Nash problem on spaces of arcs

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Abstract

This article is organized as an introduction to the Nash problem on arc spaces, and a survey of all known (to the author) results answering to the problem for particular classes. Among them are toric and stable toric varieties (Ishii, Kollár, Petrov), some classes of normal surface singularities (Nash, Plénat, Popescu-Pampu, Reguera, Lejeune-Jalabert, Morales), including two criteria (Reguera, Plénat), and few results in higher dimension (Plénat, Popescu-Pampu, Pérez). Also, three modifications are discussed - the embedded Nash problem, the Nash problem for pairs, and the local Nash problem.

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Contents

Chapter 1

Introduction

Singularities play important role in algebraic geometry. Let X be an algebraic variety over an algebraically closed field k. The geometry of its singular locus $\mathrm{Sing}(X)$ could be understood better using different algebraic, topological or geometrical objects associated with it. The space of arcs X_∞ is such an object. It reflects the local geometry of the singular locus, and on it could also be defined a powerful analytical tool, the motivic integration ([Loo02], [Vey04], [Cra04]), with numerous applications to singularities and their invariants. Introduced by J. Nash in 1968, the arc spaces have been studied actively. One of the major problems in the area is the Nash problem. Our main goal here is to review the results obtained about it, together with some modifications and generalizations of the problem.

For an algebraic variety X over k the set of k-points on space of arcs X_∞ is the set of all morphisms $\mathrm{Spec}\ k[[t]] \to X$, coming naturally with a structure of a scheme over k. There is a canonical map $\pi\colon X_\infty \to X$, sending each arc γ to $\gamma(0)$, where $0 \in \mathrm{Spec}\, k[[t]]$ is the closed point. Then $\pi^{-1}(\mathrm{Sing}(X))$ is a union of irreducible components. Such a component C is called Nash, or good component, if there exists an arc $\alpha \in C$ such that $\alpha(\eta) \notin \mathrm{Sing}(X)$, where $\eta \in \mathrm{Spec}\, k[[t]]$ is the generic point.

The essential divisors over X are, roughly speaking, those exceptional divisors in some resolution of singularities of X that appear birationally in any resolution of X. In 1968 J. Nash ([Nas95]) proved that there is an injective map $\mathcal{N}_{\mathrm{X}}$ from the set of good components of $\pi^{-1}(\mathrm{Sing}(X))$ to the set of the essential divisors over X. In it he asked if $\mathcal{N}_{\mathrm{X}}$ is always bijective. In their paper ([IK03]) Ishii and Kollár proved that this holds for toric varieties but fails in general, with a counterexample in dimension 4. Nevertheless it is quite useful to know for which classes of algebraic varieties the question is positively answered. This is called Nash problem.

The counterexample given by Ishii and Kollár works in any dimension bigger than three, but in dimensions two and three the problem is open in general. Results have been obtained for some particular cases, such as A_n ([Nas95])and D_n [Plé05b]) surface singularities, minimal singularities ([Reg95]), surface sandwiched singularities ([LJRL99]), toric varieties of arbitrary dimension ([IK03]), and some other classes ([Ish05], [Mor08], [PPP06], [GP07]).

Another case for which the Nash problem was solved ([Pet09]) is the case of stable toric varieties (STVs). They are the analogs of stable curves in the case of toric varieties, appearing for example in degenerations of abelian varieties, and have been studied by Alexeev ([Ale02]). To solve the problem in this case is helpful to put it into a new frame, the Nash problem for pairs (X, Y), where X is an algebraic variety and Y a proper closed subset with $\mathrm{Sing}(X) \subset Y$. This could be of general interest as well, and appears each time the Nash problem is asked for a class of non-irreducible algebraic varieties [GP07]). Let a Y-resolution of X be any proper birational morphism $f\colon X' \to X$ such that X' is smooth, $f|_{f^{-1}(X\setminus Y)}$ is an isomorphism, and $f^{-1}(Y)$ is of pure codimension one. Again the Nash map $\mathcal{N}_{(X,Y)}$ from the set of irreducible components of $f^{-1}(Y)$ to the set of Y-essential divisors over X is defined and injective. The Y-essential divisors are by definition the exceptional divisors of X appearing birationally in each Y-resolution of X. So we have naturally the Nash problem for pairs, asking for which pairs (X, Y) is $\mathcal{N}_{(X,Y)}$ bijective. In Chapter 4 we will obtain a positive answer to it in the case of toric pairs, i.e. pairs (X, Y) such that X is a toric variety and $Y \supset \mathrm{Sing}(X)$ is a $\mathbb{T}$-invariant subvariety.

This article is based on my thesis in 2007. The first three chapters are an introduction to Nash problem. In Chapter 2 are given the basic definitions and facts about jet and arc spaces that will be needed later. In Chapter 3 some basic material about singularities is collected. There is proved the theorem of Nash, formulated the Nash conjecture ([Nas95]), and given a counterexample proposed by Ishii and Kollár ([IK03]). At the end is introduced the Nash problem for pairs. In Chapter 4 the results about toric varieties ([IK03]) and stable toric varieties ([Pet09]) are proved, and the embedded Nash problem ([ELM04], [Ish04]) is discussed. In Chapter 5 are collected all known to the author results about normal surface singularities. These include the results of Nash about A_n ([Nas95]), and of Plénat about D_n (

[Plé05b]) rational double points, of Plénat, Popescu-Pampu ([PPP06]) about a class of non-rational surface singularities, including a couple of useful criteria. The earlier results of Reguera ([Reg95]) on minimal singularities, of Reguera, Lejeune-Jalabert ([LJRL99], [Reg06]) on sandwiched surface singularities, and a result of Morales ([Mor08]), are included as well. In Chapter 6 are discussed some other classes of varieties ([PPP08], [GP07]), and another modification of the problem, the local Nash problem ([Ish06]).

Notations: In the following, k is an algebraically closed field of arbitrary characteristic. The schemes are of finite type over k, although some results hold for arbitrary k-schemes. The varieties are reduced and irreducible schemes of finite type over k, except in Sec. 4.3 on stable toric varieties (which are not irreducible in general). For a function $f \in \mathcal{O}_X$ or an ideal $I \subset \mathcal{O}_X$, $Z(f)$ and $Z(I)$ are respectively the sets of zeros in X.

Chapter 2

Spaces of Arcs

Let X be an algebraic variety over k with singular locus $\mathrm{Sing}(X)$. There are different approaches using commutative algebra, topology, analysis, or geometry to study it. One could attach an object to $\mathrm{Sing}(X)$ or to X of an algebraic, topological, or combinatorial nature and use it as a valuable source of information for both the local and global geometry of the singular locus. Examples for this are the spaces of n-jets X_n for any $n \in \mathbb{N}$, and the space of arcs X_∞ of X. Although most of the notions in this section could be defined for an arbitrary scheme of finite type over k, we will restrict ourselves to the case of varieties over k.

In this chapter are defined the space of arcs and the spaces of n jets over an algebraic variety. The definition of a good, or Nash component is given. It is shown that X_∞ has a structure of a scheme over k, that in char(k) = 0 all components are good, and that X_∞ is irreducible when char(k) = 0 (Kolchin's theorem). The chapter finishes with an elementary example of calculations with arcs.

Most of the results in this chapter are from ([Mus06], [EM06]), ([IK03]) and ([Kol73]).

In the following R will be a k-algebra. As a topological space, the scheme $\mathrm{Spec}\, k[[t]]$ has two points, the closed one noted by 0, and the generic one noted by η.

Definition 2.0.1. *An m-jet over X with coefficients in R is a morphism $\beta : \mathrm{Spec}\, R[[t]]/(t^{m+1}) \to X$. An R-arc over X is a morphism $\alpha : \mathrm{Spec}\, R[[t]] \to X$.*

As we will see below, an arc over k could be viewed as an "infinitely small neighborhood" of the image of the closed point on some infinitesimal curve in X. Here "infinitely small" means a "neighborhood" of a closed point, containing no other closed points.

Definition 2.0.2. *For each $m \in \mathbb{N}$, define a contravariant functor from the category of schemes over k to the category of sets*

$$\begin{aligned}\mathcal{F}_m : &\mathcal{S}ch_k \to \mathcal{S}et,\\ &Z \mapsto Hom(Z \times \operatorname{Spec}(k[[t]]/(t^{m+1})), X).\end{aligned}$$

Proposition 2.0.3. *For any m the functor $\mathcal{F}_m$ is representable.*

Proof. Take $X \hookrightarrow \mathbb{A}^n$ for some n defined by an ideal $I = (f_1, f_2, \ldots f_s)$. Then any morphism $\operatorname{Spec} R[t]/(t^{m+1}) \to X$ is defined by a homomorphism $k[x_1, x_2, \ldots x_n]/I \to R[t]/(t^{m+1})$, namely by the images $y_i = \sum_{j=0}^{m} a_{i,j} t^j$ of x_i. For any l, $f_l(y_1, y_2, \ldots y_n) = 0$ is a polynomial equation in the powers of t with coefficients $g_{l,p}(\{a_{i,j}\}_{i,j})$. Then the system $\{g_{l,p} = 0\}_{l,p}$ defines $X_m \subset \mathbb{A}^{(m+1)n}$ together with $\pi^m : X_m \to X$, sending each morphism $\gamma : \operatorname{Spec} R[t]/(t^{m+1}) \to X$ to $\gamma(0) \in X$.

Take now $X = \bigcup_{i=1}^{r} U_i$ to be an arbitrary variety with an affine open covering $U_{i i\in\S}$. For any i take the corresponding jet scheme $\pi_i^m : (U_i)_m \to U_i$. Then one has

Lemma 2.0.4. *If X is an affine scheme and $U \subset X$ is open, then $U_n = (\pi^n)^{-1}(U)$.*

Proof. Let $\alpha : \operatorname{Spec} A \to \operatorname{Spec} A[t]/(t^{m+1})$ be the morphism induced by the truncation. A morphism $f : \operatorname{Spec} R[t]/(t^{m+1}) \to X$ factors set-theoretically through the embedding $U \hookrightarrow X$ iff $f \circ \alpha$ factors set-theoretically through it. □

By the above lemma $(\pi_i^m)^{-1}(U_i \cap U_j) \simeq (\pi_j^m)^{-1}(U_i \cap U_j)$ over X for each i, j, each being isomorphic over X to $(U_i \cap U_j)_m$. So they are canonically isomorphic, and this defines a scheme X_m, gluing the U_i's along these canonical isomorphisms. In this way is also defined a morphism $\pi_m : X_m \to X$. Then from the proof of Lemma 1.0.4. X_m will have the property that $\operatorname{Hom}(\operatorname{Spec} R, X_m) \simeq \operatorname{Hom}(\operatorname{Spec} R[t]/(t^{m+1}), X)$. □

Denote by X_m the scheme representing $\mathcal{F}_m$. Then $\operatorname{Hom}(\operatorname{Spec} R, X_m) \simeq \operatorname{Hom}(\operatorname{Spec} R[t]/(t^{m+1}), X)$ by definition, so the k-points on X_m correspond to morphisms $\operatorname{Spec} k[t]/(t^{m+1}) \to X$, i.e. to m-jets with coefficients in k.

Remark 2.0.5. 1) Attaching X_m to X defines a covariant functor from the category of schemes of finite type over k to itself.
2) For each $n, m \in \mathbb{N}$ there are projection morphisms $\pi_{n,m} : X_n \to X_m$ (for $m \leq n$), corresponding to the truncation homomorphisms $k[[t]]/(t^{n+1}) \to k[[t]]/(t^{m+1})$, and $\pi_n : X_n \to X$.
3). By Prop. 2.0.2. the k-points on X_n correspond to the $k[[t]]/(t^{n+1})$ -points on X taking $Z = \operatorname{Spec} k$; in particular $X_0 = X$.

Then one has a projective system of k-schemes and affine morphisms $\{(X_m, \pi_{m,m-1})\}$. It follows that its projective limit $\varprojlim(X_m)$ is a scheme over k, though not of finite type in general.

Definition 2.0.6. $X_\infty(R) := \varprojlim(X_m(R))$.

The closed points on $X_\infty(k)$ correspond to k-arcs $\operatorname{Spec} k[[t]] \to X$ because the k-points will correspond to inverse limits of k-points on
$\operatorname{Hom}(\operatorname{Spec} k[t]/(t^{m+1}), X) \simeq \operatorname{Hom}(\operatorname{Spec} k, X_m)$ (see below). The generic point ζ of an irreducible closed proper subset of $X_\infty(k)$ defines a morphism $\operatorname{Spec} K[[t]] \to X$, where $K \supset k$ is the residue field of ζ. For point $x \in X_\infty$, the corresponding arc will be denoted by α_x.

By Def. 2.0.6 the scheme structure on the space of arcs is defined. It is easy to see also that the maps $\phi_m : X_\infty \to X_(m+1)$ obtained by truncation at power m, are scheme morphisms. In the case $m = 0$ we will obtain the canonical morphism $\pi = \phi_0 : X_\infty \to X$, sending each arc $\alpha \in X_\infty$ to $\alpha(0)$, the image of the closed point of $\operatorname{Spec} k[[t]])$ in X. In this way each arc on X could be viewed as an infinitesimal neighborhood of a point on an infinitesimal curve on X.

The lemma above could be easily generalized for étale morphisms ([Mus06]). Recall ([Uen03]) that a morphism of schemes $f : X \to Y$ is called étale if for each scheme morphism $g : Z \to Y$ and each closed subscheme, defined by a nilpotent ideal, $Z_0 \hookrightarrow Z$, and arbitrary morphism $h : Z_0 \to X$ over Y, giving a commutative diagram, there is a unique morphism $\bar{g} : Z \to X$, in a commutative diagram again.

Lemma 2.0.7. *For an étale morphism of varieties $f : X \to Y$, the diagram*

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \downarrow \pi_{X,m} & & \downarrow \pi_{Y,m} \\ X & \xrightarrow{f} & Y. \end{array}$$

is Cartesian.

Proof. The R-valued points on X_m and Y_m correspond to the $R[t]/(t^{m+1})$-valued points on X and Y. By the fact that f is formally étale in a neighborhood of each point, for any commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} R & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \operatorname{Spec} R[t]/(t^{n+1}) & \longrightarrow & Y. \end{array}$$

there is a unique morphism $\bar{\alpha} : \operatorname{Spec} R[t]/(t^{n+1}) \to X$, giving again a commutative diagram. This proves the diagram in the lemma to be Cartesian. □

Remark that in the case of étale morphism f taking the projective limit of the diagram, one has by the previous lemma the corresponding morphism of arc spaces $f_\infty := \varprojlim(f_m) : X_\infty \to Y_\infty$, and a commutative diagram

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y. \end{array}$$

Proposition 2.0.8. *If X is a smooth variety of dimension n, for any $m \in \mathbb{N}$ X_m is smooth variety of dimension $(m+1)n$, and the morphism $\pi_{m,m-1}$ is locally trivial with fibers isomorphic to $\mathbb{A}^n$.*

Proof. In an open affine neighborhood $U \subset X$ of any point one could find a system of local parameters $x_1, x_2, \dots x_n$, because X is smooth. Such a system of coordinates gives an étale morphism $U \to \mathbb{A}^n$. Also, on U $dx_1, dx_2, \dots dx_n$ forms a basis for Ω^1_X, and $\pi_{m,m-1}$ corresponds to the projection $\mathbb{A}^{mn} \to \mathbb{A}^{(m-1)n}$ skipping the last n factors. Finally, we apply Lemma 2.0.4 to prove the claim in the affine case. Using the fact that a system of local parameters $x_1, x_2, \dots x_n$ defines an étale morphism and applying Lemma 2.0.7, one has that $U_m \simeq U \times \mathbb{A}^{mn}$, which end the proof.

□

Using the definition of formal completion of a scheme along closed subscheme (see [Har77], Ch.II.9) and Proposition 2.0.3., one has:

Proposition 2.0.9. *For any scheme Y of finite type over k,*

$$\operatorname{Hom}(Y, X_\infty) \simeq \operatorname{Hom}(Y \widehat{\times_{\operatorname{Spec} k} \operatorname{Spec}} \, k[[t]], X),$$

the formal completion of $Y \times_{\operatorname{Spec} k} \operatorname{Spec} k[[t]]$ to be taken along the closed subscheme $Y \times_{\operatorname{Spec} k} \{0\}$.

In particular, take $Y = X_\infty$ and the identity morphism on the left.

Corollary 2.0.10. *There is a universal family of arcs* $X_\infty \times_{\mathrm{Spec}\, k} \widehat{\mathrm{Spec}}\, k[[t]] \to X$.

Corollary 2.0.11. *For any k-algebra S,* $\mathrm{Hom}(\mathrm{Spec}\, S, X_\infty) \simeq \mathrm{Hom}(\mathrm{Spec}\, S[[t]], X)$.

If $f : X \to Y$ is a morphism of varieties, one has the corresponding $f_m : X_m \to Y_m$ by the functorial description of the space of m-jets for any m, so taking the projective limit one has:

$$\begin{array}{ccc} X_\infty & \xrightarrow{f_\infty} & Y_\infty \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X & \xrightarrow{f} & Y \end{array}$$

As remarked above, by Lemma 2.0.6. this diagram is Cartesian. If f is a closed immersion, so is f_∞. In this way, sending $X \longmapsto X_\infty$ defines a covariant functor $\mathcal{V}ar_k \to \mathcal{S}ch_k$.

