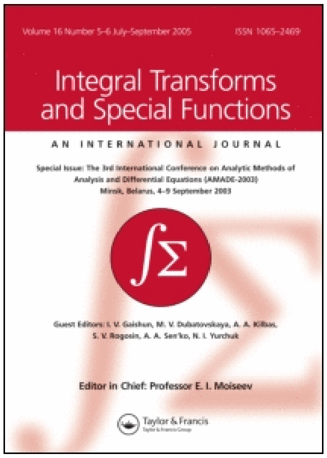

Integral Transforms and Special Functions

Commutants of the Euler operator and corresponding mean-periodic functions

Integral Transforms and Special Functions
Vol. 18, No. 2, February 2007, 117–131

Taylor & Francis
Taylor & Francis Group

Commutants of the Euler operator and corresponding mean-periodic functions

IVAN H. DIMOVSKI and VALENTIN Z. HRISTOV*

Bulgarian Academy of Sciences, Institute of Mathematics and Informatics, Section Complex Analysis, Acad. G. Bonchev Str., Block 8, 1113 Sofia, Bulgaria

(Received 17 May 2006)

The Euler operator $\delta = t(\mathrm{d}/\mathrm{d}t)$ is considered in the space $C = C(\mathbb{R}_+)$, $\mathbb{R}_+ = (0, \infty)$, and the operators $M: C \to C$ such that $M\delta = \delta M$ in $C^1(\mathbb{R}_+)$ are characterized. Next, for a non-zero linear functional $\Phi: C(\mathbb{R}_+) \to \mathbb{C}$ the continuous linear operators M with the invariant hyperplane $\Phi\{f\} = 0$ and commuting with δ in it are also characterized. Further, mean-periodic functions for δ with respect to the functional Φ are introduced and it is proved that they form an ideal in a corresponding convolutional algebra $(C(\mathbb{R}_+), *)$. As an application, unique mean-periodic solutions of Euler differential equations are characterized.

Keywords: Commutant; Riesz–Markov theorem; Invariant hyperplane; Convolutional algebra; Multiplier; Cyclic element; Mean-periodic function

Mathematics Subject Classification: 47B38; 47B37

1. Introduction

Compared with the case of differentiation operator $D = \mathrm{d}/\mathrm{d}t$ in a space C of continuous functions, the problem of characterizing the continuous linear operators $M: C \to C$ commuting with the Euler operator $\delta = t(\mathrm{d}/\mathrm{d}t)$, *i.e.* such that

$$M\delta = \delta M$$

in C^1, had not been so intensively treated as the corresponding problem for D. Here we can mention only the classical book of Levin [1, Ch. 8 and 9, Theorem 20, pp. 379–380], where δ is considered in spaces of entire functions.

In the operational calculus developed in Elizarraraz and Verde-Star [2] in fact some operators commuting with the Euler operator are found.

Due to the analogy of the considerations for δ and D, a short survey of the results for differentiation operator will be made.

*Corresponding author. Email: valhrist@bas.bg

Integral Transforms and Special Functions
ISSN 1065-2469 print/ISSN 1476-8291 online
http://www.tandf.co.uk/journals
DOI: 10.1080/10652460601092246

Bourbaki [3, Chapter 6] seems to be the first to characterize the linear continuous operators $M: C(\mathbb{R}) \to C(\mathbb{R})$ with $MD = DM$ in $C^1(\mathbb{R})$. These are the operators of the form

$$Mf(t) = \Phi_\tau\{f(t+\tau)\},$$

where Φ is a linear functional on $C(\mathbb{R})$. According to Riesz–Markov theorem ([4, Theorem 4.10.1]) Φ has the form

$$\Phi(f) = \int_\alpha^\beta f(\tau)\,\mathrm{d}\sigma(\tau),$$

where $-\infty < \alpha < \beta < \infty$ and $\sigma(\tau)$ is a Radon measure.

Delsarte [5] introduced the space of the mean-periodic functions determined by the functional Φ as the kernel space of M. For details see also Schwartz [6].

One of the authors (Dimovski [7]) had found the linear continuous operators $M: C(\mathbb{R}) \to C(\mathbb{R})$, such that the subspace $C_\Phi = \{f \in C(\mathbb{R}), \Phi(f) = 0\}$ is an invariant subspace of M and M commutes with D in C^1_Φ. It happened that these are the operators of the form

$$Mf = \mu f(t) + m * f,$$

where μ =const, $m \in C(\mathbb{R})$, and $*$ is the operation

$$(f * g)(t) = \Phi_\tau\left\{\int_\tau^t f(t+\tau-\sigma)\,g(\sigma)\,\mathrm{d}\sigma\right\}.$$

Quite natural is the question about the relationship between the two types of commutants. A partial answer is given by the following theorem (Dimovski and Skórnik [8, 9]):

The space of the mean-periodic functions determined by the functional Φ forms an ideal in the convolutional algebra $(C(\mathbb{R}), *)$.

