DESCRIPTION OF THE COMMUTANT OF COMPOSITIONS OF DUNKL OPERATORS

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Abstract. In this paper the commutant of a composition $\tilde{D} = D_1 D_2 \ldots D_n$ of Dunkl operators $D_j f(z) = \frac{d f(z)}{dz} + k_j \frac{f(z) - f(-z)}{z}$ with parameters $k_j \geq 0$, $j = 1, 2, \ldots, n$, is described using power series in the space $A_R$ of the analytic functions in the disk $D_R = \{ z \in \mathbb{C} : |z| < R \}$.

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1 Introduction

The Dunkl operator is a differential-difference operator defined in [3] in 1989 and since then many mathematicians have studied its properties and applications.

Let $A_R$ be the space of the analytic functions in the disk $D_R = \{ z \in \mathbb{C} : |z| < R \}$.

Definition 1. For $f \in A_R$, the operator $D_k : A_R \to A_R$ defined by

$$D_k f(z) = \frac{d f(z)}{dz} + k \frac{f(z) - f(-z)}{z}$$

is called the Dunkl operator with parameter $k \geq 0$. 
Definition 2. It is said that a continuous linear operator \( M \) commutes with a fixed operator \( L \), if \( ML = LM \). The set of all operators commuting with \( L \) is called the commutant of \( L \) and will be denoted by \( \text{Comm}(L) \).

M.S. Hristova describes in [5] the commutant \( \text{Comm}(D_k) \) of the Dunkl operator \( D_k \) and in [6] the commutant \( \text{Comm}(D_k^\nu) \) of arbitrary power \( n \) of \( D_k \). Here our goal is to extend this description to the case of a composition \( \tilde{D} = D_1 D_2 \ldots D_n \) of Dunkl operators \( D_j = D_{k_j} \) with arbitrary parameters \( k_j \geq 0, j = 1, 2, \ldots, n \).

2 Representation of the commutant

Theorem 3. Let \( f \) be an analytic function from \( A_R \) with a Taylor series \( f(z) = \sum_{m=0}^{\infty} a_m z^m \). Then every continuous linear operator \( M : A_R \to A_R \) commutes with the composition \( \tilde{D} = D_1 D_2 \ldots D_n \) of Dunkl operators \( D_j = D_{k_j}, k_j \geq 0, j = 1, 2, \ldots, n \), i.e. \( M \in \text{Comm}(\tilde{D}) \), if and only if it can be represented in a power series form as

\[
Mf(z) = \sum_{\mu=0}^{n-1} \sum_{m=0}^{\infty} a_m d_{m,\mu} z^\mu + \sum_{\mu=n}^{\infty} \sum_{m=[\frac{\mu}{2}]}^{\infty} a_m \left( \prod_{\nu=1}^{ \mu \left( \nu \right) \left( \nu \right) \nu + j } c_{j,\mu - \nu + j} \right) d_{m-\nu,\mu-\nu} z^\mu,
\]

where

\[
d_{m,\mu}, 0 \leq \mu \leq n - 1, m = 0, 1, 2, \ldots, \text{are arbitrary complex numbers with the only restriction the series in the representation (2) to be convergent, and } [A] \text{ denotes the integer part of the number } A.
\]

Proof. First, let us consider the action of the Dunkl operator \( D_j = D_{k_j} \) on a single power \( z^m \) of the variable \( z \in \mathbb{C} \). If the power is even, i.e. \( m = 2s \), then

\[
D_j z^{2s} = \left( \frac{dz^2}{dz} z^{2s} - k_j \left( z^2 - (-z)^2 \right) \right) = 2s z^{2s-1} \quad \text{for } s \geq 1,
\]

for \( s = 0 \).