For $\mathrm{Sing}(X) \subset X$, let $\pi_X^{-1}(\mathrm{Sing}(X)) = \bigcup_i C_i$ with C_i its irreducible components as a closed subscheme in X_∞. They will be called the components over X in the sequel.

Definition 2.0.12. *A component C_i is called Nash, or good, component if it contains an arc α, such that $\alpha(\eta) \notin \mathrm{Sing}(X)$.*

In other words, a component is Nash component if it is not contained in $(\mathrm{Sing}(X))_\infty$.

The next theorem is ([IK03], Lem. 2.12).

Theorem 2.0.13. *If* $\mathrm{char(k)} = 0$ *any component C_i over X is a Nash component.*

Proof. The claim is local, so we may assume that X is affine.
Claim: Any k-arc $\alpha : \mathrm{Spec}\, k[[t]] \to X, \alpha(0) \in \mathrm{Sing}(X)$ is a specialization of an arc $\gamma \in \pi^{-1}(\mathrm{Sing}(X))$ with $\gamma(\eta) \notin \mathrm{Sing}(X)$.
Proof: If $S := \overline{\mathrm{Im}(\alpha)}$, then S is irreducible, i.e. $\mathcal{O}_S$ is a domain (see [Reg95]). So α is defined by an injective homomorphism $\mathcal{O}_S \hookrightarrow k[[t]]$ coming from the dominant morphism on S. If $S \nsubseteq \mathrm{Sing}(X)$ we are done. Otherwise we show as a first step that α is a specialization of an arc β such that $\beta(0)$ is the generic point of S.

Take the injective homomorphism $k[[t]] \hookrightarrow k[[u,v]]$, $t \mapsto u+v$. The composition $k[[t]] \hookrightarrow k[[u,v]] \to k[[u,v]]/(v) \simeq k[[u]]$
is an isomorphism. Then define β by the homomorphism $\mathcal{O}_S \xrightarrow{\alpha} k[[t]] \hookrightarrow k[[u,v]] \hookrightarrow k((u))[[v]]$,
the middle arrow being the inclusion defined by the isomorphism above, and the last one being the natural inclusion. Then let $K := k(\{u^{1/n}\}_{n\in\mathbb{N}})$ be the algebraic closure of $k((u))$. The closed point of $\operatorname{Spec} K[[v]]$ maps by β to $\beta_*^{-1}(u)$, so that the pre-image of (u) in $k'[[t]]$ is (0). This shows that $0 \in \operatorname{Spec} K[[u]]$ is sent by β to the generic point of S. That is, α is a specialization of an arc β such that $\beta(0) = \eta_S$.

Intersecting repeatedly by hypersurfaces in X containing S, we could eventually find a subvariety $Z \supset S$, $\dim Z = \dim S + 1$. Moreover, using that X is affine we could choose Z such that X will be smooth at its generic point. If $\nu : \tilde{Z} \to Z$ is the normalization, take $S' := \nu^{-1}(S)$ with the reduced scheme structure. Then ν defines finite and surjective morphism on S', so it is étale at the generic point of S' (this holds because char(k) = 0). Then the arc $\beta : \operatorname{Spec} K[[u]] \to S$ could be lifted to $\beta' : \operatorname{Spec} K[[u]] \to S'$ by the remark after Lem. 2.0.7. Since $\tilde{Z}$ is normal, it will be smooth along the generic point of S', because the singular locus of $\tilde{Z}$ has codimension 2 or more. This shows that β' is a specialization of of an arc γ through S', mapping the generic point η to the generic point of $\tilde{Z}$. The composition $\nu \circ \gamma$ then will give β, so α is a specialization of an arc γ with the desired properties.

From this one concludes that any component will contain an arc α with $\alpha(\eta) \notin \operatorname{Sing}(X)$.

□

Remark 2.0.14. 1) If char(k) $= p > 0$, there could exist bad, that is non Nash, components over X. For example, take $X \subset \mathbb{A}^3$ defined by $x_1^p = x_2^p x_3$. Then $\operatorname{Sing}(X)$ is given by $x_1 = x_2 = 0$, and $\pi^{-1}(\operatorname{Sing}(X))$ has a non-good component ([IK03]). Indeed, take the normalization of X, which is $\tilde{X} \simeq \mathbb{A}^2$, corresponding to the ring homomorphism $(u,v) \mapsto (uv, v, u^p)$. Then $\nu^{-1}(\operatorname{Sing}(X)) = (v = 0)$ and $\nu|_{\nu^{-1}(\operatorname{Sing}(X))}$ is purely inseparable, meaning that an arc on $\operatorname{Sing}(X)$ cannot be lifted on its pre-image. Thus such an arc is not a specialization of an arc on $\operatorname{Sing}(X)$, sending the generic point into $X \setminus \operatorname{Sing}(X)$, so there is a bad component over X.

2) It is easy to see that there is a canonical isomorphism $(X \times Y)_n \simeq X_n \times Y_n$, for any $n \in \mathbb{N}$ and $n = \infty$. Equivalently, for any scheme Z, $\operatorname{Hom}(Z, X_n \times Y_n) \simeq \operatorname{Hom}(Z, X_n) \times \operatorname{Hom}(Z, Y_n)$. This holds because for a

k-algebra A, $\mathrm{Hom}(\mathrm{Spec}\, A, X_n \times Y_n) \simeq$
$\mathrm{Hom}(\mathrm{Spec}\, A, X_n) \times \mathrm{Hom}(\mathrm{Spec}\, A, Y_n)$. Applying the functorial description of X_n, Y_n to the right side, we obtain $\mathrm{Hom}(\mathrm{Spec}\, A \times \mathrm{Spec}\, k[[t]]/(t^{n+1}), X \times Y) \simeq \mathrm{Hom}(\mathrm{Spec}\, A, (X \times Y)_n)$.

The next result will be need also ([Ish04]).

Proposition 2.0.15. *If X is a toric variety, every component over X is good.*

Proof. Suppose there is a bad (i.e. a non-Nash) component $\mathcal{C}$ over X. Let $f : X' \to X$ be an equivariant resolution of singularities with $E_1, \ldots E_m$ the exceptional prime divisors. Then $\pi_{X'}^{-1}(E_i)$ are irreducible sets of $(\pi_X^{-1} \circ f_\infty)(\mathrm{Sing}(X))$.

Claim: f_∞ is surjective. Indeed, for any arc $\alpha \in X_\infty, \alpha(\eta) \in \mathrm{orb}(\tau)$ for some face τ in the fan of X. Because f is equivariant, $f^{-1}(\mathrm{orb}(\tau))$ contains subset isomorphic to $\mathrm{orb}(\tau) \times \mathbb{T}^s$, the last factor being the s-dimensional torus. But then the composition $\mathrm{Spec}\, k((t)) \to X$ of α and the canonical morphism $\mathrm{Spec}\, k((t)) \to \mathrm{Spec}\, k[[t]]$ can be lifted to X'. By the properness of f, the same holds for α.

So there exists a component of $\pi_{X'}^{-1}(E_i)$ mapped by f_∞ to $\mathcal{C}$. But this gives a contradiction since the component will contain an arc which, composed with f will send η out of $\mathrm{Sing}(X)$. The latter holds by the surjectivity of f_∞ and the smoothness of X'.

From the claim in the proof one has immediately:

Corollary 2.0.16. *For a toric variety X, X_∞ is irreducible.*

Finally, we will mention without proof an important theorem of Kolchin ([Kol73]), that could be useful in calculations with the irreducible components of closed sets in the space of arcs. In fact, it could be proved using Thm. 2.0.13 (see [Ish07]) which could be seen as a stronger version of it.

Theorem 2.0.17. (Kolchin's Theorem) *If* $\mathrm{char(k)} = 0$*, the space of arcs X_∞ is irreducible.*

Note that the Kolchin's theorem does not hold in positive characteristic, see ([NS05]).

Example 2.0.18. Here we will calculate directly the number of irreducible components of $\pi^{-1}(Y)$ for some closed subsets $Y \supset \mathrm{Sing}(X)$.

Let X be the cone defined by $xy - z^2 = 0$ in $\mathbb{A}^3_k$, char(k) $= 0$, so that $\mathrm{Sing}(X) = \{0\}$. Take $Y \subset X$ to be defined by $y = z = 0$, wanting to describe $\pi^{-1}(Y)$. Each arc α on X defines a homomorphism $\mathcal{O}_X \to K[[t]]$ determining a principal ideal in $K[[t]]$, generated by a power series A and unique up to an invertible factor. Then in the terms of these power series the space X_∞ is given by the equation $AB - C^2 = 0$, where $A = \sum a_n t^n$, $B = \sum b_n t^n$ and $C = \sum c_n t^n$. From this we have an infinite system of equations relating the coefficients:
$a_0 b_0 = c_0^2$,
$a_0 b_1 + a_1 b_0 = 2c_0 c_1$,
$a_2 b_0 + a_1 b_1 + a_0 b_2 = c_1^2 + 2c_0 c_2$,
$\cdots$
The corresponding arcs are at over Y, that is $b_0 = c_0 = 0$, so $a_0 b_1 = 0$. This gives two systems of equations:
$a_0 = 0$,
$a_1 b_1 = c_1^2$,
$\cdots$,
and
$b_1 = 0$,
$a_1 b_2 = c_1^2$,
$\cdots$
They determine two subschemes, each one isomorphic to X_∞, which by Kolchin's theorem is irreducible. Thus $\pi^{-1}(Y)$ has 2 irreducible components.

If $Y = \mathrm{Sing}(X)$ and char(k) $= 0$ using similar calculations and applying Kolchin's theorem one has that $\pi^{-1}(Y)$ is irreducible.

Chapter 3

Nash Theorem

In this chapter we are giving the basic definitions needed about singularities, in particular the notion of essential divisor over X, relating the irreducible components of $\pi^{-1}(\mathrm{Sing}(X))$ with some of these divisors using the Nash map. The Nash will be proved, the Nash conjecture formulated, and a counterexample in dimensions 4 or higher will be discussed. The main sources for this chapter are ([IK03], [Pet09]).

3.1 Singularities and Resolutions

Definition 3.1.1. *A resolution of singularities of X is a proper birational morphism $f : \tilde{X} \to X$, with $\tilde{X}$ nonsingular and such that the restriction $f|_{f^{-1}(X \setminus \mathrm{Sing}(X))}$ is an isomorphism. An exceptional prime divisor in $\tilde{X}$ is one appearing as an irreducible component of the exceptional locus $f^{-1}(\mathrm{Sing}(X))$.*

Definition 3.1.2. *Let $f_i : Y_i \to X, i = 1, 2$ be proper birational morphisms with normal Y_i's, and $E \subset Y_1$ be an prime exceptional divisor of f_1. Then $f_2^{-1} \circ f_1 : Y_1 \dashrightarrow Y_2$ is defined on a nonempty open subset E_0 of E (by Zariski's Main Theorem the indeterminacy set is of codimension ≥ 2), and the center of E in f_2 is defined to be $\overline{f_2^{-1} \circ f_1(E_0)}$. We say that E appears in f_2 if its center in f_2 is a divisor.*

This defines an equivalence relation $(E_1, f_1) \sim (E_2, f_2)$ on the set of pairs of exceptional divisors and resolutions if E_2 is the center of E_1 in f_2. Each equivalence class corresponds to a divisorial valuation on $k(X)$ and is called exceptional divisor over X.

Definition 3.1.3. *A resolution of singularities is called divisorial if the exceptional locus is of pure codimension 1.*

Example 3.1.4. A $\mathbb{Q}$-factorial variety is a normal variety such that an integer multiple of each Weil divisor is Cartier. For a $\mathbb{Q}$-factorial variety X every resolution of singularities is divisorial.

Definition 3.1.5. *An essential divisor of X is an exceptional divisor E over X whose center in each resolution of singularities of X is an irreducible component of the exceptional locus. E is divisorially essential if its center in every divisorial resolution is a divisor.*

For any resolution $f : Y \to X$ there is a bijection between the set of the essential divisors over X and the set of the essential components of f (that is, the components of the exceptional locus whose center is an irreducible component of the exceptional locus of any resolution), which are centers of essential divisors.

With each good resolution of singularities one associates the dual complex of the resolution. It is a triangulated space that bears important topological information about the singularity ([Ste]). A good resolution is one whose exceptional set is a divisor with simple normal crossings.

Definition 3.1.6. *The dual complex, associated with a good resolution of singularities $f : X' \to X$ is the incidence complex Δ of the exceptional locus Z of f. The vertices Δ_i of Δ correspond to the divisors E_i in the exceptional locus $Z = \sum a_i E_i$. The edges $\Delta_{ij}^{(k)}$ connecting Δ_i and Δ_j correspond to the irreducible components of of the intersection $E_i \cap E_j$, etc. In the case of* $\dim X = 2$ *one has the dual graph of the resolution.*

In the following chapters we will work with some classes of singularities which we define now.

Definition 3.1.7. *Let X be a normal variety over a field of characteristic 0. X has rational singularities if for some (and then for any) resolution of singularities $f : \tilde{X} \to X$ one has $R^n f_*(\mathcal{O}_{\tilde{X}}) = 0$ for any $n > 0$.*

In the case of normal surface singularities it is enough to check $R^1 f_*(\mathcal{O}_{\tilde{X}}) = 0$ only. There are some important subclasses of this class.

Definition 3.1.8. *A sandwiched singularity is any normal 2-dimensional singularity $(X, 0)$ such that there is a proper birational morphism $X \to Y$, where Y is a nonsingular surface.*

If $f : X' \to X$ is a resolution of singularities for normal surface X and $\{E_i\}_{i \in I}$ are the exceptional components, among all divisors $Z = \sum_i a_i E_i$ with $a_i > 0$ for all i such that $Z.E_i \leq 0 \ \forall i$, there is the minimal one (w.r.t. the coefficient a_i for any i) ([Art66]). It is called the fundamental cycle of X.

Definition 3.1.9. *A sandwiched singularity is called minimal if its fundamental cycle Z is reduced, i.e. $Z = \sum E_i$ with E_i the irreducible components of the exceptional locus of the minimal resolution.*

The classes defined above for surfaces satisfy ([Spi90]):
$\{\text{rational}\} \supsetneqq \{\text{sandwiched}\} \supsetneqq \{\text{minimal}\} \supsetneqq \{\text{cyclic quotient}\}$.

3.2 Nash theorem and Nash problem

The following is an easy lemma that will be used to define the Nash map and to prove its injectivity ([IK03]).

Lemma 3.2.1. *Let $f : \tilde{X} \to X$ be a resolution of singularities. For each Nash component C_i over X and any $\alpha \in C_i$ with $\alpha(\eta) \notin \mathrm{Sing}(X)$, α could be lifted uniquely to an arc $\alpha' \in \tilde{X}_\infty$.*

Proof. We apply the valuative criterion of properness to the diagram (where $K \supset k$):

$$\begin{array}{ccc} \mathrm{Spec}\, K((t)) & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow f \\ \mathrm{Spec}\, K[[t]] & \xrightarrow{\alpha} & X. \end{array}$$

in which f is a proper morphism which is isomorphism outside $\mathrm{Sing}(X)$, and $\alpha(\eta) \notin \mathrm{Sing}(X)$. Then α lifts to a unique arc $\alpha' \in Y_\infty$, such that $f \circ \alpha' = \alpha$.

Now we can formulate an important result obtained in 1968 and published in 1995 ([Nas95]).

Theorem 3.2.2. (Nash) *There is a map, associating each Nash component with an essential divisor over X. It is always an injective map. In particular, the set of good components is finite.*

Proof. Given a resolution $f : Y \to X$, take the corresponding commutative diagram

$$\begin{array}{ccc} Y_\infty & \xrightarrow{f_\infty} & X_\infty \\ \downarrow \pi_Y & & \downarrow \pi_X \\ Y & \xrightarrow{f} & X, \end{array}$$

and let $\{E_i\}$ be the prime exceptional divisors of f. For any j, $\pi_Y^{-1}(E_j)$ is irreducible, because Y is nonsingular and E_j is irreducible. Let $M_j \subset \pi_Y^{-1}(E_j)$ be the open set of all arcs $\gamma \in Y_\infty$ such that $\gamma(\eta) \notin \cup_i E_i$. Restricting f_∞ we obtain a map $F_\infty : \cup_i M_i \to \cup_i C_i^\circ$, where $C_i^\circ := \{\alpha \in C_i | \ \alpha(\eta) \notin \mathrm{Sing}(X)\}$.

Any $\alpha_x \in C_i^\circ$ defines a morphism $\mathrm{Spec}\, K[[t]] \to X$ with $K \supset k$ the residue field at the corresponding point $\alpha_x \in X_\infty$. By Lem. 3.0.6 it can be lifted to an arc $\alpha' : \mathrm{Spec}\, K[[t]] \to Y$. Thus $F_\infty(\alpha') = \alpha$, so F_∞ is surjective. Then for any j there is a unique $i(j)$ such that the generic point $\xi_{i(j)}$ of $M_{i(j)}$ is mapped to the generic point ζ_j of C_j°. Let α'_j be the lifting of the arc α_j corresponding to ζ_j, and let $\beta_{i(j)} \in Y_\infty$ be the the arc corresponding to $\xi_{i(j)}$.

Claim: $\alpha'_j = \beta_{i(j)}$. Indeed, if the residue field of α'_j is K and the residue field of $\beta_{i(j)}$ is L, then $K \hookrightarrow L$, which gives a morphism $g : \mathrm{Spec}\, L[[t]] \to \mathrm{Spec}\, K[[t]]$. Then $\beta_{i(j)} = \alpha'_j \circ g$. Thus $K = L$, so $\alpha'_j = \beta_{i(j)}$ and $\beta_{i(j)}(0) = \pi_Y(\xi_{i(j)})$ is the generic point on $E_{i(j)}$.

