \in L^1[a,b]$ is a function with essentially bounded variation on $[a,b]$;

b) the operator N admits the representation

$$Nf(x) = \frac{d}{dx} \int_a^x n(x+a-t) f(t) dt\,, \tag{9}$$

where the integral is to be understood in the Lebesgue sense.

The only non-trivial point is the proof of the fact that if n is a function with essentially bounded variation, and f is a Lebesgue integrable function, then the convolution product $n * f$ is an absolutely continuous function on $[a,b]$. Such result cannot be seen in Mikusiński and Ryll-Nardzewski [5]. It is proved by N. Bozhinov in his book [1], p.137.

In the same way as in the case of $C[a,b]$ the characterization result, given by Theorem 5, can be stated in an equivalent form:

Theorem 6. *The commutant of I^{α}_{a+} in $L^1[a,b]$ is a commutative algebra, isomorphic to the convolution algebra $(\mathrm{ess}\,BV, \widetilde{*})$ of the functions of $L^1[a,b]$ with essentially bounded variation with the convolution product*

$$(f \,\widetilde{*}\, g)\,(x) = \frac{d}{dx}\int_a^x f(x+a-t)g(t)dt\,. \qquad (10)$$

as the multiplication. The algebraic and topological isomorphism is given by the mapping $N \mapsto n(x) = N\{1\}$.

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