Similar results for the Pommiez operator $\Delta f(z) = [f(z) - f(0)]/z$ are presented by Dimovski and Hristov [10].

An interesting historical survey about commutants of differentiation operator and related operators like the Euler one can be found in the book of Korobeinik [11].

2. General commutant

THEOREM 2.1 *A linear continuous operator $M : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M: C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ commutes with $\delta = t(\mathrm{d}/\mathrm{d}t)$ in $C^1(\mathbb{R}_+)$ iff it admits a representation of the form*

$$(Mf)(t) = \Phi_\tau\{f(t\tau)\} \tag{1}$$

with a continuous linear functional $\Phi: C(\mathbb{R}_+) \to \mathbb{C}$.

Proof Consider the one-parameter family T^τ, $0 < \tau < \infty$, of the shift operators defined by

$$(T^\tau f)(t) := f(t\tau), \quad 0 < \tau < \infty. \tag{2}$$

Each of them commutes with $\delta = t(\mathrm{d}/\mathrm{d}t)$ in $C^1(\mathbb{R}_+)$. Indeed,

$$(\delta T^\tau f)(t) = tf'(t\tau)\tau = t\tau f'(t\tau) = (\delta f)(t\tau) = (T^\tau \delta f)(t).$$

■

LEMMA 2.2 *A linear operator $M\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M\colon C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ commutes with $\delta = t(\mathrm{d}/\mathrm{d}t)$ in $C^1(\mathbb{R}_+)$ iff M commutes with T^τ for all τ, $0 < \tau < \infty$.*

Proof First a 'multiplicative' version of the Taylor formula is needed. Let f be a polynomial and g be the function defined by

$$g(x) = f(\mathrm{e}^x).$$

Then

$$f(t\tau) = g(\ln(t\tau)) = g(\ln t + \ln \tau).$$

Denote $x = \ln t$ and $\xi = \ln \tau$, *i.e.* $t = \mathrm{e}^x$ and $\tau = \mathrm{e}^\xi$, and apply the usual Taylor formula for g:

$$f(t\tau) = g(x + \xi) = \sum_{n=0}^{\infty} \frac{g^{(n)}(x)}{n!}\xi^n. \tag{3}$$

Then

$$g'(x) = \frac{\mathrm{d}g(x)}{\mathrm{d}x} = \frac{\mathrm{d}g(\ln t)}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\mathrm{d}f(t)}{\mathrm{d}t}\cdot\frac{\mathrm{d}\mathrm{e}^x}{\mathrm{d}x} = f'(t)\mathrm{e}^x = tf'(t) = (\delta f)(t). \tag{4}$$

Further,

$$g''(x) = (\delta^2 f)(t), \ldots, g^{(n)}(x) = (\delta^n f)(t), \ldots \tag{5}$$

Substituting (4) and (5) in (3) gives the desired 'multiplicative' Taylor formula:

$$(T^\tau f)(t) = f(t\tau) = \sum_{n=0}^{\infty} (\delta^n f)(t)\frac{(\ln \tau)^n}{n!}. \tag{6}$$

It is true for arbitrary polynomial $f(t)$.

Now suppose that M commutes with the Euler operator δ, *i.e.* $M\delta = \delta M$. Then, for every τ, $0 < \tau < \infty$, (6) implies

$$\begin{aligned}(MT^\tau f)(t) &= M\sum_{n=0}^{\infty}(\delta^n f)(t)\frac{(\ln \tau)^n}{n!} = \sum_{n=0}^{\infty}(M(\delta^n f))(t)\frac{(\ln \tau)^n}{n!}\\ &= \sum_{n=0}^{\infty}(\delta^n(Mf))(t)\frac{(\ln \tau)^n}{n!} = (T^\tau Mf)(t).\end{aligned}$$

In order to prove the opposite implication, suppose $MT^\tau = T^\tau M$ for every τ, $0 < \tau < \infty$, and for arbitrary polynomial $f(t)$, and reverse the order in the last chain of equalities as follows:

$$\sum_{n=0}^{\infty}(M(\delta^n f))(t)\frac{(\ln \tau)^n}{n!} = (M(T^\tau f))(t) = (T^\tau(Mf))(t) = \sum_{n=0}^{\infty}(\delta^n(Mf))(t)\frac{(\ln \tau)^n}{n!}.$$

The sums have to coincide for every τ and hence the coefficients of $(\ln \tau)^n$ are equal for arbitrary n. For $n = 1$, it follows that

$$(M(\delta f))(t) = (\delta(Mf))(t). \tag{7}$$

Assuming that (7) is true for polynomials, it follows that it is true for arbitrary $f \in C^1(\mathbb{R}_+)$ since f could be approximated by polynomials. The proof of the lemma is completed. ■

Proof of Theorem 2.1 It is a matter of a direct check to show that the operators of the form (1) commute with δ and here only the proof of the necessity is needed.