By Lem. 3.2.1 there is a map $\psi : \cup_i C_i^\circ \to \cup_j E_j$ sending each α to $\alpha'(0)$ (here E_j are the exceptional components). Then $\psi(\alpha_i) = \alpha'_i(0) = \beta_{i(j)}(0)$. This shows that $\psi(\beta_j)$ is the generic point of an exceptional component $E' := E_{i(j)} \subset Y$.

In fact, this is an essential component. Let $E' \subset Y$ correspond to a Nash component $C \subset \pi_X^{-1}(\mathrm{Sing}(X))$. Let $\tilde{Y}$ be the normalization of $\mathrm{Bl}_E(Y)$, and $\tilde{E} \subset \tilde{Y}$ be the prime exceptional divisor dominating E. Then $\tilde{E}$ corresponds to C by the argument in the proof of Lem. 3.2.1. Take another resolution of singularities $f' : Y' \to X$ and let E' be the irreducible component corresponding to $\mathcal{C}$. This means that E' is the center of $\tilde{E}$ on Y', so E is an essential component.

This defines a map from the set of Nash components to the set of essential divisors over X, sending to each good component C_i a unique essential divisor $E_{i(j)}$. To finish the proof we need to show that this map is injective. Suppose $j_1 \neq j_2$, but $E_{i(j_1)} = E_{i(j_2)}$. Then they will have equal generic points, so their images under F_∞ will be equal as well, which gives two different components over X with equal generic points - a contradiction.

Definition 3.2.3. *The map above from the set of good components to the set of essential divisors over X is called the Nash map, noted in the sequel by $\mathcal{N}_{\mathrm{X}}$.*

In the same paper ([Nas95]) Nash asked the following as a question, which became later a conjecture:

Nash conjecture. The map $\mathcal{N}_{\mathrm{X}}$ is always bijective.

There is a counterexample of an affine hypersurface of dimension 4 over a field of characteristic different from 2 and 3, which has one Nash component over its singular locus but two essential divisors ([IK03]). The conjecture remains open in dimensions 2 and 3 in general, although for some classes of surfaces and higher dimensional varieties a positive answer has been obtained (Chapters 5 and 6).

Here is the idea of the counterexample. Let $x \in \mathrm{Sing}(X)$, $h : Z \to X$ be proper birational morphism and partial resolution (i.e. h is an isomorphism over $X \setminus \{x\}$), and let $F \subset Z$ be the exceptional divisor of h. Further, assume that $z \in \mathrm{Sing}(Z)$ with $z \in F$ having the property that any arc $\gamma : \mathrm{Spec}\, k[[t]] \to Z$ at z is contained in a wedge at z.

Definition 3.2.4. *For arbitrary variety X and a field extension $K \supset k$ a morphism $\Gamma : \mathrm{Spec}\, K[[s,t]] \to X$ is called a K-wedge on X.*

Suppose that there exists an essential divisor E over X whose center on Z is z (the center could be a point because the resolution is partial). Then all arcs in Z_∞ that could be lifted to E are arcs at z, and if such an arc is contained in a wedge Γ at z, one could move it to Z_∞ so that its closed point moves along $\Gamma^{-1}(F) \subset \mathrm{Spec}\, k[[s,t]]$. Then any arc that could be lifted to E would be a limit of arcs over some component of F. It follows that E does not correspond to an irreducible component of $\pi^{-1}(\mathrm{Sing}(X))$, and if one could pick up E to be essential, this gives a counter example to the Nash conjecture.

For the following lemma we note that a smooth formal curve at $z \in Z$ is given by a surjective homomorphism $\widehat{\mathcal{O}}_{(Z,z)} \to k[[t]]$, where the completion is at the maximal ideal at z. Similarly, a smooth surface germ is given by a surjection $\widehat{\mathcal{O}}_{(Z,z)} \to k[[s,t]]$. The corresponding linear map $\mathfrak{m}/\mathfrak{m}^2 \to (t)/(t)^2$ defines a point on the exceptional divisor of $\mathrm{Bl}_z\, Z$, and the second linear map $\mathfrak{m}/\mathfrak{m}^2 \to (s,t)/(s,t)^2$ defines a line on that divisor.

Lemma 3.2.5. *Let $Z \subset \mathbb{A}^n$ be a hypersurface with singularity at 0, defined by $F = F_s + F_{s+1} + \cdots = 0$, where F_i is the i-th homogeneous part of F.*

Define $Z' := (F_s = 0) \subset \mathbb{P}^{n-1}$, and suppose $z \in Z'$ is such that $z \in L \subset Z'$, where L is a line, Z' is smooth along it and $H^1(L, N_{L|Z'}) = 0$. If ϕ is a smooth formal curve at 0 with tangent line defined by z, then it could be extended to a smooth surface germ at z with tangent space defined by (the points on) L.

Theorem 3.2.6. *Let $V \subset \mathbb{P}^{n-1}$ be a hypersurface, such that*
i) V is not ruled;
ii) through a general point there is a line $L \subset V$ with $H^1(L, N_{L|V} = 0$.
Suppose that X is variety such that:
1) $0 \in \operatorname{Sing}(X)$, $p : \tilde{X} \to X$ is a partial resolution, and $x \in p^{-1}(0)$;
2) $x \in \tilde{X}$ is a hypersurface singularity with a tangent cone whose projectivization is isomorphic to V;
3) $p^{-1}(0)$ is a Cartier divisor.
Then in $Bl_x(\tilde{X})$ there is an essential divisor over $p^{-1}(0)$ which does not correspond to any component over X.

We skip the proofs of both the lemma and the theorem, which are too technical. We would apply them to create a counter example ([IK03]).

Proposition 3.2.7. (Ishii - Kollar) *Let k be an algebraically closed field with $\operatorname{char}(k) \neq 2, 3$, and let $X \subset \mathbb{A}^5$ be defined by*

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^6 = 0.$$

Then there is one irreducible component and two essential divisors over X.

Proof. First, note that $\pi^{-1}(\operatorname{Sing} X) \subset X_\infty$ is isomorphic to X_∞. This could be seen using calculations with power series defined by the arcs, similar to those shown at the end of Ch. 2. Then by Kolchin's theorem one concludes that X has only one Nash component.

Take $\tilde{X} := Bl_0(X)$ and apply Thm. 3.0.10. Then $E := p^{-1}(0)$ is Cartier divisor and $\tilde{X}$ has an isolated singularity which is a cone over $V := (y_1^3 + \cdots + y_5^3 = 0) \subset \mathbb{P}^4$. Indeed, take $x_i y_j = x_j y_i, i, j = 1, \ldots 5$ defining $Bl_0(X)$. For $y_5 = 1$ the equation becomes $x_5^3 y_1^3 + \ldots + x_5^3 y_4^3 + x_5^6$, giving the cone above. Also, by ([Kol96]) every line L on a smooth cubic surface V satisfies $H^1(L, N_{L|V}) = 0$.

Moreover, V is not birationally ruled. This holds because a smooth cubic in $\mathbb{P}^4$ is not birational to $\mathbb{P}^3$ over a field of $\operatorname{char}(k) \neq 2)$ ([Mur73]). Then V is not ruled, because if it is birational to $S \times \mathbb{P}^1$, there would be a degree 2 rational map $\mathbb{P}^3 \dashrightarrow V$ ([Kol96]). Having char(k) $\neq 2$, the composition $\mathbb{P}^3 \dashrightarrow S \times \mathbb{P}^1 \to S$ gives a dominant separable map, defined out of a codimension 2 subset. By Castelnuovo-Enriques criterium for rationality S is rational iff $H^0(S, \Omega_S^2) = H^0(S, \Omega_S^1) = 0$, which holds in this case. But then V would be birationally ruled - a contradiction.

Now take a blow up of $\tilde{X}$ at its singular point. It is easy to see that it is smooth, giving a resolution of singularities for X with two exceptional divisors, one of them isomorphic to V, and the other is E', the strict transform of E. E' is essential because by the Nash theorem the irreducible component over X corresponds to one essential divisor, and this could be only E'. The other divisor is essential as well, because of the fact ([Abh56], Rem. 4.4) that if E is an exceptional divisor of birational morphism $X \to Y$ with Y smooth, then E is birationally ruled. As noted in ([Nas95], Sec. Essential components), this implies that any nonruled exceptional divisor of a resolution is an essential component. This holds for V, so it is an essential divisor as well. This proves that the example above is indeed a counter example to the Nash conjecture.

Remark 3.2.8. This example could be easily generalized any dimension greater than or equal to 4 (just take $X \times Z$ for any Z which is not uniruled), but can not be used to create a counter example in dimensions 2 and 3 ([IK03], Rem. 4.4). It appears useful to know for which classes of algebraic varieties the Nash conjecture holds, so the following problem quite naturally arises:

Nash problem. For which X is the map $\mathcal{N}_X$ bijective?

The next three chapters contain all results (known to the author), which give the answer for some particular classes of varieties. In Ch. 4 will be seen that it could be helpful in some cases to work in the frame of the Nash problem for pairs ([Pet09].

Let (X, Y) be a pair of an algebraic variety X and a proper closed subset $Y \supset \mathrm{Sing}(X)$.

Definition 3.2.9. *A proper birational morphism $f : X' \to X$ with X' smooth such that $f|_{X' \setminus f^{-1}(Y)}$ is an isomorphism on $X \setminus Y$, and $f^{-1}(Y)$ of pure codimension 1, will be called a Y-resolution of X. The birational class of any prime divisor on X' with center appearing in any Y-resolution of X as a divisor will be called a Y-essential divisor over X, or an essential divisor on (X, Y).*

Definition 3.2.10. *A good, or Nash, component for the pair* (X, Y) *is an irreducible component of* $\pi_X^{-1}(Y)$ *that contains an arc* α *such that* $\alpha(\eta) \notin Y$.

In the absolute case, i.e. when $Y = \mathrm{Sing}(X)$, the essential and divisorially essential divisors are different notions (although no explicit example is known). Our definition is the log analogue of the latter case. In [IK03] it is shown that for toric varieties the two notions coincide. In the case of toric pairs we obtain a similar result in Chapter 4.

Let C be a good component for (X, Y), let $\alpha \in C$ be an arc such that $\alpha(\eta) \notin Y$, and let $f' : X' \to X$ be any Y-resolution of singularities. By the valuative criterion of properness, α lifts to a unique arc $\alpha' \in X'_\infty$, such that $f \circ \alpha' = \alpha$. Moreover, if α is the generic point of a component C, $\alpha'(0)$ is the generic point on some Y-exceptional prime divisor E' on X'.

Theorem 3.2.11. *For any pair* (X, Y) *and any* Y*-resolution* $X' \to X$ *there is a map*

$$\mathcal{N}_{(X,Y)} : \{\textit{good components of } (X, Y)\} \to \{\textit{essential divisors of } (X, Y)\},$$

and it is injective.

Proof (Sketch of the proof)**.** The proof is the same as the proof of Thm. 3.2.2. For any Y-resolution $f' : X' \to X$ and for any good component C of (X, Y), let z be its generic point. By the remark above there is a component of $(f' \circ \pi_{X'})^{-1}(Y)$, i.e. an irreducible component of $\pi_{X'}^{-1}(E')$ for some Y-exceptional component $E' \subset X'$, whose generic point is sent to z by f_∞. To show that it is also an essential component, one takes another Y-resolution $f'' : X'' \to X$. Then E' appears in f'', so by Def. 3.2.9 it is Y-essential. This defines the map $\mathcal{N}_{(X,Y)}$. It is an injective map because if C' is another good component with z' its generic point, then the lifts of both in X' will be the generic point of the same Y-essential divisor. Taking $f'_\infty(z')$ gives a generic point of the same good component, which contradicts the choice of z, z'.

We will call again $\mathcal{N}_{(X,Y)}$ the Nash map. The following question arises naturally in this new context.
Nash problem for pairs. For which pairs (X, Y) is the map $\mathcal{N}_{(X,Y)}$ bijective?

Later in Ch. 4 will be proved that for a toric variety X and a proper $\mathbb{T}$-invariant closed subset $Y \supset \mathrm{Sing}(X)$, the answer of this problem is positive. Such a pair (X, Y) will be called in the the following a toric pair. It is an example when the Nash problem for pairs appears naturally.

Chapter 4

Nash Problem - Toric Varieties

In this chapter is proved a theorem of Ishii, Kollár ([IK03]) claiming the positive answer to Nash problem in the case of toric varieties. Then is proved a result of Ishii ([Ish04]) giving answer to the embedded Nash problem ([ELM04]) for toric varieties. Finally is obtained the solution of the Nash problem for pairs ([Pet09]) in the case of toric pairs, which gives a positive answer to the Nash problem in the case of STVs.

4.1 Toric varieties

Remind some standard definitions and notations from toric geometry.

All cones are strongly convex rational polyhedral cones in some $\mathbb{R}^n = N_{\mathbb{R}}$ where $N \simeq \mathbb{Z}^n$ is the lattice. Each σ defines an affine toric variety X with natural action of the torus $\mathbb{T} := k^{*n}$. The cone σ is called regular (nonsingular) if the generating lattice vectors of its rays are part of some basis for the lattice, otherwise it is singular. The dual cone $\check{\sigma}$ is in the dual space $M_{\mathbb{R}}$, with M the dual lattice of N. Its lattice points $\check{\sigma} \cap N$ correspond to monomials generating the ring of regular functions on X. Each face $\tau \prec \sigma$ defines an orbit $\mathrm{orb}(\tau) \subset X$ under the action of $\mathbb{T}$. Its closure is a $\mathbb{T}$-invariant subvariety of codimension equal to $\dim(\tau)$. In particular, the rays correspond to $\mathbb{T}$-invariant divisors. By τ° is denoted the interior of the cone τ.

Let X be an affine toric variety defined by a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$. We will prove that the Nash map $\mathcal{N}_X$ is bijective, which answers the Nash problem for arbitrary toric variety.

We will need some definitions.

Definition 4.1.1. *For any $a, b \in N \cap \sigma$ define a partial order $a \leq b$ iff $b \in a + \sigma$. This is equivalent to $\langle a, u \rangle \leq \langle b, u \rangle$ for all $u \in \check{\sigma}$.*

Definition 4.1.2. *Let W be the set of the minimal elements in $\bigcup_{\tau\ singular}(\tau^\circ \cap N)$.*

Definition 4.1.3. *For a toric variety defined by a fan Σ, and v a primitive vector on a ray $\rho \in \Sigma$, define $D_v := \overline{\mathrm{orb}(\rho)}$ to be the divisor corresponding to v.*

An arc α with $\alpha(\eta) \in \mathbb{T}$ defines a homomorphism $\alpha^* : \operatorname{Spec} k[M \cap \check{\sigma}] \to K[[t]]$, giving a unique homomorphism $\alpha^* : k[M] \to K((t))$ for some field extension $K \supset k$. Thus one has a group homomorphism $M \to \mathbb{Z}$ by $u \mapsto \mathrm{ord}_t(\alpha^*(x^u))$. Any such homomorphism is determined by some $v \in N$ such that $\langle v, u \rangle = \mathrm{ord}_t(\alpha^*(x^u))$ for all $u \in M$. Denote it by v_α. One has $\langle v_\alpha, u \rangle \geq 0$ for all $u \in M \cup \check{\sigma}$, so $v_\alpha \in N \cup \sigma$.

The main result in this chapter is the following theorem([IK03]):

Theorem 4.1.4. (Ishii-Kollár) *For X an affine toric variety the Nash map $\mathcal{N}_X$ is bijective.*

This result gives a positive answer to the Nash problem in the case of arbitrary toric variety. The idea of the proof is to define and prove the injectivity of some maps on finite sets shown in the diagram below. Then one proves that their composition equals identity, so that each map in the composition is bijection. The proofs of all propositions and lemmas follow with minor changes ([IK03], Sec. 3).

$$\begin{array}{ccccc} & \mathrm{W} & \xrightarrow{F_1} & \left\{ \begin{array}{c} \text{the components of} \\ \pi^{-1}(\mathrm{Sing}(X)) \end{array} \right\} \\ & & & \downarrow \mathcal{N}_X \\ & F_2 \uparrow & & \left\{ \begin{array}{c} \text{essential divisors} \\ \text{over } X \end{array} \right\} \\ & & & \cap \\ \left\{ \begin{array}{c} \text{toric divisorially essential} \\ \text{divisors over X} \end{array} \right\} & & \supset & \left\{ \begin{array}{c} \text{divisorially essential} \\ \text{divisors over } X \end{array} \right\} \end{array}$$

In this section X is affine toric variety defined by a cone σ, and τ is a face of σ.

Proposition 4.1.5. *For any $v \in W$ there is an arc $\alpha \in X_\infty$ with $\alpha(0) \in \bigcup_{\tau singular} \mathrm{orb}(\tau)$, $\alpha(\eta) \in \mathbb{T}$, such that $v = v_\alpha$.*

Lemma 4.1.6. *1) For $\alpha \in X_\infty$ with $\alpha(\eta) \in \mathbb{T}$, one has $\alpha(0) \in \operatorname{orb}(\tau)$ iff $v_\alpha \in \tau^\circ$.*
2) For any subdivision Σ of σ defining a toric morphism $f : X' \to X$, any arc α with $\alpha(\eta) \in \mathbb{T}$ could be lifted to an arc $\alpha' \in X'_\infty$.
3) For $\tau \in \Sigma$, $\alpha'(0) \in \operatorname{orb}(\tau)$ iff $v_\alpha = v_{\alpha'} \in \tau^\circ$.