If the power is odd, i.e. \( m = 2s + 1 \), then

\[
D_j z^{2s+1} = \frac{dz^{2s+1}}{dz} z^{2s+1} - k_j \left( z^{2s+1} - (-z)^{2s+1} \right) = (2s+1)z^{2s} + 2k_j z^{2s} = (2s+1+2k_j)z^{2s}.
\]

The two representations can be combined in one formula:

\[
D_j z^m = \left\{ \begin{array}{ll}
           c_{j,m} z^{m-1}, & \text{if } m \geq 1, \\
           0, & \text{if } m = 0.
        \end{array} \right.
\]

Next, if the composition \( \tilde{D} = D_1 D_2 \ldots D_n \) is considered, its action on an arbitrary power \( m \) of the variable \( z \) can be expressed as

\[
\tilde{D} z^m = \left\{ \begin{array}{ll}
           c_{n,m} c_{n-1,m-1} \ldots c_{1,m-n+1} z^{m-n} = \left( \prod_{j=1}^{n} c_{j,m-n+j} \right) z^{m-n}, & \text{for } m \geq n, \\
           0, & \text{for } 0 \leq m \leq n - 1.
        \end{array} \right.
\]
Now consider an arbitrary operator $M$ from the commutant $\text{Comm}(\tilde{D})$. Let us represent its action again on an arbitrarily fixed power $z^m$ by the power series

$$Mz^m = \sum_{\mu=0}^{\infty} d_{m,\mu}z^\mu, \quad m = 0, 1, 2, \ldots$$

(6)

Here the coefficients $d_{m,\mu}$ are unknown, but they will be determined below.

In order to analyze the commutation $M\tilde{D} = \tilde{D}M$, we start by expressing $M\tilde{D}z^m$ and $\tilde{D}Mz^m$ for arbitrarily fixed power $z^m$:

$$M\tilde{D}z^m = \begin{cases} M_{m,n} \ldots c_{1,m-n+1}z^{m-n} = \sum_{\mu=0}^{\infty} c_{n,m} \ldots c_{1,m-n+1}d_{m-n,\mu}z^\mu & \text{for } m \geq n, \\ 0 & \text{for } 0 \leq m \leq n - 1. \end{cases}$$

(7)

$$\tilde{D}Mz^m = \tilde{D}\sum_{\mu=0}^{\infty} d_{m,\mu}z^\mu = \sum_{\mu=0}^{\infty} d_{m,\mu}\tilde{D}z^\mu$$

$$= \sum_{\mu=n}^{\infty} d_{m,\mu}c_{n,\mu} \ldots c_{1,\mu-n+1}z^{\mu-n} = \sum_{\mu=0}^{\infty} d_{m,\mu+n}c_{n,\mu+n} \ldots c_{1,\mu+1}z^\mu.$$ 

(8)

In the last formula $\mu - n$ was replaced by a single letter $\mu$ for convenience.

We want to have $M\tilde{D}f(z) = \tilde{D}Mf(z)$ for every $f \in \mathcal{A}(R)$. By the uniqueness theorem for analytic functions this will be true if and only if for every $m \geq 0$ one has $M\tilde{D}z^m = \tilde{D}Mz^m$, i.e. if the expressions in (7) and (8) coincide.

Let us consider first the case $0 \leq m \leq n - 1$. Then one must have

$$0 = \sum_{\mu=0}^{\infty} d_{m,\mu+n}c_{n,\mu+n} \ldots c_{1,\mu+1}z^\mu.$$

By the uniqueness theorem the power series on the right is zero if and only if all its coefficients are equal to zero, i.e. $d_{m,\mu+n}c_{n,\mu+n} \ldots c_{1,\mu+1} = 0$ for every $\mu = 0, 1, 2, \ldots$. But all $c_{j,\mu+j}$, $1 \leq j \leq n$, are different from zero and hence it is necessary to have

$$d_{m,\mu+n} = 0, \quad 0 \leq m \leq n - 1, \quad \mu = 0, 1, 2, \ldots.$$

This can be written in a better way if $\mu + n$ is replaced by a single index $\mu$:

$$d_{m,\mu} = 0, \quad 0 \leq m \leq n - 1, \quad \mu \geq n.$$ 

(9)

Now a recurrent formula for arbitrary $m \geq n$ will be found.