If M commutes with δ, then by the lemma

$$MT^{\tau} f(t) = T^{\tau} Mf(t), \quad 0 < \tau < \infty. \tag{8}$$

Applying the symmetry property

$$(T^{\tau} f)(t) = f(t\tau) = f(\tau t) = (T^{t} f)(\tau) \tag{9}$$

to the right hand side of (8) gives

$$(M(T^{\tau} f))(t) = (T^{t}(Mf))(\tau). \tag{10}$$

Define the linear functional Φ as

$$\Phi\{f\} := (Mf)(1).$$

Then, substituting 1 for t in (10) and taking into account that T^1 is the identity operator, one has

$$(M(T^{\tau} f))(1) = (T^{1}(Mf))(\tau) = (Mf)(\tau).$$

The left hand side is the value of the functional Φ for the function $g(t) = (T^{\tau} f)(t)$, and hence

$$(Mf)(\tau) = \Phi_{\sigma}\{(T^{\tau} f)(\sigma)\} = \Phi_{\sigma}\{(T^{\sigma} f)(\tau)\}.$$

Using (2) and (9), this is in fact the desired representation (1) of the commutant of δ with τ for t, and with the dumb variable σ instead of τ. This completes the proof. ∎

The abundance of the operators, commuting with δ in $C(\mathbb{R}_+)$ given by Theorem 2.1, is in sharp contrast to the set of linear operators commuting with δ in $C(\Delta)$, where Δ is a segment $[a, b] \subset \mathbb{R}_+$. Then the only such operators are the trivial ones:

$$Mf(t) = cf(t), \quad c = \text{const}.$$

Such a result for differentiation operator $\mathrm{d}/\mathrm{d}x$ is shown by Kahane [12]. The corresponding result for the Euler operator δ will be stated in the following theorem.

THEOREM 2.3 *Let $[a, b] \subset \mathbb{R}_+$. Then a continuous linear operator $M\colon C[a, b] \to C[a, b]$, such that $M\colon C^1[a, b] \to C^1[a, b]$, commutes with the Euler operator δ in $C^1[a, b]$ if and only if it is an operator of the form*

$$Mf(t) = cf(t),$$

with a constant c.

Proof Let $[a, b]$ be an arbitrary segment of $\mathbb{R}_+$ and let $M\delta = \delta M$ in $C^1[a, b]$. Consider the substitution $t = e^x$ as the transformation

$$Sf(t) = f(e^x) =: \tilde{f}(x). \tag{11}$$

Obviously S: $C[a, b] \to C[\ln a, \ln b]$ and S: $C^1[a, b] \to C^1[\ln a, \ln b]$. Then, denoting $D := d/dt$, one has as in (4)

$$S\delta f(t) = f'(e^x) = DSf(t). \tag{12}$$

It is supposed that

$$M\delta f(t) = \delta Mf(t).$$

Applying S on the left hand side and using (12) yields

$$SM\delta f(t) = S\delta Mf(t) = DSMf(t). \tag{13}$$

Denoting by $\tilde{M}$ the operator

$$\tilde{M} = SMS^{-1}. \tag{14}$$

It is easily seen from (13) and (12) that

$$\tilde{M}D\tilde{f}(x) = D\tilde{M}\tilde{f}(x). \tag{15}$$

This means that the conditions of Kahane's theorem [12] are fulfilled for the operator $\tilde{M}$ in $C[\ln a, \ln b]$ and the result is that

$$\tilde{M}\tilde{f}(x) = c\tilde{f}(x), \quad c = \text{const},$$

which in view of (11) and (14) gives also the desired

$$Mf(t) = cf(t), \quad c = \text{const}.$$

■

3. A general convolution related to the Euler operator

Basic for the theory of differentiation operator d/dt considered in a space $C(\Delta)$ of continuous functions on an interval Δ is the operation

$$(f * g)(t) = \Phi_\tau \left\{ \int_\tau^t f(t + \tau - \sigma) g(\sigma)\, d\sigma \right\}, \tag{16}$$

where Φ is a linear functional on $C(\Delta)$. Its properties are studied in detail in [13]. The operation (16) is bilinear, commutative, and associative in $C(\Delta)$. It generalizes the classical Duhamel convolution

$$(f * g)(t) = \int_0^t f(t - \tau) g(\tau)\, d\tau \tag{17}$$

when the functional Φ in (16) is $\Phi(f) = f(0)$.