Proof. First is proved the lemma. The claim 2) holds by the proof of Lem. 3.2.1, and 3) follows from 1) taking $U_\tau = \bigcup_{\tau' \prec \tau} \operatorname{orb}(\tau')$ in the place of X. To prove 1) note that $\tau^\circ = \tau \setminus \bigcup_{\tau' \precneqq \tau} \tau'$ and $\operatorname{orb}(\tau) = U_\tau \setminus \bigcup_{\tau' \precneqq \tau} U_{\tau'}$. So it is enough to prove that $v_\alpha \in \tau$ iff $\alpha(0) \in U_\tau$. But $v_\alpha \in \tau$ is equivalent to $\langle v_\alpha, u\rangle \geq 0$ for any $u \in M \cap \check{\sigma}$. The latter holds iff $\alpha^* : k[M \cap \check{\sigma}] \to k[[t]]$ could be extended to $k[M \cap \check{\tau}]$, i.e. iff α factors through U_τ. Finally, note that $\alpha(\eta) \in \mathbb{T} \subset U_\tau$, so this holds iff $\alpha(0) \in U_\tau$.

Next is proved the proposition. Define a homomorphism $\alpha^* : k[M \cap \check{\sigma}] \to k[[t]]$ by $\alpha^*(x^u) := t^{\langle u,v\rangle}$ (one has $\langle u, v\rangle \geq 0$ for all $u \in M \cap \check{\sigma}$). Then extend it to $\alpha^* : k[M] \to k((t))$, and let α be the arc defined by the homomorphism α^*. Thus $\alpha(\eta) \in \mathbb{T}$ and $v = v_\alpha$. For $v \in W$ there is a singular face τ such that $v \in N \cap \tau^\circ$. Then by the lemma $\alpha(0) \in \operatorname{orb}(\tau) \subset W$.

For the next proposition we need an auxiliary lemma.

Lemma 4.1.7. *For a k-algebra A and a family of arcs on X, $\alpha : \widehat{\operatorname{Spec} A \times \operatorname{Spec}} k[[t]] \to X$, let $\alpha_c : \operatorname{Spec} k(c)[[t]] \to X$ be the arc induced by α for each $c \in \operatorname{Spec} A$, $k(c)$ being the residue field at c. Take $\alpha_c(\eta) \in \mathbb{T}$ for any $c \in \operatorname{Spec} A$. Then $\operatorname{Spec} A \to N \cup \sigma, c \mapsto v_{\alpha_c}$ is upper semi-continuous, i.e. the sets $\{c \in \operatorname{Spec} A : v_{\alpha_c} \leq v$ are open in $\operatorname{Spec} A$ for any $v \in N \cap \sigma$. In particular, for w minimal in W there is an open non-empty $U_w \subset \operatorname{Spec} A$ such that for any c in it, $v_{\alpha_c} = w$.*

Proof. Take $\alpha^* : k[M \cap \check{\sigma}] \to A[[t]]$ such that $\alpha^*(x^u) = a_0 + a_1 t + a_2 t^2 + \cdots,\ a_i = a_i^u \in A$. A point c is in U_w iff $\langle v_{\alpha_c}, u\rangle \leq \langle w, u\rangle$ for any $u \in M \cap \check{\sigma}$. Then for any u of the generating set for $M \cap \check{\sigma}$ there is an index i such that $a_i(c) = a_i^u(c) \neq 0$. So the set U_w is a finite union of complements of zero loci on $\operatorname{Spec} A$, thus is open.

Proposition 4.1.8. *For any $v \in W$ there is a good component C and (by Cor. 2.0.9) a family of arcs $\alpha : C \times \widehat{\operatorname{Spec}} k[[t]] \to X$ parametrized by C, such that on a non-empty open $U \subset C$ one has $v_{\alpha_c} = v$ for any $c \in U$. Here $\alpha_c : \operatorname{Spec} k(c)[[t]] \to X$ is the arc in the family α, indexed by c, with $k(c)$ the* *residue field at c. Define a map from W to the set of good components over X (see the diagram above) $F_1(v) := C$. Then F_1 is injective.*

Proof. By Prop. 2.1.5, for any $w \in W$ there is an arc $\alpha : \operatorname{Spec} k[[t]] \to X$ with $\alpha(0) \in \bigcup_{\tau singular} \operatorname{orb}(\tau), \alpha(\eta) \in \mathbb{T}$ and $v_\alpha = w$. There exists a good component C_i defining a family α_i (by Cor. 2.0.9) which contains α, i.e. for some k-point $z \in C, \alpha_{iz} = \alpha$. Then $\alpha_i(C_i \times \{0\}) \subset \bigcup_{\tau singular} \operatorname{orb}(\tau)$ and $\alpha_{ic}(\eta) \in \mathbb{T}$ for all c in some open subset $C' \subset C$. By Lem. 4.1.7 there is an open $C'' \subset C'$ such that $v_{\alpha c} = w$ for all $c \in C''$. The last statement follows because w is minimal, i.e. C is uniquely determined.

Lemma 4.1.9. $\mathcal{N}_X \circ F_1(v) = D_v$.

Proof. By Prop. 4.1.8 the generic point of $F_1(v)$ corresponds to an arc α with $v_\alpha = v$. Let $f : X' \to X$ be a toric divisorial resolution of X, with β a lifting of α on X'. Then $\mathcal{N}_X \circ F_1(v)$ will be an exceptional divisor with generic point $\beta(0)$. It corresponds to a ray τ in the subdivision of σ corresponding to f, so from Lem. 4.1.16 that $v = v_\alpha = v_\beta \in \tau^\circ$. Then this exceptional divisor will be D_v.

The remaining step for completing the proof of Thm. 4.1.4 is the following:

Proposition 4.1.10. *Define*
F_2 : {*toric divisorially essential divisors over* X} $\to \bigcup_{\tau singular}(N \cap \tau^\circ)$
by $F_2(D_v) := v$. *Then* F_2 *is injective and* $\operatorname{Im}(F_2) \subset W$.

Proof. First is proved that if a primitive $v \in W$ is not minimal there exists a divisorial resolution of X in which D_v is not a component of the exceptional locus. To see this, construct a regular subdivision of σ defining a resolution of singularities $f : X' \to X$ with $\mathbb{R}^{\geq 0} v$ not among the rays of the corresponding fan.

Because $v = a + b$ for some $a \in W, b \in N \cap \sigma \setminus \{0\}$, there are two cases:
1) $a, b \in W$;
2) $a \in W$, b is on a ray of σ.
Indeed, if $b \notin W$ then $b \in \tau$ for a non-singular face $\tau = \operatorname{Cone}(e_1, \dots e_r)$ of σ. Having $b = \sum_{i=1}^r c_i e_i$, take $c_1 \neq 0$, and ρ the minimal face containing both $a, \sum_{i=2}^r c_i e_i$. Then ρ is singular because $a \in \rho$ and $a \in \bigcup_{\tau singular} \tau^\circ$. Also, $a + \sum_{i=2}^r c_i e_i \in \rho^\circ \subset \bigcup_{\tau singular} \tau^\circ$. Replacing a by $a + \sum_{i=2}^r c_i e_i$ and b by $c_1 e_1$, 2) holds.

Take a minimal regular subdivision of Cone(a, b) (it always exists), and let Cone(a', b') be the 2-dimensional regular cone in it containing v in its interior. If could be constructed a regular subdivision of σ containing this cone, the claim is proved. At least one of a', b' is in W, say a'. Define the fan Σ obtained by σ by a star subdivision with center a'. For the cone a', b' again there are the cases 1) and 2). If 1) holds, take Σ' to be the star subdivision of Σ with center b'. If 2) holds and Σ is not simplicial, take γ to be a minimal dimensional cone in the fan that is not simplicial. Then pick up an integral point in its interior and take the star subdivision of Σ at it. As a result γ will be divided in simplicial cones, and continuing in this way one would have a simplicial subdivision Σ''. If it is not regular, take a cone in it Cone$(c_1, \dots c_s)$ of maximal multiplicity (i.e. maximal volume of the corresponding unit parallelotope P). By Minkowski's theorem, since $\mathrm{vol}(P) > 1$, there is $c \in P \cap N \setminus \{0\}$. Take the star subdivision of Σ'' with center c. Continuing the procedure one obtains a regular subdivision Σ'''. Moreover, all subdivisions above produce a divisor as exceptional set. Because in any subdivision above Cone(a, b) did not change, in case 1) Σ' and in case 2) Σ''' will contain it. All regular cones have not been changed in any subdivision as well. As D_v does not appear in any of these regular subdivisions, one has the resolution needed. The injectivity of F_2 is a direct consequence of its definition.

Next is proved Thm. 4.1.4. using the diagram of sets and maps drawn above.

Proof. By Prop. 4.1.8 F_1 is injective, by Thm. 3.0.22 $\mathcal{N}_X$ is injective, and by Prop. 4.1.10 F_2 is injective, so is their composition. By Lem. 4.1.9 and Prop. 4.1.10 it is id_W, which proves the Nash map is indeed bijective.

From the theorem and its proof the following holds:

Corollary 4.1.11. *Over a toric variety X*
1) Every toric divisorially essential divisor is essential.
2) The number of the essential divisors and the number of good components over X are both equal to $|W|$.

This gives a positive answer to the Nash problem in the case of toric varieties.

4.2 Embedded Nash problem

The main result of this section was obtained by Ishii ([Ish04]) on the embedded Nash problem, proposed by Ein, Lazarsfeld and Mustata ([ELM04]). It gives an answer to it in the case of toric varieties.

Let X be an affine variety, and $\mathfrak{p} \subset k[X]$ an ideal.

Definition 4.2.1. *The n-th contact locus of $\mathfrak{p}$ is*
$\mathrm{Cont}^{\mathrm{n}}(\mathfrak{p}) := \{\alpha \in X_\infty : \ \min_{f\in\mathfrak{p}} \mathrm{ord}_t(\alpha^*(f)) = n\}$.

This set is a cylinder set, i.e. a pre-image by some ψ_m of a locally closed subset of X_m (one could take any $m \geq n$). Then it is a union of irreducible components $\mathrm{Cont}^{\mathrm{n}}(\mathfrak{p}) = \bigcup \mathcal{C}_i$. When X is smooth each of them is a cylinder set as well. Indeed, for a smooth affine variety X, let $\mathrm{Cont}^{\mathrm{n}}(\mathfrak{p})_m := \{\alpha \in X_m : \ \min_{f\in\mathfrak{p}} \mathrm{ord}_t(\alpha^*(f)) = n\}$ for any $m \geq n$. This is a closed subset of a smooth affine variety and has the same number of irreducible components for any m, because for $m' \geq m''$ the jets in $X_{m'}$ are obtained by truncations from the jets in $X_{m''}$. This holds for $m'' = \infty$ as well.

In the case of a singular variety X this does not hold in general, i.e. there could be components of $\mathrm{Cont}^{\mathrm{n}}(\mathfrak{p})$ that are not cylinders.

Definition 4.2.2. *A divisorial valuation on $k(X)$ is $n.\mathrm{val}_D$ where D is a divisor on some normal X' birationally equivalent to X, and $n \in \mathbb{N}$.*

In ([ELM04]) is shown that for any smooth X over $\mathbb{C}$ an irreducible cylinder $\mathcal{C} \subset X_\infty$ such that $\overline{\pi(\mathcal{C})} \neq X$ defines a divisorial valuation. It could be defined first as $\mathrm{ord}\,\alpha^*(f)$ for general $\mathbb{C}$-valued $\alpha \in \mathcal{C}$ and any rational function f in a neighborhood of $\pi(\alpha)$ whose domain intersects $\pi(\mathcal{C})$, and then to be extended in a natural way on the whole $k(X)$.

This means that in the case of smooth X each component $\mathcal{C}_i$ defines a divisorial valuation, which is not true in general for singular X. The next question then appears naturally for any affine variety X ([ELM04]):
Embedded Nash problem. Which divisorial valuations correspond to ireducible components of $\mathrm{Cont}^{\mathrm{n}}(\mathfrak{p})$?

Let X be a toric variety and $\mathfrak{a}$ a $\mathbb{T}$-invariant ideal in $k[X]$. Here is given the solution of the embedded Nash problem proposed by Ishii in this case ([Ish04]). With no loss of generality one could take X to be affine toric variety defined by a cone σ.

Remark 4.2.3. Since $\mathbb{T}$ acts on X, it induces an action of T_∞ on X_∞, and using the claim in the proof of Prop. 2.0.14 one has that $T_\infty \subset X_\infty$ as an open orbit. By ([Ish04], Cor. 4.4) the orbits $\mathbb{T}_\infty.\alpha \subset X_\infty$ are in bijection with the points $v \in N \cap \sigma$. Such an orbit corresponding to $v \in N \cap \sigma$ is denoted by $T_\infty(v)$.

Next proposition with proof skipped, explains how the orbits are related in terms of the corresponding vectors v ([Ish04], Prop. 4.8).

Proposition 4.2.4. *For any two orbits contained in $X_\infty(0)$, $T_\infty(v_1) \subset \overline{T_\infty(v_2)}$ iff $v_2 \leq_\sigma v_1$.*

Remind that if τ is a face of σ, $X(\tau) := \overline{\mathrm{orb}(\tau)}$.

Definition 4.2.5. *For $\tau \prec \sigma$,*
$X_\infty(\tau) := \{\alpha \in X_\infty : \ \alpha \textit{ factors through } X(\tau), \textit{but does not factor through } X(\rho) \ for \rho \not\prec \tau\} = \{\alpha \in X_\infty : \alpha(\eta) \in \mathrm{orb}(\tau)\}$.

Theorem 4.2.6. *With the assumptions made, any irreducible component of* $\mathrm{Cont^n}(\mathfrak{a})$ *is equal to* $\overline{T_\infty(v)}$ *for some* $v \in V_n := \{v' \in \sigma \cap N : \ \min_{x^u \in \mathfrak{a}} \langle u, v\rangle = n\}$, *such that* v *is minimal w.r.t. the partial order* $\leq_\sigma$. *This defines a bijection between the minimal elements in* V_n *and the set of irreducible components of* $\mathrm{Cont^n}(\mathfrak{a})$.

In particular, it follows in the case of arbitrary toric variety that all components of $\mathrm{Cont^n}(\mathfrak{p})$ correspond to some divisorial valuations.

Proof. The proof is based on the next two lemmas.

Lemma 4.2.7. *For any $n \in \mathbb{N} \setminus \{0\}$ and any $v \in N \cap \sigma$, either $T_\infty(v) \subset$* $\mathrm{Cont^n}(\mathfrak{n})$, *or* $T(v) \cap \mathrm{Cont^n}(\mathfrak{n}) = \emptyset$.

Proof. For any $\alpha \in T_\infty(v)$, $\alpha \in \mathrm{Cont^n}(\mathfrak{a})$ iff $n = \min_{x^u \in \mathfrak{a}} \mathrm{ord}_t\, \alpha^*(x^u)$. For any cone $\tau \prec \sigma$ such that $T_\infty(v) \subset X_\infty(\tau)$ and $u \notin \tau^\perp$, define $\langle v, u\rangle := \infty$. Then $\min_{x^u \in \mathfrak{a}} \mathrm{ord}_t\, \alpha^*(x^u) = \min_{x^u \in \mathfrak{a}} \langle v, u\rangle$, so the claim holds.

Lemma 4.2.8. *If $T_\infty(v) \subset$ $\mathrm{Cont^n}(\mathfrak{a})$ and $T_\infty(v) \subset X_\infty(\tau)$, for $\tau \neq \{0\}$ there exists $v' \in N \cap \check{\sigma}$ such that $T_\infty(v') \subset X_\infty(0)$, $T_\infty(v') \subset$ $\mathrm{Cont^n}(\mathfrak{a})$, and $T_\infty(v) \subset \overline{T_\infty(v')}$.*

Proof. There is a point $v' \in N \cap \sigma$ with the same image as v under the projection
$N_{\mathbb{R}} \to N_{\mathbb{R}}/\mathbb{R}.\tau$, because v belongs to the image of $N \cap \sigma$. Then for all $u \in \check{\sigma} \cap \tau^{\perp}$ one has $\langle u, v \rangle = \langle u, v' \rangle$. Put as above $\langle u, v \rangle = \infty$ if $u \in \check{\sigma} \setminus \tau^{\perp}$. Take $v'' \in N \cap \tau^{\circ}$, so that for any $m \geq n$ and $u \in N \cap \check{\sigma} \setminus \tau^{\perp}$ one has $\langle u, mv'' \rangle \geq n$. Take $w := v' + mv''$. For any $u \in \tau^{\perp} \cap \check{\sigma} \cap M$, $\langle u, w \rangle = \langle u, v' \rangle = \langle u, v \rangle$, while for any $u \in (\check{\sigma} \setminus \tau^{\perp}) \cap M$, $\langle u, w \rangle > n$. Thus $\min_{x^u \in \mathfrak{a}} \langle u, v \rangle = \min_{x^u \in \mathfrak{a}} \langle u, w \rangle = n$, and $T_{\infty}(w) \subset \mathrm{Cont}^{\mathrm{n}}(\mathfrak{a})$. Moreover, under the projection the image of w is v, so $\overline{T_{\infty}(w)} \supset T_{\infty}(v)$ (see [Ish04], Thm. 4.11).