Comparing the first line in (7) with (8), we get by the uniqueness theorem that

$$c_{n,m} \ldots c_{1,m-n+1}d_{m-n,\mu} = d_{m,\mu+n}c_{n,\mu+n} \ldots c_{1,\mu+1}, \quad m \geq n, \mu \geq 0.$$ 

Replacing $\mu$ by $\mu - n$ we have

$$c_{n,m} \ldots c_{1,m-n+1}d_{m-n,\mu-n} = c_{n,\mu} \ldots c_{1,\mu-n+1}d_{m,\mu}, \quad m \geq n, \mu \geq n.$$ 

(10)

But all constants $c_{j,\mu-n+j}$, $1 \leq j \leq n$, are different from zero and we obtain the desired recurrent formula

$$d_{m,\mu} = \frac{c_{n,m} \ldots c_{1,m-n+1}}{c_{n,\mu} \ldots c_{1,\mu-n+1}} d_{m-n,\mu-n} = \left(\prod_{j=1}^{n} \frac{c_{j,m-n+j}}{c_{j,\mu-n+j}}\right) d_{m-n,\mu-n}, \quad m \geq n, \mu \geq n.$$ 

(11)
Now this important recurrent formula (11) will be used for expressing any coefficient $d_{m, \mu}$ with $m \geq n$ and $\mu \geq n$ by a coefficient $d_{p, q}$, where either $0 \leq p \leq n - 1$ or $0 \leq q \leq n - 1$.

Let us remind that in the sequel $[A]$ will denote the integer part of a number $A$.

In the case $\left\lfloor \frac{m}{n} \right\rfloor < \left\lfloor \frac{\mu}{n} \right\rfloor$ one can apply $\left\lfloor \frac{m}{n} \right\rfloor$ times the recurrent formula (11) and then

$$d_{m, \mu} = \left( \prod_{j=1}^{n} c_{j, m-n+j} \right) d_{m-n, \mu-n} = \left( \prod_{j=1}^{n} c_{j, m-n+j} \right) \left( \prod_{j=1}^{n} c_{j, \mu-2n+j} \right) d_{m-2n, \mu-2n}$$

$$= \ldots = \left( \prod_{\nu=1}^{\lceil \frac{\mu}{n} \rceil} \prod_{j=1}^{\lfloor \frac{m}{n} \rfloor} c_{j, m-\nu n+j} \right) d_{m-[\frac{\mu}{n}]n, \mu-[\frac{\mu}{n}]n}. \quad (12)$$

Here $m - \left\lfloor \frac{m}{n} \right\rfloor n \leq n - 1$, i.e. the first index is the remainder when $m$ is divided by $n$, and $\mu - \left\lfloor \frac{\mu}{n} \right\rfloor n \geq n$. Then by our first observation (9) the coefficient $d_{m-[\frac{\mu}{n}]n, \mu-[\frac{\mu}{n}]n}$ must be zero. Therefore (12) gives

$$d_{m, \mu} = 0, \quad \text{for } \left\lfloor \frac{m}{n} \right\rfloor < \left\lfloor \frac{\mu}{n} \right\rfloor. \quad (13)$$

In the other case, when $\left\lfloor \frac{m}{n} \right\rfloor \geq \left\lfloor \frac{\mu}{n} \right\rfloor$, one can apply $\left\lfloor \frac{\mu}{n} \right\rfloor$ times the recurrent formula (11) to get

$$d_{m, \mu} = \left( \prod_{j=1}^{n} c_{j, m+n-j} \right) d_{m-n, \mu-n} = \left( \prod_{j=1}^{n} c_{j, m+n-j} \right) \left( \prod_{j=1}^{n} c_{j, \mu-2n+j} \right) d_{m-2n, \mu-2n}$$

$$= \ldots = \left( \prod_{\nu=1}^{\lfloor \frac{\mu}{n} \rfloor} \prod_{\nu=1}^{\lfloor \frac{\mu}{n} \rfloor} c_{j, m-\nu n+j} \right) d_{m-[\frac{\mu}{n}]n, \mu-[\frac{\mu}{n}]n}. \quad (14)$$

Now the second index $\mu - \left\lfloor \frac{\mu}{n} \right\rfloor n$ is the remainder when $\mu$ is divided by $n$.