In ref. [7] it is shown that any operator of the commutant of d/dt in $C(\Delta)$ with an invariant hyperplane $C_\Phi(\Delta) = \{f \in C(\Delta), \Phi(f) = 0\}$ has the form $Mf(t) = \mu f(t) + (m * f)(t)$ with $\mu = \text{const}$ and $m \in C(\Delta)$.

In order to extend this result to the Euler operator an analogue of the operation (16) is needed. In the literature only the analogue

$$(f * g)(t) = \int_1^t f\left(\frac{t}{\tau}\right) g(\tau) \frac{\mathrm{d}\tau}{\tau}$$

of the Duhamel convolution (17) is known (see [2]).

DEFINITION 3.1 *The analytic function*

$$E(\lambda) = \Phi_\tau(\tau^\lambda) \tag{18}$$

is said to be the Euler indicatrix of the functional Φ.

It is also convenient to denote for the rest of this article

$$\varphi_\lambda(t) = \frac{t^\lambda}{E(\lambda)} = \frac{t^\lambda}{\Phi_\tau(\tau^\lambda)}. \tag{19}$$

Here a 'multiplicative variant' of (16) is proposed.

THEOREM 3.2 *Let* Φ *be a continuous non-zero linear functional on* $C(\mathbb{R}_+)$. *Then the operation*

$$(f * g)(t) = \Phi_\tau\left\{\int_\tau^t f\left(\frac{t\tau}{\sigma}\right) g(\sigma) \frac{\mathrm{d}\sigma}{\sigma}\right\} \tag{20}$$

is a separately continuous, bilinear, commutative, and associative operation in $C(\mathbb{R}_+)$ *such that*

$$\Phi(f * g) = 0. \tag{21}$$

Proof According to Riesz–Markov theorem ([4, Theorem 4.10.1])

$$\Phi\{f\} = \int_\alpha^\beta f(\tau)\,\mathrm{d}\sigma(\tau)$$

with $\Delta = [\alpha, \beta] \subset \mathbb{R}_+$ and a Radon measure $\sigma(t)$. Hence, (20) is a separately continuous operation in $C(\Delta)$.

The bilinearity and the commutativity of the operation (20) are almost evident, whereas the associativity of (20) is by no means obvious and needs a proof.

Let $f(t) = t^\mu$ and $g(t) = t^\nu$. Then

$$\begin{aligned}\{t^\mu\} * \{t^\nu\} &= \Phi_\tau\left\{\int_\tau^t \frac{(t\tau)^\mu}{\sigma^\mu}\sigma^\nu \frac{\mathrm{d}\sigma}{\sigma}\right\} = t^\mu \Phi_\tau\left\{\tau^\mu \int_\tau^t \sigma^{\nu-\mu-1}\,\mathrm{d}\sigma\right\} \\ &= t^\mu \Phi_\tau\left\{\tau^\mu \frac{t^{\nu-\mu} - \tau^{\nu-\mu}}{\nu - \mu}\right\} = \frac{E(\mu)t^\nu - E(\nu)t^\mu}{\nu - \mu}.\end{aligned}$$

Using this expression, it follows that

$$(\{t^\mu\} * \{t^\nu\}) * \{t^\varkappa\} = \{t^\mu\} * (\{t^\nu\} * \{t^\varkappa\}) \tag{22}$$

because both sides of (22) have one and the same symmetric form

$$t^\mu \frac{E(\nu)E(\varkappa)}{(\mu-\nu)(\mu-\varkappa)} + t^\nu \frac{E(\varkappa)E(\mu)}{(\nu-\varkappa)(\nu-\mu)} + t^\varkappa \frac{E(\mu)E(\nu)}{(\varkappa-\mu)(\varkappa-\nu)}$$

with respect to μ, ν, and $\varkappa$. Then, (22) differentiated m, n, and k times with respect to μ, ν, and $\varkappa$ correspondingly, gives

$$(\{t^{\mu}(\ln t)^m\} * \{t^{\nu}(\ln t)^n\}) * \{t^{\varkappa}(\ln t)^k\} = \{t^{\mu}(\ln t)^m\} * (\{t^{\nu}(\ln t)^n\} * \{t^{\varkappa}(\ln t)^k\}).$$