By Lem. 4.1.18 $\mathrm{Cont}^{\mathrm{n}}(\mathfrak{a})$ is an union of orbits. By Lem. 4.1.19 each component of $\mathrm{Cont}^{\mathrm{n}}(\mathfrak{a})$ is the closure of an orbit corresponding to some $v \in N \cap \sigma$, such that $\min_{x^u \in \mathfrak{a}} \langle u, v \rangle = n$. By Prop. 4.1.15 and the fact that $T_{\infty}(v) \subset X_{\infty}(0)$ defines a divisorial valuation ([Ish04], Prop. 5.7), the proof of the theorem is completed.

4.3 Stable toric varieties

For more details about the definitions and results appearing in this section, see [Ale02].

Definition 4.3.1. *A connected algebraic variety X over k (not necessarily irreducible) with an action by a torus $\mathbb{T}$ on X is called stable toric variety, or STV, if it satisfies the following conditions:*
i) X is seminormal;
ii) there are only finitely many orbits by the action;
iii) for each $x \in X$ the stabilizer $T_x \subset \mathbb{T}$ is a subtorus.

The stable toric varieties are analogs of stable curves in the case of toric varieties. They appear for example when working with degenerations of abelian varieties. Here is given briefly an idea for their classification. By [Ale02] each affine STV X defines a face-fitting complex of cones Σ with a reference map to $\Lambda_{\mathbb{R}}$, where $\mathbb{Z}^n \cong \Lambda \subset \mathbb{R}^n$ is a lattice. This means that there exists a connected topological space $|\Sigma| = \cup \sigma_i$ and a finite-to-one map $\rho : |\Sigma| \to \Lambda_{\mathbb{R}}$, identifying each σ_i with a lattice cone. Since Σ is a face-fitting complex the minimal faces of all σ_i are equal to the same linear subspace $F_{\min} \subset \Lambda_{\mathbb{R}}$. Then every σ_i is a pre-image of a strictly convex cone in $\Lambda_{\mathbb{R}}/F_{\min}$. Moreover, X is a union of (ordinary) toric varieties X_{σ_i} glued in the way the complex Σ is glued from σ_i, that is $X_{\sigma_i} \cap X_{\sigma_j} = \cup_{\tau\subset\sigma_i\cap\sigma_j} X_\tau$. Any projective polarized STV is glued from affine STVs in the *é*tale topology.

Example 4.3.2. Let us take the complex of cones consisting of two cones in the plane, corresponding to the first and the third quadrants, with their faces. Then the corresponding STV will consist of two affine planes joined at the origin. If consider the complex of the first and the second quadrants, with their faces, the STV corresponding will be simply two planes intersecting along a line. Taking the complex of the first quadrant, with all its faces, and a ray from the origin in the third quadrant, the corresponding STV will be a plane and a line intersecting it transversally. The first two constructions give examples of equidimensional STVs.

Our first goal would be to prove the analogue of Thm. 4.1.4 in the case of pairs. We note that the analog of Prop. 2.0.15 holds for pairs, so that one has:

Lemma 4.3.3. *For a toric pair (X, Y) all components of $\pi^{-1}(Y)$ are good.*

Proof. If $f : X' \to X$ is an equivariant Y-resolution of X, the induced morphism $f_\infty : X'_\infty \to X_\infty$ is surjective. The reason is that for any arc $\alpha \in X_\infty$, $\alpha(\eta) \in \operatorname{orb}(\tau)$ for some τ in the fan defining X. Because f is equivariant, the pre-image of $\operatorname{orb}(\tau)$ contains a product $\operatorname{orb}(\tau) \times T'$ for some torus T' of dimension less than $\dim X$. Then the restriction of α on $k((t))$ lifts to X' which, by the valuative criterion of properness, gives lifting of α itself.

Next, suppose that an irreducible component C of $\pi_X^{-1}(Y)$ is not a good component for (X, Y). For any equivariant Y-resolution f as above with E_i the irreducible components of $f^{-1}(Y)$, $\pi_{X'}^{-1}(E_i)$ are the irreducible components of $(\pi_X \circ f_\infty)^{-1}(Y)$. By the same argument, there exists i such that $\pi_{X'}^{-1}(E_i)$ will be mapped to C. But the pre-image of E_i contains an arc which sends η outside Y, which contradicts the choice of C.

Next, we will prove the key result needed to answer the Nash problem in the case of STVs. It is the equivalent of Thm. 4.1.4 in the relative case, and its proof follows the same idea.

Theorem 4.3.4. *For X a toric variety over k and $Y \subset X$ a T-invariant proper closed subset containing* $\operatorname{Sing}(X)$*, the Nash map $\mathcal{N}_{(X,Y)}$ is bijective.*

First, some remarks. One wants to relate somehow the sets of Y-irreducible components and Y-essential divisors using the combinatorial data defining X. The question is local, so without loss of generality X could be taken to be an affine toric variety defined by a cone $\sigma \subset N_{\mathbb{R}}$. Next, denoting by $I_Y \subset k[X]$ the ideal corresponding to Y, one has $\pi^{-1}(Y) = \cup_{n\geq 1} \mathrm{Cont}^{\mathrm{n}}(I_Y)$. By Thm. 2.14, for O_1, O_2 the orbits in X_∞ corresponding to points $v_1, v_2 \in N \cap \sigma$, one has $\overline{O_1} \supset O_2$ iff $v_1 \leq_\sigma v_2$. Since Y is T-invariant, it corresponds to a finite union of faces $\tau_1, ..., \tau_s \subset \sigma$ such that $\mathrm{orb}(\tau_i) \subset Y$.

Define $W^{\geq 0} := N \cap (\cup_i \tau_i)$ and $W^0 := \{v \in N \cap (\cup_i \tau_i) : \exists\ x^u \in I_Y \text{ s.t. } \langle v, u \rangle = 0\}$. Then $W^{\geq 0} \backslash W^0$ will contain exactly the lattice points corresponding to the orbits in X_∞, contained in $\pi^{-1}(Y)$. But this set is actually $\cup_i(\tau_i^\circ \cap N)$, where τ_i° is the relative interior of the cone τ_i for each i. Indeed, if $I_Y = (x^u)$ is principal, the difference of sets is just $(N \cap (\cup_i \tau_i)) \setminus H_u$, where H_u is the hyperplane in $N_{\mathbb{R}}$ defined by u. In general, if $I_Y = (x^{u_1}, ..., x^{u_r})$, then $\cap_{j=1,...,r}(N \cap (\cup_i \tau_i)) \setminus H_{u_j}) = (N \cap (\cup_i \tau_i)) \setminus (\cup_j H_{u_j})$, which is in fact $W^{\geq 0} \setminus W^0$.

Another remark to make is about subdivisions of the cone σ. One needs regular subdivisions, that is subdivisions into regular cones corresponding to resolutions of singularities $f : X' \to X$ with $f^{-1}(Y)$ a divisor. If $\mathcal{F}$ is the map of fans corresponding to f, then for each ν in the set of cones defining Y, $\mathcal{F}^{-1}(\nu)$ is a union of cones having either a ray ρ with $\mathrm{orb}(\rho) \subset Y$, or a ray $\rho' \subset \mathcal{F}^{-1}(\nu \setminus \cup_{\rho \text{ ray in } \nu} \rho)$. In the latter case ρ' is not a ray of ν but of the fan obtained by the subdivision $\mathcal{F}$.

Definition 4.3.5. *Define*

$$W := \{v \in N \cap (\cup_i \tau_i^\circ) : v \textit{ is minimal w.r.t. } \leq_\sigma\} \subset N \cap \sigma.$$

Lemma 4.3.6. *There exists an injection F_1 from W to the set $\mathcal{C}$ of irreducible components of $\pi^{-1}(Y)$.*

Proof. For each element $w \in W$ there exists an arc α with $\alpha(0) \in Y$ and $\alpha(\eta) \in T$, defined by a ring homomorphism $\alpha^* : k[X] \to k[[t]]$ taking $\alpha^*(x^u) := t^{\langle w, u \rangle}$. That is, for $w \in \tau_i^\circ$, $\alpha(0) \in \mathrm{orb}\, \tau_i$, and α factors through $U_{\tau_i} = \mathrm{Spec}[\tau_i^* \cap M]$. There exists a face τ containing w such that $\alpha(0) \in \mathrm{orb}(\tau) \subset Y$. This holds because for the dual cone $\tau^* := \{u \in \check{\sigma} | \langle u, v \rangle = 0,\ \forall\ v \in \tau\}$ one has $\langle w, u \rangle \geq 0$ for all $u \in \tau^*$. Thus α^* extends to U_τ. Then α defines a point in a good component C of (X, Y). According to Prop. 4.1.8 there is a non-empty open subset $U \subset C$, such that for all

$\gamma \in U$ the corresponding lattice point will be w. Since Y is a T-invariant, $\pi^{-1}(Y)$ is T_∞-invariant, hence $T_\infty(w)$ is a dense orbit in C, i.e. $\overline{T_\infty(w)} = C$. So there is a well defined injective map $F_1 : W \to \mathcal{C}$.

Definition 4.3.7. *Define a toric Y-divisorially essential divisor to be a divisor which appears in each toric Y-resolution of (X, Y).*

The next lemma is a modification of Prop. 4.1.10, with some minor details in the proof skipped.

Lemma 4.3.8. *There exists a map*
$F_2 : \{$toric Y-divisorially essential divisors$\} \to W$,
defined by $F_2(D_v) := v$, and it is injective.

Proof. In any Y-resolution of X defined by a fan Σ, each exceptional divisor corresponds to a ray $\rho \in \Sigma$ which either subdivides some face τ among the faces corresponding to Y of σ, or coincides with it (i.e. $\rho = \tau$). It is defined either by some primitive vector $w \in \tau^\circ$, or by some primitive vector $w \in \rho^\circ$. The main fact to be proved is that if a primitive vector $w \in N \cap (\cup \tau_i^\circ)$ is not minimal, the divisor D_w defined by it does not appear in some Y-resolution. For this will be constructed a Y-regular subdivision Σ of σ, corresponding to a Y-resolution, in which the ray $\rho = \mathbb{R}^{\geq 0}.w$ does not appear.

Take such non-minimal w. There are $n_1 \in N \cap (\cup \tau_i^\circ), n_2 \in (N \cap \sigma) \setminus \{0\}$ such that $w = n_1 + n_2$. Then either

1. $n_1, n_2 \in N \cap (\cup \tau_i^\circ)$, or

2. $n_1 \in N \cap (\cup \tau_i^\circ)$, and n_2 could be taken on a ray of σ.

The reason is that if $n_2 \notin N \cap (\cup \tau_i^\circ)$ one has $n_2 \in \gamma$ for a non-singular face γ generated by primitive vectors $p_1, ...p_s$ (recall that $\mathrm{Sing}(X) \subset Y$, hence every singular face of σ appears among the τ_i's). Then for $n_2 = \sum_{i=1}^s b_i p_i$ there is, say, a non-zero coefficient before p_1. Let δ be the minimal face of σ containing n_1 and $\sum_{i=2}^s a_i p_i$. But $\mathrm{orb}(\delta) \subset Y$ because δ contains n_1 (so δ will be among the τ_i's), and $n_1 + \sum_{i=2}^s a_i p_i \in \delta^\circ$. Replacing n_1 by $n_1 + \sum_{i=2}^s a_i p_i$ and n_2 by $a_1 p_1$, (2) holds.

Take the minimal regular subdivision of $\mathrm{Cone}(n_1, n_2)$ (it exists for any 2-dimensional cone), and let $\mathrm{Cone}(u, v)$ be the cone in this subdivision containing w in its interior. At least one of u, v should be in $N \cap (\cup \tau_i^\circ)$. Indeed, if $n_1 \in \tau_1^\circ$ and $n_1 + n_2 = w \in \tau_2^\circ$ then τ_1 is a face of τ_2 and $\mathrm{Cone}(u, v) \subset \mathrm{Cone}(n_1, n_2)$.

If (1) holds, take the star subdivision Σ' of Σ with center u, and then take the star subdivision of Σ'' of Σ' with center v. The last one could be completed to a regular Y-subdivision, with the corresponding fan not containing the ray ρ.

If (2) holds, take the same Σ' as in (1). If Σ' is not simplicial, take cone μ of minimal dimension with a lattice vector in its interior, and the corresponding star subdivision of μ. Continue this way to obtain a simplicial subdivision Σ_1 with exceptional locus of pure codimension 1. If it is not regular, take a cone β with maximal multiplicity. The volume of the polytope $P = \sum a_j q_j$ generated by the primitive vectors q_j on the rays of β is bigger than 1. This polytope contains a non-zero lattice point m not on any of its edges, so one can take the star subdivision of β with center m. Its exceptional locus is a divisor and the volume of the corresponding polytopes will decrease or remain the same. Repeating this procedure for each cone with maximal multiplicity bigger than 1, will be obtained a regular subdivision Σ_2 containing a $\mathrm{Cone}(u, v)$ whose exceptional locus is of pure codimension 1. If necessary, subdivide by rays few more of the faces different from $\mathrm{Cone}(u, v)$ to obtain at the end a Y-regular subdivision Σ_3. This is possible since in both (1) and (2), the ray defined by one of u, v was used in the subdivision. Any of this subdivisions did not change $\mathrm{Cone}(u, v)$ which was regular by definition, so $\rho \notin \Sigma_3$. All subdivisions did not change the regular cones in Σ as well, and this defines the needed Y-resolution of X.

The injectivity of F_2 is obvious by its definition.

So there are maps:

$$\begin{array}{ccc}
\mathrm{W} & \xrightarrow{F_1} & \left\{ \begin{array}{c} \text{the components of} \\ \pi^{-1}(Y) \end{array} \right\} \\
 & & \downarrow \mathcal{N}_{(X,Y)} \\
F_2 \uparrow & & \left\{ \begin{array}{c} \text{Y-essential divisors} \\ \text{of } X \end{array} \right\} \\
 & & \cap \\
\left\{ \begin{array}{c} \text{toric Y-divisorially} \\ \text{essential divisors of X} \end{array} \right\} & \supset & \left\{ \begin{array}{c} \text{Y-divisorially essential} \\ \text{divisors of } X \end{array} \right\}
\end{array}$$

These maps satisfy the following

Lemma 4.3.9. $F_2 \circ \mathcal{N}_{(X,Y)} \circ F_1 = \mathrm{id}_W$.

Proof. The map $\mathcal{N} \circ F_1 : W \to \{Y-\text{essential divisors of } X\}$ maps a point w to D_w, because the generic point of $F_1(w)$ could be lifted by a toric divisorial resolution $f : X' \to X$ to an arc $\gamma \in X'_\infty$ by the same argument as in the remark preceding Thm. 2.8. So $\gamma(0)$ is the generic point on $(\mathcal{N}_{(X,Y)} \circ F_1)(w)$. Hence the corresponding exceptional divisor defined by a ray ρ contains $\gamma(0)$ and satisfies $\rho = D_w$. Finally, apply Lem. 4.6.

Now one proves Thm. 4.2.4 by applying Lemmas 4.2.6, 4.2.8 and 4.2.9. From the diagram above one has:

Corollary 4.3.10. *For a toric variety X with a proper closed subset $Y \supset \mathrm{Sing}(X)$,*
i) the set of the Y-essential divisors, the set of Y-divisorially essential divisors and the set of toric Y-divisorially essential divisors over X coincide;
ii) $\pi^{-1}(Y)$ has finitely many components.

Now let X be a stable toric variety, affine or polarized projective. As the observations are local, without loss of generality take it to be affine. This means that it corresponds to a complex Σ of rational polyhedral cones. The singular locus of $X = X_\Sigma$ is a union of two sets: the union of intersection loci $\cup_{i \neq j}\{X_i \cap X_j\}$, and the union of singular loci $\cup_i \mathrm{Sing}(X_i)$. Moreover, there is a normalization map $\nu : \coprod_i X_i \to X_\Sigma$ such that for each i, $\nu^{-1}(\mathrm{Sing}(X)) \cap X_i = \mathrm{Sing}(X_i) \cup (\cup_{j \neq i}(X_i \cap X_j))$. This gives a closed subset $Y_i \subset X_i$. Also, each essential divisor over X_Σ becomes a Y-essentail divisor for the pair (X_Σ, Y), where $Y := \cup_i \nu(Y_i)$. Then the answer of Nash problem for X is obtained naturally from the answer of Nash problem for each pair (X_i, Y_i). This follows from the Prop. 4.3.11 and Prop. 4.3.12 below. We remind (Prop. 2.0.14) that over a toric variety all irreducible components are good.

Define Ω to be the disjoint union of the sets of irreducible components of $\pi_i^{-1}(Y_i)$, where $\pi_i : X_{i_\infty} \to X_i$.

Proposition 4.3.11. *There is a one-to-one correspondence between the set of irreducible components of $\pi^{-1}(\mathrm{Sing}(X))$ and Ω.*

Proof. Let α be the generic point of a component C of $\pi^{-1}(\mathrm{Sing}(X))$. Let $\nu_\infty : \widetilde{X}_\infty \to X_\infty$ be the map of arc spaces corresponding to the normalization map $\nu : \widetilde{X} \to X$. Then $\nu_\infty^{-1}(\alpha)$ contains the generic points of some components of $\pi_i^{-1}(Y_i)$ for some i's. By the description of affine stable toric varieties it follows that i is unique. Taking the restriction $\pi|_{\pi^{-1}(X_i)}$ one has that C is unique as well. This defines an injective map between the sets above.