Let us combine (13) and (14) as

$$d_{m, \mu} = \begin{cases} 0 & \text{for } \left\lfloor \frac{m}{n} \right\rfloor < \left\lfloor \frac{\mu}{n} \right\rfloor, \\ \left( \prod_{\nu=1}^{\lfloor \frac{\mu}{n} \rfloor} \prod_{\nu=1}^{\lfloor \frac{\mu}{n} \rfloor} c_{j, m-\nu n+j} \right) d_{m-[\frac{\mu}{n}]n, \mu-[\frac{\mu}{n}]n} & \text{for } \left\lfloor \frac{m}{n} \right\rfloor \geq \left\lfloor \frac{\mu}{n} \right\rfloor. \end{cases} \quad (15)$$

This important formula shows that all coefficients $d_{m, \mu}$ with $0 \leq \mu \leq n - 1$ can be chosen arbitrarily, and then all other coefficients $d_{m, \mu}$ with $\mu \geq n$ are either equal to zero or can be expressed by some of the arbitrarily chosen $d_{\nu, \mu}$ with $0 \leq \nu \leq n - 1$.

The recurrent relation (15) allows a representation of $Mz^m$ as a polynomial of degree at most $\left( \left\lfloor \frac{m}{n} \right\rfloor + 1 \right) n - 1$:

$$Mz^m = \sum_{\mu=0}^{n-1} d_{m, \mu} z^\mu + \sum_{\mu=n}^{\lfloor \frac{\mu}{n} \rfloor n-1} \left( \prod_{\nu=1}^{\lfloor \frac{\mu}{n} \rfloor} \prod_{\nu=1}^{\lfloor \frac{\mu}{n} \rfloor} c_{j, m-\nu n+j} \right) d_{m-[\frac{\mu}{n}]n, \mu-[\frac{\mu}{n}]n} \cdot z^\mu. \quad (16)$$

Finally, the action of an operator $M \in \text{Comm}(\hat{D})$ on some analytic function $f(z) = \sum_{m=0}^{\infty} a_m z^m$ is
\( Mf(z) = M \sum_{m=0}^{\infty} a_m z^m = \sum_{m=0}^{\infty} a_m M z^m \)  

\[
\sum_{m=0}^{\infty} a_m \left( \sum_{\mu=0}^{n-1} d_{m,\mu} z^\mu \right) + \sum_{\mu=n}^{\infty} \left( \prod_{\nu=1}^{n} c_{j,\mu-\nu+n+j} \right) d_{m-[\frac{n}{\mu}]-[\frac{n}{\mu}] n} . z^\mu.
\]

It is natural to interchange the two sums in order to have a standard power series representation of (17):

\[
Mf(z) = \sum_{\mu=0}^{n-1} \sum_{m=0}^{\infty} a_m d_{m,\mu} z^\mu + \sum_{\mu=n}^{\infty} \sum_{m=0}^{\infty} a_m \left( \prod_{\nu=1}^{n} c_{j,\mu-\nu+n+j} \right) d_{m-[\frac{n}{\mu}]-[\frac{n}{\mu}] n} . z^\mu.
\]

This is in fact the desired representation (2) and thus, we proved the necessity, i.e. if \( M \in \text{Com}(\tilde{D}) \), then the operator \( M \) must be of the form (2).

Now, let us check the sufficiency, i.e. if an operator \( M \) has the form (2), then it commutes with the composition \( \tilde{D} = D_1 D_2 \ldots D_n \) of the Dunkl operators \( D_j = D_{k_j}, j = 1, 2, \ldots, n \), i.e. \( M \tilde{D} = \tilde{D} M \). It is enough to verify this for all powers \( z^m, m = 0, 1, 2, \ldots \), since they form a basis of the space of the analytic functions \( A_R \). In fact, for arbitrarily fixed \( m \) we can use the representation (16) instead of the general expression (2).

In the case \( 0 \leq m \leq n - 1 \) the representation (16) reduces to the first sum and \( M z^m = \sum_{m=0}^{n-1} d_{m,\mu} z^\mu \). Now we calculate \( \tilde{D} M z^m \) and \( M \tilde{D} z^m \):

\[
\tilde{D}(M z^m) = \tilde{D} \sum_{m=0}^{n-1} d_{m,\mu} z^\mu = \sum_{m=0}^{n-1} d_{m,\mu} \tilde{D} z^\mu = \sum_{m=0}^{n-1} d_{m,\mu} 0 = 0;
\]

\[
M(\tilde{D} z^m) = M 0 = 0,
\]

i.e. \( \tilde{D} M z^m = M \tilde{D} z^m = 0 \). Here we used the second case in (5).

In the case \( m \geq n \) use the first line in (5) and next use (16) with \( z^{m-n} \) to represent \( M \tilde{D} z^m \):

\[
M \tilde{D} z^m = M \left( \prod_{j=1}^{n} c_{j,m-n+j} \right) z^{m-n} = \left( \prod_{j=1}^{n} c_{j,m-n+j} \right) M z^{m-n}
\]

\[
= \left( \prod_{j=1}^{n} c_{j,m-n+j} \right) \left( \sum_{\mu=0}^{n-1} d_{m,\mu} z^\mu + \sum_{\mu=n}^{\infty} \left( \prod_{\nu=1}^{n} c_{j,\mu-\nu+n+j} \right) d_{m-[\frac{n}{\mu}]-[\frac{n}{\mu}] n} . z^\mu \right).
\]

To represent the inverse commutation \( \tilde{D} M z^m \), apply \( \tilde{D} \) to (16):

\[
\tilde{D} M z^m = \sum_{\mu=0}^{n-1} d_{m,\mu} \tilde{D} z^\mu + \sum_{\mu=n}^{\infty} \left( \prod_{\nu=1}^{n} c_{j,\mu-\nu+n+j} \right) d_{m-[\frac{n}{\mu}]-[\frac{n}{\mu}] n} \cdot \tilde{D} z^\mu.
\]
The first sum will vanish because the second case in (5) gives $\tilde{D} z^{\mu} = 0$ for $0 \leq \mu \leq n - 1$. Now use (5) for $\mu \geq n$:

$$
\tilde{D} M z^m = \sum_{\mu = n}^{[\frac{n}{2}]} \frac{\prod_{j=1}^{[\frac{n}{2}]} \prod_{j=1}^{[\frac{n}{2}]} c_{j,m-\nu+n+j}}{\prod_{j=1}^{[\frac{n}{2}]} \prod_{j=1}^{[\frac{n}{2}]} c_{j,m-\nu+n+j}} d_{m-[\frac{n}{2}]n,\mu-[\frac{n}{2}]n} \left( \prod_{j=1}^{[\frac{n}{2}]} c_{j,m-\nu+n+j} \right) z^{\mu-n}. \tag{21}
$$

It is suitable to separate the sum as $\sum_{\mu = n}^{2n-1} \left( \prod_{j=1}^{[\frac{n}{2}]} c_{j,m-\nu+n+j} \right) \left( \prod_{j=1}^{[\frac{n}{2}]} \prod_{j=2}^{[\frac{n}{2}]} c_{j,m-\nu+n+j} \right) d_{m-[\frac{n}{2}]n,\mu-[\frac{n}{2}]n} z^{\mu-n}$. In the first sum the whole denominator will be canceled with the product in brackets since $[\frac{n}{2}] = 1$, but in the second sum, after canceling $\prod_{j=1}^{n} c_{j,m-\nu+n+j}$, the denominator will have $n$ factors less than the numerator (without $\nu = 1$):

$$
\tilde{D} M z^m = \sum_{\mu = n}^{2n-1} \left( \prod_{j=1}^{[\frac{n}{2}]} c_{j,m-\nu+n+j} \right) d_{m-n,\mu-n} z^{\mu-n} \tag{22}
$$

$$
+ \sum_{\mu = 2n}^{(\frac{n}{2})+n-1} \left( \prod_{j=1}^{[\frac{n}{2}]} c_{j,m-\nu+n+j} \right) \left( \prod_{j=1}^{[\frac{n}{2}]} \prod_{j=2}^{[\frac{n}{2}]} c_{j,m-\nu+n+j} \right) d_{m-[\frac{n}{2}]n,\mu-[\frac{n}{2}]n} z^{\mu-n}. \tag{23}
$$