Conversely, take any element $C_i \in \Omega$. Its image $\nu_\infty(C_i)$ will be an irreducible component of $\pi^{-1}(\mathrm{Sing}(X))$, because there is an open subset $U_i \subset C_i$ such that $\nu_\infty(U_i)$ contains no arc which is the image of an arc in another component.

It would be nice to have a similar claim for the Y-essential divisors over (X, Y), which are in fact the essential divisors over X. Take Ξ to be the disjoint union of the sets of Y_i-essential divisors for (X_i, Y_i).

Proposition 4.3.12. *The set of essential divisors over X is in one-to-one correspondence with Ξ.*

Proof. Let $f' : X' \to X$ be any divisorial resolution of X. By the universal property of the normalization f' factors through ν. Take for any i, $f'_i : X'_i \to X_i$ to be a Y_i-resolution of (X_i, Y_i). Then $\coprod_i f'_i$ is a resolution of $\coprod_i X_i$. So it defines a birational map $\psi : X' \dashrightarrow \coprod X'_i$. Pick up an essential divisor $D \subset X'$. The closure of $\psi(D)$ gives an Y_i-essential divisor over (X_i, Y_i) for some i, defining an element of Ξ, because $\nu \circ (\coprod_i f'_i)$ is a resolution for X. An injective map on the set of essential divisors of X is defined. It is also surjective because the restriction $f'|_{f'^{-1}(X_i)}$ is a Y_i-resolution for (X_i, Y_i), for any i.

By Prop. 4.2.11 and Prop. 4.2.12, to prove the bijection of $\mathcal{N}_X$ is enough to show that the set Ω is in bijection with the set Ξ. But this follows from Thm. 4.1 applied to each X_i, and the remark at the beginning of this section. This gives a positive answer for Nash problem in the case of STVs ([**?**]10.1002):

Theorem 4.3.13. *For X an equidimensional STV there is a bijection between the set of the irreducible components of $\pi^{-1}(\mathrm{Sing}(X))$ and the set of essential divisors over X.*

Example 4.3.14. Take as example $X := (x_1x_2 - x_3x_4 = 0) \subset \mathbb{A}^4$ over a field with characteristic 0 and $Y = (x_1 = x_3 = 0) \supset \mathrm{Sing}(X) = \{0\}$. Then Y is defined by the union of the cone σ, corresponding to $\mathrm{Sing}(X)$), and a facet τ where $Y = \mathrm{orb}(\tau)$. The intersection $(\sigma^\circ \cup \tau^\circ) \cap N$ will have one minimal element, namely the minimal element in $N \cap \tau$. There is only one irreducible component of $pi^{-1}(Y)$. Indeed, any k-arc in X_∞ corresponds to a $k[[t]]$-point on X. If $(\sum a_it^i, \sum b_it^i, \sum c_it^i, \sum d_it^i)$ is such a point on $\pi^{-1}(Y)$, then $\sum a_it^i . \sum b_it^i = \sum c_it^i . \sum d_it^i$ with $a_0 = c_0 = 0$. Comparing

the coefficients at the corresponding powers of t, one has:
$a_1 b_0 = c_1 d_0$,
$a_1 b_1 + a_2 b_0 = c_1 d_1 + c_2 d_0$,
$\dots$
This gives the same system defining X_∞, which by Kolchin's theorem is irreducible. Thus $\pi^{-1}(Y)$ has one component, as expected.

Chapter 5

Nash Problem - Surfaces

In this chapter will be discussed the results in the case of surfaces, all of them giving a positive answer to the Nash problem. We start with the cases of A_n and D_n singularities ([Nas95], [Plé05b]), including a couple of useful criteria about the partial order that could be defined on the set of the families of arcs over the exceptional components in some resolution of singularities of X ([PPP06]). Then are proved the results of Reguera about minimal surface singularities ([Reg95]), and Lejeune-Jalabert and Reguera about sandwiched surface singularities ([LJRL99], [Reg06]). At the end is given example of Plénat, Popescu-Pampu for a class of non-rational surface singularities with a positive answer to the Nash problem ([PPP06]), and similar results of Morales ([Mor08]).

5.1 A_n and D_n surface singularities

Let k be a field of characteristic 0. The case of A_n singularities had been explained by Nash in 1968 ([Nas95]).

Proposition 5.1.1. *Over each A_n singularity there are n irreducible components (in X_∞). In particular, the Nash problem has a positive answer in this case.*

Proof. We follow the original idea of Nash. Such a singularity is defined by an equation $xy = z^{n+1}$ in $\mathbb{A}^3$. Each arc at 0 is defined by a triple of power series $a = \sum a_i t^i, b = \sum b_i t^i, c = \sum c_i t^i$, giving $ab = c^{n+1}$. For some $1 \leq j \leq n$ the first j coefficients in a, the first $n+1-j$ coefficients in b, and

c_0 are 0. This defines n good components (or families of arcs, as Nash calls them). Indeed, any arc in the i-th component is not a limit of arcs in the j-th component for $i \neq j$, and the number of components is at most n, the number of the essential divisors.

Next, we give the results of Plénat about D_n singularities ([Plé05b]).

Definition 5.1.2. *Let $U \subset X$ be an open affine, $\phi : X' \to X$ a resolution of singularities, E_i an exceptional divisor such that $f^{-1}(U) \cap E_i \neq \emptyset$, and $f \in \mathcal{O}_U$. Define $\mathrm{ord}_{E_i}(f)$ to be the coefficient a_i before E_i in $\phi^*(Z(f)) = \sum_k a_k E_k + Z(f')$, $Z(f')$ being the strict transform of $Z(f)$. Let $N_i^\circ(X') := \{\alpha \in X'_\infty : \alpha(0) \in E \setminus \cup_{j \neq i} E_j,\ \alpha'$ intersecting transversally $E_i\}$, where α' is the strict transform of α, and define N_i to be its closure.*

It is clear that any N_i is irreducible subset of X'_∞. The following criterion ([PPP06], [Reg95]) gives for a normal surface singularity X a sufficient condition for a family of arcs over an exceptional component in a resolution of X not to be contained in the closure of another family. It was proved in arbitrary dimension in ([Plé05a]).

Criterion 5.1.3. *Let $(X, 0)$ be a normal surface singularity with minimal resolution $\phi : (\tilde{X}, E) \to (X, 0)$. If E_i, E_j are exceptional prime divisors for ϕ with N_i, N_j the irreducible sets corresponding to them, and there is an $f \in \mathcal{O}_U$ such that $\mathrm{ord}_{E_i}(f) < \mathrm{ord}_{E_j}(f)$, then $N_i \nsubseteq N_j$.*

Proof. Let $(x_1, \ldots, x_n)$ be a system of coordinates on some affine open neighborhood of 0, where $(X, 0)$ is embedded in $(\mathbb{A}^n, 0)$. Then an arc over 0 is defined by power series $\alpha = (x_1(t), \ldots, x_n(t))$, $x_i(t) = \sum_{k=1}^\infty a_{ik} t^k$ for $i = 1, \ldots, n$. For each E_k let $U_{f,k} \subset N_k$ be the open subset of all arcs α on $\tilde{X}$ which meet E_k transversely at a smooth point of $Z(f \circ \phi)$. For such an α one has $\mathrm{ord}_{E_k}(f) = \mathrm{ord}_t(f \circ \alpha)$, i.e. the first $\mathrm{ord}\, E_k - 1$ coefficients vanish. This defines a closed subscheme $Z_{f,k} \subset (X, 0)_\infty$, so one has $N_k \subset Z_{f,k}$. Now by $\mathrm{ord}_{E_i}(f) < \mathrm{ord}_{E_j}(f)$ meaning that $U_{f,i} \cap Z_{f,j} =$, one has $N_i \nsubseteq Z_{f,j}$, as well as $N_i \nsubseteq N_j$.

Next we give another criterion ([Plé05a]) that could be used when we have a morphism between normal surface singularities. Let $p : (X, x) \to (Y, y)$ be a dominant birational morphism between isolated surface singularities. Let $f : (X', \{E_i\}_{i \in I}) \to (X, x), g : (Y', \{G_j\}_{j \in J}) \to (Y, y)$ be their minimal resolutions. Then p defines a birational morphism $p' : (X', \{E_i\}_i) \to (Y', \{G_j\}_j)$

in a commutative diagram together with p, f, g, and p' could be factored in a sequence of blow-ups. Because both f, g are minimal resolutions for each G_j there is a unique $E_j = p'(G_j)$, i.e. $I = J \cup I'$ and all $E_i, i \in I'$ are contracted into points $a_{i,k} \in G_k, k \in J$.

Criterion 5.1.4. *If $N_i(Y, y) \nsubseteq N_j(Y, y)$, then $N_i(X, x) \nsubseteq N_j(X, x)$.*

Proof. For the proof we need two lemmas:

Lemma 5.1.5. *The induced map $p'_\infty : N_i^\circ(X, x) \to N_i^\circ(Y, y)$ is dominant.*

Proof. Having $p'_\infty(N_i^\circ(X, x)) = \{\alpha \text{ transverse to } E_i \setminus \cup_{j \neq i} E_j, \alpha(0) \neq a_{i,k} \forall k\}$, shows that the image of $N_i^\circ(X, x)$ is dense in $N_i^\circ(Y, y)$, and the statement follows.

Lemma 5.1.6. *For S a non-singular surface, $C \subset S$ a rational curve, and $F \subset C$ a finite set of points, the set of all arcs $\{\alpha$ transverse to $C \setminus F\}$ is dense in the set of all arcs $\{\alpha$ transverse to $C\}$.*

Proof. It holds because the first set is an open non-empty subset of the second set, which is irreducible.

To finish the proof of the criterion, by the previous lemma $p'_\infty(N_i^\circ(X, x))$ is dense in $N_i^\circ(Y, y)$. By $N_i(Y, y) \nsubseteq N_j(Y, y)$ there exists $\gamma \in p'_\infty(N_i^\circ(X, x))$ and a neighborhood V_γ of γ which does not intersect $N_j^\circ(Y, y)$. Then $p'^{-1}_\infty(\gamma) \in N_i^\circ(X, x)$ and $V := p'^{-1}_\infty(V_\gamma)$ is an open neighborhood of the pre-image of γ. Thus $V \cap N_j^\circ(X, x) = \emptyset$, that is $N_i(X, x) \nsubseteq N_j(X, x)$.

These criteria will be used to prove the main results in this chapter. We start with the case of D_n singularities. Such a singularity $(X, 0)$ is defined by $F := z^2 - x(y^2 + x^{n-1} = 0)$ in $\mathbb{A}^3$. An arc at $0 \in X$ is defined by $(x(t), y(t), z(t))$ satisfying $F(x(t), y(t), z(t)) = 0$, where $x(t) = \sum_i a_i t^i, y(t) = \sum_i b_i t^i, z(t) = \sum_i c_i t^i$. The last equality defines the relations between the coefficients which generate an ideal I.

The proof of the next result is based on long computations, so we give the explanation of the basic idea and the main steps ([Plé05b], for details see ([Pl4])).

Theorem 5.1.7. (Plénat) *In the case of D_m, $(m \geq 4)$, surface singularities over $\mathbb{C}$, the Nash problem has positive answer.*

Proof. (Sketch) Take the minimal desingularisation of X and let E_i be the prime exceptional divisors. Let N_i be as in Def. 5.1.2, so that $(X,0)_\infty = \cup_{i\in I} N_i$. By the definition there are as many irreducible sets N_i as prime exceptional divisors E_i in the minimal resolution, and one proves that $\overline{N_i} \nsubseteq \overline{N_j}, i \neq j$ using Crit. 5.1.3.

Start with the case $m = 2n$. Denote by $N_i(q)$ the family of the truncations of order q of the arcs in N_i. It is enough to prove for some q that $N_i(q) \nsubseteq N_j(q)$ for $i \neq j$. Take $q = 4n - 1$.

In the dual graph of the resolution denote the main branch by $E_1, \dots E_{2n-2}$, and the other two branches by $E_1, \dots, E_{2n-2}, E_{2n-1}$ and $E_1, \dots, E_{2n-2}, E_{2n}$. Each vertex has weight -2. Now apply Crit. 5.1.3. to the functions $x, y, z, y - ix^{n-1}, y + ix^{n-1}$. This gives the following relations:
1) $N_{2n-1-k}(4n-1) \nsubseteq N_{2n-1-l}(4n-1), 1 \leq l < k \leq 2n-2$,
2) $N_{2n-1-k}(4n-1) \nsubseteq N_{2n-1}(4n-1), N_{2n}(4n-1)$,
3) $N_{2n-1}(4n-1), N_{2n}(4n-1) \nsubseteq N_{2n-1-k}(4n-1), 1 \leq k \leq 2n-2$,
4) $N_{2n-1}(4n-1) \nsubseteq N_{2n}(4n-1)$,
5) $N_{2n}(4n-1) \nsubseteq N_{2n-1}(4n-1)$.
The remaining non-inclusions are:
6) $N_{2n-1-l}(4n-1) \nsubseteq N_{2n-1-k}(4n-1), 1 \leq l < k \leq 2n-2$;
7) $N_{2n-1-k}(4n-1) \nsubseteq N_{2n-1}, N_{2n}$;
they are proved as special cases.

Let for each $k = 1, \dots 2n$, P_k be the ideal of the relations for the coefficients defining $N_k(4n-1)$, i.e. $Z(P_k) = N_k(4n-1) \subset (X,0)_{4k-1}$. Suppose that $N_k(4n-1) \nsubseteq N_l(4n-1)$, i.e. $P_l \nsubseteq P_k$. One needs to prove the opposite, i.e. $P_k \nsubseteq P_l$.

For the coordinate functions x, y, z one has
$a_1, \dots, a_{\mathrm{ord}_{E_l}(x)-1}, b_i, \dots, b_{\mathrm{ord}_{E_l}(y)-1}, c_1, \dots, c_{\mathrm{ord}_{E_l}(z)-1} \in P_l$. Take the ideal
$I_l = I \cap \mathbb{C}[a_1, \dots c_{4n-1}] \cap (a_1, \dots, a_{\mathrm{ord}_{E_l}(x)-1}, b_i, \dots, b_{\mathrm{ord}_{E_l}(y)-1}, c_1, \dots, c_{\mathrm{ord}_{E_l}(z)-1})$.
Here I is the ideal in the ring of coefficients corresponding to $(X,0)_\infty(4n-1)$. Moreover, $a_{\mathrm{ord}_{E_l}(x)}, b_{\mathrm{ord}_{E_l}(y)}, c_{\mathrm{ord}_{E_l}(z)} \notin P_l$. This could be done for any P_l, in particular for P_k.

Definition 5.1.8. *For M submodule of a free R-module F, J ideal in R, the saturation* $(M : J) := \{a \in F |\ a.J \subset M\} \subset F$. *Then* $(M : J^\infty) := \bigcup_i (M : J^i)$. *For $d \in R$,* $(M : d) := \{a \in F |\ a.d \in M\}$ *and* $(M : d^\infty) := \bigcup_i (M : d^i)$.

The most technical part of the proof is the following claim ([Pl4]):
(i) For $d \in M_k$, $(I_k : d^\infty) \subset P_k$.

Next is proved the non-containing of ideals.

Suppose that $P_k \subset P_l \subset \mathbb{C}[a_i, \ldots, c_{4n-1}]$, and let $\nu : \mathbb{C}[a_i, \ldots, c_{4n-1}]/(P_k \cap P_l) \to S$ be the normalization of $\mathbb{C}[a_i, \ldots, c_{4n-1}]/(P_k \cap P_l)$. Then $P_k/(P_l \cap P_k) \subset P_l/(P_l \cap P_k) \subset \mathbb{C}[a_1, \ldots, c_{4n-1}]/(P_k \cap P_l)$. Moreover, there is a prime ideal Q_k such that $P_k/(P_l \cap P_k) \subset Q_k \subset S$, and for any $Q_l \subset S$ under $P_k/(P_l \cap P_k)$ in the following diagram

$$\begin{array}{ccc} P_k/(P_k \cap P_l) & \hookrightarrow & P_l/(P_k \cap P_l) \\ \downarrow & & \downarrow \\ Q_k & \hookrightarrow & Q_l \end{array}$$

the inclusion $(I_k : (d)^\infty).S \subset Q_l$ does not hold. By (i) this gives a contradiction proving the case for D_2n singularities.

In the case of D_{2n+1} singularities the proof is similar, with only change that Crit. 5.1.3. being applied to the functions $x, y, z, z + ix^n$.

Though some attempts have been made in the cases of E_6, E_7, E_8 surface singularities, they remain open.

5.2 Minimal and sandwiched surface singularities

In this section are given the results of Reguera and Lejeune-Jalabert ([Reg95], [LJRL99]) on minimal and sandwiched surface singularities respectively. The proof of the first result follows closely ([Plé05a]). Though it could be obtained as a direct corollary of the second result, it is given here as another application of Crit. 5.1.3.

Theorem 5.2.1. *([Reg95]) The Nash problem has a positive answer for minimal surface singularities.*

Proof. Let $(X, 0)$ be a minimal singularity (Def. 3.1.9), and Γ the dual graph of its minimal resolution. Take $z_1, \ldots z_k$ to be the extremes of the graph Γ, i.e. its ending vertices, $\{x_i\}_1^n$ to be the vertices corresponding to the divisors $\{E_i\}$, $w_i := E_i^2$, and γ_i to be the number of edges at x_i.