It remains to replace $\mu$ by $\mu + n$ and $\nu$ by $\nu + 1$:

$$
\tilde{D} M z^m = \sum_{\mu = 0}^{n-1} \left( \prod_{j=1}^{n} c_{j,m-n+j} \right) d_{m-n,\mu-n} z^{\mu} \tag{24}
$$

$$
+ \sum_{\mu = n}^{(\frac{n}{2})+n-1} \left( \prod_{j=1}^{[\frac{n}{2}]} c_{j,m-n+j} \right) \left( \prod_{j=1}^{n} \prod_{j=1}^{n} c_{j,m-(\nu+1)n+j} \right) d_{m-[\frac{n}{2}]n,\mu-[\frac{n}{2}]n} z^{\mu}. \tag{25}
$$

After the obvious simplifications this representation of $\tilde{D} M z^m$ coincides with the representation (19) of $M \tilde{D} z^m$ which proves the sufficiency of (2) and thus the theorem.

3. Particular cases

Example 1. Let us note that as a simplest particular case of the Dunkl operator, when all parameters $k_j$, $j = 1, 2, \ldots, n$, of the Dunkl operators $D_j = D_{k_j}$ are taken to be 0, one can have the $n$-th power $D^n$ of the classical differentiation operator $D f(z) = D f(z) = \frac{df(z)}{dz}$. Then $c_{j,m} = m, j = 1, 2, \ldots, n$, and Theorem 3 describes the commutant of $D^n$ as:

$$
M f(z) = \sum_{\mu = 0}^{n-1} \sum_{m = 0}^{\infty} a_m d_{m,n} z^{\mu} + \sum_{\mu = n}^{\infty} \sum_{m = [\frac{m}{2}]}^{\infty} a_m \frac{m - [\frac{m}{2}] n + 1}{\mu - [\frac{m}{2}] n + 1} d_{m-[\frac{m}{2}]n,\mu-[\frac{m}{2}]n} z^{\mu}. \tag{24}
$$

Similar results for $D$, its powers, and generalizations of $D$ are given by some Russian mathematicians. In particular, in [4] (5.1) one can find such theorem and also additional bibliography.

Example 2. If we take not a composition, but a single Dunkl operator with parameter $k > 0$, i.e. $n = 1$, then the representation of Comm($D_k$) given by M.S. Hristova from [5] is obtained:

$$
M f(z) = \sum_{m = 0}^{\infty} a_m d_{m,n} z^{\mu} + \sum_{\mu = 1}^{\infty} \sum_{m = \mu}^{\infty} a_m \frac{c_m \cdots c_{m-\mu+1}}{c_{\mu} \cdots c_1} d_{m-\mu} z^{\mu}. \tag{25}
$$
Example 3. If \( n \geq 1 \) is arbitrary, but all parameters of the Dunkl operators \( D_j = D_{k_j} \) in the composition \( \tilde{D} = D_1 D_2 \ldots D_n \) are equal, i.e. \( k_1 = k_2 = \ldots = k_n = k > 0 \), then our result reduces to the representation due to M.S. Hristova in [6]:

\[
Mf(z) = \sum_{\mu=0}^{n-1} \sum_{m=0}^{\infty} a_m d_{m,\mu} z^\mu + \sum_{\mu=n}^{\infty} \sum_{m=0}^{\infty} a_m \frac{c_m \ldots c_{m-[\frac{\mu}{n}]+1} d_{m-[\frac{\mu}{n}]+1} d_{m-[\frac{\mu}{n}]-[\frac{\mu}{n}]-[\frac{\mu}{n}]}}{c_{\mu} \ldots c_{\mu-[\frac{\mu}{n}]+1}} \cdot z^\mu.
\] (26)

Final notes. A different description of the commutant \( \text{Comm}(D_k) \) of the first power of the Dunkl operator in the space of the continuous functions on the real line \( \mathbb{R} \) is given in [2], based on the convolutional approach (see Dimovski [1]). It depends on an arbitrary continuous linear functional \( \Phi : C(\mathbb{R}) \to \mathbb{C} \). Note, that Theorem 3 also allows in the case of composition of \( n \) Dunkl operators to choose arbitrarily \( n \) systems of constants \( d_{m,\mu} \), \( 0 \leq \mu \leq n-1 \), \( m = 0, 1, 2, \ldots \).

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References