Let x, y be different vertices, with N_x, N_y the closures of the corresponding families associated with the divisors corresponding to x, y (Def. 5.1.2). If then neither is contained in the other would prove that there are n irreducible components over X, and the Nash map would be bijective. The strategy is to find a cycle $C = \sum a_i E_i$ with $a_x < a_y$ (which are the coefficients before the prime exceptional divisors, corresponding to x, y), and $C.E_j \leq 0, \forall j$. Then by Artin's theorem ([Art66]) there exists $f \in \mathcal{O}_{X,0}$ such that $Z(f) = C + \sum D_i$ with $D_i \nsubseteq \cup E_j$. Then one has $a_x = \mathrm{ord}_{E_x}(f) < \mathrm{ord}_{E_y} = a_y$, so by Crit. 5.1.3 $N_x \nsubseteq N_y$. Repeating this for all pairs of different E_i, E_j will prove the claim.

Each minimal singularity is rational, so by the properties of Γ there is a unique way to connect x and y by a subgraph Γ', which is a non-branched tree (called "bamboo"), and continue it to some extremes:
$z_1 - \ldots - x - \ldots - y - \ldots z_2$.
For convenience re-enumerate the vertices in Γ':
$x_1 := z_1, \ldots, x_k := x, \ldots, x_l := y, \ldots x_m := z_2$.
Define C as follows. Put $a_i := i, \forall x_i \in \Gamma'$, and take the complement $\Gamma \setminus \Gamma' = \cup \Gamma_r$. Here all Γ_r are disjoint connected subgraphs (trees) of Γ. To define the cycle C one has to determine the coefficients before the divisors corresponding to vertices in Γ_r. For each r there is a vertex $x_{j(r)}$ to which the subgraph Γ_r is attached. Define the coefficient before all divisors corresponding to vertices in Γ_r for any r to be $j(r)\ \forall r$.
Claim: $C.E_i \leq 0$ for any i. Indeed,
1) for $i = 1,\ C.E_1 = w_1.1 + 2.1 \leq 0$, because the index before E_2 was taken to be 2;
2) for $1 < i < m,\ C.E_i = w_i.i + (i-1) + (i+1) + (\gamma_i - 2).i = (w_i + \gamma_i).i$;
3) for $i = m,\ C.E_m = (w_m + \gamma_m).(m-1)$, and $\gamma_m = 1$;
4) for $D_i \in \Gamma_r,\ C.D_i = (w_i + \gamma_i).j(r)$.
In all these cases $C.D_i \leq 0$ because $w_i + \gamma_i \leq 0$ by the minimality of the singularity.

So it was shown that for any couple of different exceptional divisors E_x, E_y, by the Artin's theorem ([Art66]) and Crit. 5.1.3, one has $N_x \nsubseteq N_y$. This gives as many components over $(X, 0)$ as essential divisors, i.e. the Nash problem has positive answer in the case of minimal surface singularities.

Remark 5.2.2. There is also another proof ([Plé05a]), using Crit. 5.1.4 and a theorem of Spivakovsky ([Spi90], Prop. 1.13).

Now we want to discuss the Nash problem for a sandwiched singularity $(X, 0)$. Starting with a result of Reguera ([Reg06]), giving a necessary and sufficient condition for an essential divisor to be in $\mathrm{Im}(\mathcal{N}_X)$ for arbitrary X over am uncountable algebraically closed k of characteristic 0, then using a result of Lejeune-Jalabert and Reguera, as a corollary one has the positive answer to the problem.

Let X be an algebraic variety. For E an essential divisor in some resolution of singularities $f: Y \to X$, the generic point of $\pi_Y^{-1}(E)$ is send by $f_\infty : Y_\infty \to X_\infty$ to the generic point of a closed subscheme (see Thm. 3.0.11).

Definition 5.2.3. *Denote this subscheme by $N_E \subset \pi_X^{-1}(\mathrm{Sing}(X))$. It does not depend on the resolution, because $\mathcal{N}_X(C) = C$ for any component C of $\pi_X^{-1}(\mathrm{Sing}(X))$.*

Any K-wedge $\phi : \operatorname{Spec} K[[s,t]] \to X$ could be viewed as a $K[[s]]$-point h_ϕ on X_∞. The arc $h_\phi(0)$ is called the special arc, and $h_\phi(\operatorname{Spec} K((s)))$ is called the generic arc of ϕ.

Theorem 5.2.4. *([Reg95]) For E_i an essential divisor over X, let z_i be the generic point of $N_i := N_{E_i}$ with residue field k_i. The following are equivalent:*
1) $E_i \in \mathcal{N}_X$;
2) For any resolution $p : Y \to X$ and any field extension $K \supset k_i$, any K-wedge ϕ on X with $h_\phi(0) = z_i$ and $h_\phi(\operatorname{Spec} K((s))) \in \pi_X^{-1}(\mathrm{Sing}(X))$ lifts to Y;
3) There exists a resolution p satisfying 2).

Proof. 1) $\Longrightarrow$ 2) Take $\eta = \operatorname{Spec} K((t))$ to be the generic point of $\operatorname{Spec} K[[t]]$, so that $z_i = h_\phi(0)$ is a specialization of $z = h_\phi(\eta)$. There exists E_0 a component of the exceptional locus of p so that $z \in N_0$ (Thm. 3.0.11). Thus $z_i \in \overline{\{z\}} \subset N_0$. By 1), $N_i = N_0$, i.e. $z_i = z_0 = z$. Both $h_\phi(0), h_\phi(\eta)$ could be lifted to Y, and ϕ lifts to Y.
2) $\Longrightarrow$ 3) is obvious.
3) $\Longrightarrow$ 1) Assume $N_i \subsetneq N_j$ and $p : Y \to X$ is a resolution satisfying 2). Then $(N_j)_{z_i} \subset X_\infty$ corresponds to a morphism $\operatorname{Spec} \mathcal{O}_{N_j,z_i}[[t]] \to X$ by the functorial description of X_∞ (see Ch. 2). Here $\mathcal{O}_{N_j,z_i}$ is the localization of the local ring of N_j with its reduced structure at the generic point z_i of N_i. One would like to construct, using the three lemmas below, a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} k((s))[[t]] & \xrightarrow{h_z} & X \\ \downarrow & & \uparrow \\ \operatorname{Spec} k[[s,t]] & \longrightarrow & \operatorname{Spec} \mathcal{O}_{N_j,z_i}[[t]] \\ \uparrow & & \uparrow \\ \operatorname{Spec} k[[t]] & \longrightarrow & \operatorname{Spec} k_i[[t]] \end{array}$$

such that $h_{z_i} : \operatorname{Spec} k_i[[t]] \to X$ would be the composition of the vertical arrows on the right. Looking for $z \in N_j \setminus N_i$, take $K \supset k_i$ to be a field extension. At the upper rectangle of the diagram, $h_z(0)$ is not in the center of E_i on Y (denoted by E_i as well). By 2) applied for p the wedge $\operatorname{Spec} k[[s,t] \to X$ lifts to Y. Thus the generic point of the essential component $E_i \subset Y$ is a specialization of $\tilde{h}_z(0) \notin E_i$ ($\tilde{h}_z$ being the lift of h_z on Y), which gives a contradiction.

To construct the diagram above one needs the next lemmas. Without loss of generality, assume $X \subset \mathbb{A}^N$.

Lemma 5.2.5. *For E_i an essential component of Y, and $P \subset \mathcal{O}_{X_\infty}$ the prime ideal of N_i, the following hold:*
i) $N_i^\circ(Y) \subset N_i$ is non-empty;
ii) There exists $G_0 \in \mathcal{O}_{X_\infty} \setminus P$, and a finitely generated ideal $I \subset \mathcal{O}_{X_\infty}$ such that $\sqrt{I_{G_0}} = P_{G_0}$.

Proof. One may assume $E_i \subset Y$ to be a divisor by blowing-up it if necessary. Let $U \subset Y$ be an affine open set such that $U \cap E_i \neq \emptyset$, so that E_i is defined on U by a single equation $\updownarrow = 0$. Let v_i be the divisorial valuation on $k(X)$ with center E_i, and $f_i \in \mathcal{O}_X,\ i = 1, \dots m$ be such that $X \dashrightarrow U$ is defined by $y_i := f_i/f_0 = 0,\ i = 1, \dots m$. Take also $p \in k[x_1, \dots x_{m+1}]$ such that
(*) $l(f_1/f_0, \dots f_m/f_0) = p(f_0, \dots f_m)/f_0^a$, $a \in \mathbb{N}$.

Note that the generic point z_i of E_i is a specialization of $\tilde{h}_{z_i}(0)$, so that one has $b_k := \operatorname{ord}_t h_{z_i}^*(f_k) < \infty,\ k = 0, \dots m$. Then any $x \in X_\infty$ with $\operatorname{ord}_t h_x^*(f_k) < \infty,\ k = 0, \dots m$ lifts to an arc $\tilde{h}_x \in Y_\infty$ (Lem. 3.0.10). For such an x, $\operatorname{ord}_t h_x^*(p(f_0, \dots f_m)) = 1 + ab_0$ is equivalent to $\operatorname{ord}_t h_x^*(l) = 1$ by (*). Define $\Omega := \{x \in X_\infty |\ \operatorname{ord}_t h_x^*(f_k) = b_k,\ k = 0, \dots m\ \ \operatorname{ord}_t h_x^*(p(f_0, \dots f_m)) = 1 + ab_0\}$. Then $\Omega \neq \emptyset$ (by the definition of b_k) is an open subset of N_i contained in $N_i^\circ(Y)$. So $\Omega = D(G_0) \cap C$ with $G_0 \in \mathcal{O}_{X_\infty} \setminus P$, $D(G_0) := \{G_0 \neq 0\}$, and $C = Z(I)$ for finitely generated $I \subset P \subset \mathcal{O}_{X_\infty}$. Then $N_i \cap D(G_0) = C \cap D(G_0)$, from which ii) holds by the Nullstellensatz. Here is used the fact that k is uncountable, in which case a point in X_∞ is closed iff it is a k-point ([Ish04], Pr.2.10). Applying the same argument, just replacing $D(G_0)$ with U, to the finite affine open cover by the sets U of the set of nonsingular points in E_i of $p^{-1}(\operatorname{Sing}(X))_{red}$, i) holds. In particular, $b_k = v_i(f_k),\ k = 0, \dots m$.

Let for any $n \in \mathbb{N}$, $\mathcal{O}_n$ be the structure ring of $\overline{\psi_n(X_\infty)}$ (remind that $\psi_n : X_\infty \to X_n$ is the canonical map corresponding to the n-truncations of power series). Let also $P_n \subset \mathcal{O}_{X_n}$ be the prime ideal defining it. Then $\mathcal{O}_n \subset \mathcal{O}_{n+1}$ and $P_n = P \cap \mathcal{O}_n$.

The next lemma is purely technical, so we omit the proof (see [Reg06], Lem. 3.2).

Lemma 5.2.6. *There exists $G \in \mathcal{O}_{X_\infty} \setminus P$ such that the localization P_G is finitely generated.*

Finally, one needs one more auxiliary lemma.

Lemma 5.2.7. *If $E_i \neq E_j$ are essential divisors over X, then*
i) $\widehat{\mathcal{O}_{X_\infty,z_i}}$ is Noetherian;
ii) $N_i \subset N_j \Longrightarrow \dim \widehat{\mathcal{O}_{X_\infty,z_i}} \geq 1$.

Proof. i) $\widehat{\mathcal{O}_{X_\infty,z_i}}$ is a complete local ring with maximal ideal $P\widehat{\mathcal{O}_{X_\infty,z_i}}$. By the previous lemma, P is finitely generated, and so is $P\widehat{\mathcal{O}_{X_\infty,z_i}}$.

ii) Take $R := \mathcal{O}_{N_j,z_i}$ so that $M := P\mathcal{O}_{N_j,z_i}$ will be its maximal ideal, which is finitely generated. By the condition $N_i \neq N_j$, so R is not a field. But it is a localization of a domain by a prime ideal, so is a domain itself. Then $M^n \neq (0)$ for any n, and by i), $\widehat{R}$ is a Noetherian ring. Suppose $\dim \widehat{R} = 0$, that is the only prime ideal in R is its maximal ideal. By Thm. 2.14 of ([Eis95], p. 74) this is equivalent to R being an Artinian ring. This means $M^n \widehat{R} = 0$ for some n. So by the definition of $\widehat{R}$, $M^n = M^{n+1} = M.M^n$ and by the Nakayama's lemma $M^n = 0$, a contradiction.

Next we will construct the diagram above applying the curve selection lemma, but first we need a definition.

Definition 5.2.8. *An irreducible $N \subset X_\infty$ is called generically stable if there exists an affine open $W \subset X_\infty$ so that $N \cap W$ is a nonempty closed subset of W_0 with defining ideal which is a radical of a finitely generated ideal.*

Lemma 5.2.9. *(Curve selection lemma, [Reg06]) If $N \subsetneq N' \subset X_\infty$ are two irreducible subsets, N is generically finite, z is its generic point, and k_z its residue field, there exists a morphism $\phi : \operatorname{Spec} K[[t]] \to N'$, with $K \supset k_z$ an algebraic extension, such that $\phi(0) = z$, $\phi(\eta) \in N' \setminus N$.*

Applying this lemma to $\operatorname{Spec} \widehat{\mathcal{O}_{N_j,z_i}}$ gives a morphism $\psi : \operatorname{Spec} k[[s]] \to \operatorname{Spec} \widehat{\mathcal{O}_{N_j,z_i}}$ with $\psi(0)$ the closed point in $\operatorname{Spec} \widehat{\mathcal{O}_{N_j,z_i}}$, and $\psi(\eta)$ different from that closed point. Here $k \supset k_i$ is a finite extension. Then the composition of ψ and $\operatorname{Spec} \widehat{\mathcal{O}_{N_j,z_i}} \to \operatorname{Spec} \mathcal{O}_{N_j,z_i}$ induces the morphism $\operatorname{Spec} k[[s,t]] \to \operatorname{Spec} \mathcal{O}_{N_j,z_i}[[t]]$ in the diagram above. The other morphisms and the commutativity are straightforward.

Now we are ready to apply Thm. 5.2.4 to the case of sandwiched surface singularities. For this we use the following statement ([LJRL99], Thm. 7):

Theorem 5.2.10. (Lejeune-Jalabert, Reguera) *Let ϕ be a wedge on a sandwiched surface singularity $(X,0)$ centered at a general arc α. Then ψ lifts to the minimal desingularization of $(X,0)$.*

This gives a positive answer to Nash problem for sandwiched surface singularities.

5.3 Non-rational surface singularities

Following Plénat and Popescu-Pampu we are going to use the criteria in Sec. 5.1 to obtain an answer of the Nash problem in a case of a class of non-rational surface singularities ([PPP06]).

Let in the sequel $(X,0)$ be a normal surface singularity with minimal resolution $p : (X', E) \to (X,0)$, with $\{E_i\}_{i \in I}$ its exceptional components.

Definition 5.3.1. *In the real vector space V_p with basis $\{E_i\}_i$ define the fundamental half-spaces of p to be $H_{i,j} := \{\sum_{i \in I} a_i E_i : a_i < a_j\}$.*

Definition 5.3.2. *The Lipman semigroup associated with p is $L(p) \subset V_p$ such that*
$L(p) := \{D \neq 0 : D.E_i \leq 0 \ \forall i\}$. The strict Lipman semigroup is its subsemigroup $L^\circ(p) := \{D \neq 0 : D.E_i < 0 \ \forall i\}$.

We will need a criterion for $D \in L^\circ(p)$ to be the exceptional part (i.e. with support in the exceptional locus of p) of a divisor of the form $Z(f \circ p)$ ([PPP06]), the proof of which is skipped.

Proposition 5.3.3. *Let D be an effective divisor with support on the exceptional locus of p, satisfying $(D + E_i + K_{X'})E_j + 2\delta_{i,j} \leq 0$, where $\delta_{i,j}$ is the Kronecker symbol. Then there exists $f \in \mathfrak{m}_0$ such that the exceptional part of $Z(f \circ p)$ is D.*

The following two conditions on $(X,0)$ depend on the intersection matrix of E only.
A) $L^\circ(p) \cap H_{i,j} \neq \emptyset$ for all $i \neq j$.
B) $E \in L^\circ(p)$.

Proposition 5.3.4. *A) follows from B), but is not equivalent to it.*

Proof. Take $(X,0)$ satisfying A) and $n \in \mathbb{N}, n \gg 0$. Then $nE + E_j \in L^\circ(p)$ for any j. But $n.E + E_j \in H_{i,j}$, so the A) is satisfied.

Let now $(X,0)$ be an A_n singularity, $n > 3$. Then $E = \sum E_i, E_i^2 = -2$ for all i, $E_i.E_{i+1} = 1$ for $i = 1, \ldots n-1$, and $E_i.E_j = 0$ for all i,j such that $|i-j| > 1$. Then $E.E_i = 0$ for $i = 2, \ldots n-1$, i.e. $E \notin L^\circ(p)$. Thus B) is not satisfied.

Let $a_k := nk - (k-1)k/2$ for $k = 1, \ldots n$. Then the divisors $\sum a_k E_k$ and $\sum a_{n+1-k} E_k$ are in $L^\circ(p)$. Moreover, each $H_{i,j}$ contains exactly one of them, proving that A) holds.

Proposition 5.3.5. *Suppose that $(X,0)$ satisfies A). Then for any $i \neq j$ there is $f \in \mathfrak{m}_0 \subset \mathcal{O}_{X,0}$ with $\mathrm{ord}_{E_i}(f) < \mathrm{ord}_{E_j}(f)$.*

Proof. Take $D = \sum a_k E_k \in L^\circ(p)$ with $a_i < a_j$. For $n \gg 0$ one has $(nD + E_k + K_{X'})E_j + \delta j, k \leq 0$ for all j, k, so by Prop. 5.3.3 there exists $f \in \mathfrak{m}_0$ such that the exceptional part of $Z(f \circ p)$ is nD. But then $\mathrm{ord}_{E_i}(f) = na_i < na_j = \mathrm{ord}_{E_j}$.

The next is a result of Plénat and Popescu-Pampu.

Theorem 5.3.6. *If $(X,0)$ satisfies the condition A), the Nash map $\mathcal{N}_X$ is bijective.*

Proof. By Crit. 5.1.3 and Prop. 5.3.5 $N_i \nsubseteq N_j$ for any $i \neq j$. Also, $\pi_X^{-1}(0) = \cup_i N_i$, so $N_i, i \in I$ are the irreducible components over X. Thus $\mathcal{N}_X$ is bijective.

Next, we will show that in the class of singularities for which the preceding theorem holds, there exists an infinite subset of non-rational singularities. Denote by $\Gamma(E)$ the dual graph of the resolution p, the vertices corresponding to $E_i, i \in I$, each of them with weight E_i^2, and each couple of distinct E_i, E_j joined by $E_i E_j$ edges. With $n(E_i)$ we denote the number of edges at E_i, so each loop at E_i counts for 2.

Next result characterizes the rational singularities in the class of all singularities satisfying B), showing that there are many non-rational singularities in this class.

Proposition 5.3.7. *Let the condition B) holds for* $(X, 0)$*. It is a rational singularity iff*
1) $\Gamma(E)$ *is tree;*
2) For all $i, E_i \simeq \mathbb{P}^1$*;*
3) For all $i, |E_i|^2 > n(E_i)$*.*

Proof. $\Longrightarrow$) : The conditions 1) and 2) are among the general properties of rational singularities. Also, $E.E_i = E_I^2 + n(E_i) = -|E_i^2| + n(E_i) < 0$, so 3) follows.
$\Longleftarrow$: We cite a result of Spivakovsky ([Spi90], Ch.II, Rem. 2.3, see also [Lê00], Thm. 5.3), from which the conditions 1), 2), 3) imply that the singularity is minimal, and in particular, rational.

Remark 5.3.8. The conditions 1) and 2) hold for any rational singularity. In fact, the rationality could be described in a purely combinatorial way.

Definition 5.3.9. *A weighted graph is a tree with couples of integers* (w_i, g_i) *attached to each vertex* E_i*. The numbers* w_i *are called weights, and* g_i *are called genera of the vertices* E_i*.*

By a theorem of Grauert ([TT04], Thm. 3.3) any weighted graph which defines a negative definite symmetric form is the dual graph of a normal surface singularity.

Definition 5.3.10. *A weighted graph* Γ *is called rational if:*
1) It is a tree;
2) It defines a negative definite symmetric form;
3) The genera of all vertices are 0*;*
4) If $Z = \inf\{D = \sum a_i E_i | \ D.E_i \leq 0 \ \forall i\}$*, which is non-empty by 2), then* $p(Z) := 1/2(Z.Z + \sum a_i(w_i - 2)) + 1 = 0$*.*

In 4) above, inf is defined to be $\sum m_i E_i$ where $m_i := \inf_{D \in S} a_i$. By ([TT04], Thm. 3.5), each rational tree is the dual graph of a rational surface singularity.

It is known that the normal minimal surface singularities are characterized by the conditions 1), 2) in Prop. 5.3.7, and

3)' $|E_i^2| \geq n(E_i)$ for all i.
We see that 3) in Prop. 5.3.7 is equivalent to B) in Prop. 5.3.4, supposed the singularity is rational. So taking an abstract weighted graph which gives a negative definite symmetric form and satisfying the condition B), but violating some among the conditions 1) and 2) above, one would have a non-rational singularity. This could be seen by taking the weights of the vertices to be negative enough, for an infinite family of graphs (e.g. increasing the number of vertices or the weights).

Theorem 5.3.11. (Plénat, Popescu-Pampu) *There exist infinitely many normal non-rational surface singularities satisfying the condition B) for which the Nash problem has a positive answer.*

Finally, there is a result of Morales ([Mor08].

Definition 5.3.12. *If $(X, 0)$ is a normal surface singularity, $\pi : X' \to X$ its minimal resolution, $a_{ij} := E_i E_j$ for $E_1, \dots E_n$ the exceptional divisors. The dual graph of the intersection matrix $A = (a_{ij}$ is defined to be with vertices $E_1, \dots E_n$. For $i \neq j$ there exists an edge between E_i, E_j iff $a_{ij} \neq 0$.*

The result uses the fact that any symmetric negatively defined matrix which defines a connected graph is the intersection matrix of some resolution of a normal surface singularity (using a theorem of Grauert). Then the Nash numerical conditions are defined.

Definition 5.3.13. *Let $(X, 0)$ be a normal surface singularity, $\pi : X' \to X$ its minimal resolution with $E_1, \dots E_n$ the exceptional divisors. The numerical Nash condition for (i, j) is satisfied if*
(NN_{ij}) There exists $E = \sum n_k E_k$, $n_i \in \mathbb{N}^*$ with $n_i < n_j$ and $-E.E_m \geq 2K_{X'}.E_k$, $k = 1, \dots n$.
The numerical Nash condition (NN) is satisfied for $(X, 0)$ if NN_{ij} holds for all $i \neq j$.

Using a result of Morales and Crit. 5.1.3 one has:

Proposition 5.3.14. *If NN_{ij} holds for $(X, 0)$, then $N_i \not\subseteq N_j$. In particular, if NN holds, the Nash problem has positive answer for $(X, 0)$.*

Definition 5.3.15. *For the intersection matrix A define $C(A)_i := \sum_{j=1}^n a_{ij}$.*

A leaf of a graph is any vertex connected with exactly one other vertex. The main result (the proof is omitted here), is the following one.

Theorem 5.3.16. (Morales) *In the previous notation, if $A = (a_{ij})$ is the intersection matrix of the minimal resolution $\pi : X' \to X$, such that the dual graph Γ is a tree with $C(A)_i \leq 0$ for any vertex $E_i \in \Gamma$, $C(A)_k < 0$ for any leaf E_k iff (NN) holds.*

The case of minimal singularities discussed above (Thm. 5.2.1), could be obtained now as a corollary, these singularities being exactly the subclass of the rational singularities for which the conditions of the theorem hold. Moreover, the theorem does not put any restriction on the topological type of the exceptional components, so it extends to some non-rational singularities ([Mor08], Thm. 5). Also, the condition (NN) holds for A_n singularities, but does not hold for D_n and E_6, E_7, E_8 surface singularities.
There is also a result of Fernández-Sánchez claiming the equivalence of the problem for primitive and for sandwiched surface singularities ([FS05]), which was not considered here.

Chapter 6

Nash Problem - Higher Dimension and Generalizations

The first section contains a result of Plénat, Popescu-Pampu about a class of threefolds, and a result of González Pérez for the class of quasi-ordinary hypersurface singularities, for both of which Nash problem has a positive answer. In the second section is discussed briefly another modification of the Nash problem, the local Nash problem, proposed by Ishii. The style in this chapter is different from the previous two, in particular most of the proofs are omitted. The main sources are ([PPP08], [GP07], [Ish06]).

6.1 Higher dimension

There is a construction proposed by Plénat and Popescu-Pampu, of infinitely many families of threefolds for which the Nash problem admits a positive answer. This was the first result of this kind obtained in dimension 3. It is highly technical, but the results used for it could be very useful. So they are given with full proofs, with the main idea of the construction sketched.

Working with ample divisors one needs a result that the ampleness of a vector bundle over a non necessary irreducible variety could be tested on its irreducible components ([Laz04]):

Proposition 6.1.1. *Let L be a line bundle on a projective variety X, not necessary irreducible. Then L is ample on X iff its restriction on each irreducible component of X is ample.*

Let $(X,0)$ be an irreducible normal germ of complex analytic variety with $\mathrm{Sing}(X) = \{0\}$, and let $p : (\tilde{X}, \{E_i\}_{i\in I} \to (X,0)$ be a divisorial resolution, i.e. a proper bi-meromorphic morphism with the usual properties. Define $\sigma(p) := \bigoplus_{i\in I} \mathbb{R}_+ E_i$ to be the cone of effective $\mathbb{R}$-divisors supported by the exceptional locus of p.

Definition 6.1.2. *For each $i \neq j$ define $\sigma_{ij}(p) := \{\sum a_i E_i \in \sigma(p) : \ a_i \leq a_j\}$.*

Then one has the next criterion ([PPP08]):

Theorem 6.1.3. *Fix $i \in I$ and suppose that for any $j \neq i$, $\sigma_{ij}(p)$ contains in its interior and integral divisor F_{ij} such that $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is generated by its global sections. Then E_i is in $\mathrm{Im}(\mathcal{N}_X)$. In particular, E_i is essential.*

Proof. Consider $\mathcal{O}_{\tilde{X}}(-F_{ij})$ as a subsheaf of $\mathcal{O}_{\tilde{X}}$, containing the holomorphic functions vanishing at the exceptional locus of p at least as many times as the corresponding coefficients of F_{ij}. As $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is generated by its global sections, there is a function $f_{ij} \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(-F_{ij}))$ such that $Z(f_{ij})$ has exceptional part equal to F_{ij}.

Given an isomorphism by p outside $\mathrm{Sing}(X) = \{0\}$, there is a function g_{ij} on X, such that g_{ij} is holomorphic on $X \setminus \{0\}$, continuous on X, $g_{ij}(0) = 0$, and $p^*(g_{ij}) = f_{ij}$. But $(X,0)$ is a normal germ, meaning that any bounded holomorphic function on $X \setminus \{0\}$ extends to a holomorphic function on X. This means that $g_{ij} \in \mathfrak{m}_0$.

By definition, $\mathrm{ord}_{E_i}(g_{ij}) < \mathrm{ord}_{E_j}(g_{ij})$, so by Crit. 5.1.3 $N_i \not\subseteq N_j$. This holds for any pair $j \neq i$, so N_i is a Nash component, i.e. $E_i \in \mathrm{Im}(\mathcal{N}_X)$.

There are some corollaries from the theorem (using the same notation).

Corollary 6.1.4. *Fix $i \in I$, and let for each $j \neq i$ the cone $\sigma^\circ_{ij}(p)$ contain an integral divisor F_{ij} such that $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is an ample sheaf when restricted to each component of the exceptional locus. Then $E_i \in \mathrm{Im}(\mathcal{N}_X)$, and E_i is an essential component on $\tilde{X}$.*

Proof. Let $L(p)$ be the lattice generated by $\{E_i\}_{i\in I}$. Ampleness is an open condition w.r.t. the topology on $L(p)$, i.e. the one induced by the corresponding real vector space. By Prop. 6.1.3, $\mathcal{O}_{\tilde{X}}(-F_{ij})$ is ample when restricted to the exceptional locus of p, so it is an ample sheaf on an open neighborhood $U \in \tilde{X}$ of the locus. Thus there is an integer $n_{ij} > 0$ such that $-n_{ij}F_{ij}$ is very ample, in particular the sheaf $\mathcal{O}_{\tilde{X}}(-n_{ij}F_{ij})$ is generated by its global sections.

As $n_{ij}F_{ij} \in \sigma_{ij}^{\circ}(p)$, by Thm. 6.1.5 the claim follows.

Then one has:

Corollary 6.1.5. *If for each $i \neq j$ the cone σ_{ij} contains an integral divisor F_{ij} with $\mathcal{O}_{\tilde{X}}(-F_{ij})$ an ample sheaf when restricted to each component of the exceptional locus, those components are precisely the essential components on $\tilde{X}$, and $\mathcal{N}_X$ is bijective.*

Definition 6.1.6. *An algebraic surface S is geometrically ruled over a curve C if S is the total space of locally trivial $\mathbb{P}^1$-bundle over C.*

These are used to construct infinitely many families of threefolds with bijective Nash map. The result is technical and the details are omitted (see [PPP08], Sec. 5). The idea is to construct a smooth threefold T by gluing along open sets the total spaces of suitable line bundles over two geometrically rulled surfaces S_1, S_2. Both surfaces are defined by compactification of the total space of suitable line bundles over an irreducible smooth projective curve C. When glued S_1, S_2 meet transversely along a curve isomorphic to C. Then one contracts a divisor with two components inside T by Grauert's criterion of contractability ([PPP08], Thm. 3.5). An important fact is that C is an arbitrary smooth projective curve.

Next is a result of González Pérez about quasi-ordinary hypersurface singularities. It is another example when Nash problem for pairs (or relative Nash problem) appears naturally.

In the rest of the section char(k) = 0.

Definition 6.1.7. *A quasi-ordinary hypersurface singularity is defined by $\operatorname{Spec} k[[x_1, \dots x_n]][y]/(f)$ where $f \in \operatorname{Spec} k[[x_1, \dots x_n]][y]$ is a quasi-ordinary polynomial, i.e. a Weierstrass polynomial whose discriminant with respect to y is $x_1^{a_1} \dots x_n^{a_n} g$, with $g \in k[[x_1, \dots x_n]]$ a unit.*

Theorem 6.1.8. (González Pérez) *For $(X, 0)$ a reduced germ of a quasi-ordinary hypersurface singularity, the Nash map $\mathcal{N}_X$ is bijective.*

Proof. Let $f = \prod_{i=1}^{k} f_i$ be the polynomial defining $(X, 0)$, f_i being irreducible quasi-ordinary polynomials corresponding to the irreducible components X_i of $(X, 0)$. Take $B_i := X_i \cap \operatorname{Sing}(X) = \operatorname{Sing}(X_i) \cup \bigcup_{j \neq i} X_i \cap X_j$.

Then $\pi^{-1}(\mathrm{Sing}(X)) = \coprod i = 1^k \pi_{X_i}(B_i)$, and
$\{$Nash components over $X\} \subset \coprod_i \{$Nash components of $(X_i, B_i))\}$ ([Ish06], the proof of Lem. 4.11).
Next is proved that $\{$essential divisors over $X\} \subset \coprod_i \{B_i-$essential divisors over $X_i\}$.
Take a B_i-resolution $\phi_i : Y_i \to X_i$. Then it defines
$Y := \coprod_i Y_i \to \coprod_i X_i \to X$,
so that the composition $Y \to X$ is a Sing(X)-resolution of X in the sense of Def. 3.0.18.

Let E be an essential divisor over X. Its center on Y is an irreducible component of $\phi^{-1}(\mathrm{Sing}(X)) = \coprod_i \phi_i^{-1}(B_i)$, i.e. an irreducible component of $\phi_i^{-1}(B_i)$ for some i. The following lemma ([GP07]) holds:

Lemma 6.1.9. *For each i, $\nu_i^{-1}(B_i)$ is a germ of $\mathbb{T}$-invariant closed set at the closed orbit of the toric singularity $\tilde{X}_i$.*

From it any irreducible component of $\nu_i^{-1}(B_i)$ is $\overline{\mathrm{orb}_{\tilde{X}}(\tau)}$ for some face $\tau \prec \sigma$, where $\check{\sigma}$ is the positive quadrant of $\mathbb{R}^d$ for some d. By (Thm. 2, [GP07]) it follows that the Nash map $\mathcal{N}_{(X_i,B_i)}$ is bijective for this particular i, and the claim holds.

6.2 Generalizations and modifications of the problem

We discussed above some modifications of Nash problem, as Nash problem for pairs in Chapter 3 and the embedded Nash problem in Chapter 4. Here we would like also to introduce briefly the local Nash problem, proposed by Ishii, and her results on it. The following could be formulated for reduced k-schemes, but we take X to be an algebraic variety over k.

Definition 6.2.1. *Let $x \in X$ be a point (not necessary closed), and $C \subset \pi_X^{-1}(x)$ an irreducible component. It is called good, or Nash component of (X, x)), if $C \not\subseteq (\mathrm{Sing}(X))_\infty$.*

Let $f : X \to Y$ be a resolution of singularities, $f^{-1}(x) = \bigcup_i E_j$ with E_j non-singular divisors. Let $\{C_i\}_{i \in I}$ be the set of local Nash components for (X, x). Using the valuative criterion for properness it is easy to see that $f_\infty : \bigcup_j \pi_Y^{-1}(E_j) \to \bigcup_i C_i$ is injective outside $(\mathrm{Sing}(X))_\infty$ and dominant. Thus for each i there is a unique $j(i)$ such that $\pi_Y^{-1}(E_{j(i)})$ is dominant to

C_i. Then by (Lem. 2.14, [Ish06]) $E_{j(i)}$ is essential over (X, x). This defines the local Nash map $\mathcal{LN}$ from the set of local Nash components to the set of essential divisors over (X, x), and it is obviously injective.
Local Nash problem. For which (X, x) is the local Nash map $\mathcal{LN}$ bijective?

In her paper ([Ish06], Thm. 3.3) Ishii proves that this problem has a positive answer in the case of affine toric variety. Also, she obtains the local version of Thm. 6.1.8 above, proving it first for a larger class, the so called analytically pre-toric singularities (the details are skipped here). The irreducible case of this claim was obtained by González Pérez ([GP03]):

Proposition 6.2.2. *For a quasi-ordinary singularity (X, x) the local Nash map is bijective.*

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