Next, passing to the limits $\mu \to +0$, $\nu \to +0$, and $\varkappa \to +0$, one gets

$$(\{(\ln t)^m\} * \{(\ln t)^n\}) * \{(\ln t)^k\} = \{(\ln t)^m\} * (\{(\ln t)^n\} * \{(\ln t)^k\}).$$

But the bilinearity of (20) implies for arbitrary polynomials P, Q, and R

$$(\{P(\ln t)\} * \{Q(\ln t)\}) * \{R(\ln t)\} = \{P(\ln t)\} * (\{Q(\ln t)\} * \{R(\ln t)\}).$$

To finish this proof, note that if $t \in \mathbb{R}_+$ then $\ln t$ covers the whole real line $\mathbb{R}$. Then Weierstrass' theorem allows any function in $C(\mathbb{R}_+)$ to be approximated almost uniformly by polynomials of $\ln t$, $t > 0$, *i.e.* by a sequence uniformly convergent to the function on each segment $[a, b] \subset \mathbb{R}_+$. Due to the continuity of the functional Φ, the desired equality holds for every $f, g, h \in C(\mathbb{R}_+)$

$$(f * g) * h = f * (g * h).$$

The second statement (21) of the theorem can be checked as follows: The function

$$h(t, \tau) = \int_{\tau}^{t} f\left(\frac{t\tau}{\sigma}\right) g(\sigma) \frac{\mathrm{d}\sigma}{\sigma}$$

is antisymmetric with respect to t and τ, *i.e.* $h(t, \tau) = -h(\tau, t)$, and, hence

$$\begin{aligned} \Phi\{f * g\} &= \Phi_t\{(f * g)(t)\} = \Phi_t \Phi_\tau\{h(t, \tau)\} \\ &= \Phi_t \Phi_\tau\{-h(\tau, t)\} = -\Phi_t \Phi_\tau\{h(\tau, t)\} \\ &= -\Phi_\tau \Phi_t\{h(\tau, t)\} = -\Phi_t \Phi_\tau\{h(t, \tau)\} = -\Phi\{f * g\}. \end{aligned} \tag{23}$$

Here, the Fubini property of the functional Φ is used, *i.e.* the possibility of interchanging of Φ_t and Φ_τ. At the end, t and τ are also interchanged, since they are 'dumb' variables in the expression. Thus, the last chain of equalities gives $2\Phi\{f * g\} = 0$ and $\Phi\{f * g\} = 0$ holds. ■

4. The commutant of δ in an invariant hyperplane

In this section, another commutant of δ will be described. Here, it is supposed that the operators $M: C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ preserve $C^1(\mathbb{R}_+)$, *i.e.* $M: C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$, and additionally they preserve invariant also a hyperplane

$$C_\Phi := \{f \in C(\mathbb{R}_+) : \Phi\{f\} = 0\}, \tag{24}$$

i.e. $M: C_\Phi \to C_\Phi$, where $\Phi: C(\mathbb{R}_+) \to \mathbb{C}$ is an arbitrary non-zero linear functional.

The main result of this section is the explicit representation $Mf = \mu f + m * f$ of any linear continuous operator $M: C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M: C_\Phi \to C_\Phi$ and commuting with $\delta = t(\mathrm{d}/\mathrm{d}t)$ in $C_\Phi^1 := C_\Phi \cap C^1(\mathbb{R}_+)$.

To this end some auxilliary results will be considered.

LEMMA 4.1 *A linear operator* M: $C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ *with* M: $C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ *and* M: $C_\Phi(\mathbb{R}_+) \to C_\Phi(\mathbb{R}_+)$ *commutes with the Euler operator* δ *in* $C^1_\Phi(\mathbb{R}_+)$ *iff* M *commutes with* L_λ *in* $C(\mathbb{R}_+)$, *where* L_λ *is the right inverse in* $C(\mathbb{R}_+)$ *of the perturbed Euler operator* $\delta_\lambda = \delta - \lambda$, *satisfying the boundary condition* $\Phi(L_\lambda f) = 0$.

Proof First an explicit expression for L_λ will be found. Let λ be such that $E(\lambda) \neq 0$. Then

$$L_\lambda f(t) = \int_1^t \left(\frac{t}{\tau}\right)^\lambda f(\tau)\frac{\mathrm{d}\tau}{\tau} - \frac{t^\lambda}{E(\lambda)}\Phi_\tau\left\{\int_1^\tau \left(\frac{\tau}{\sigma}\right)^\lambda f(\sigma)\frac{\mathrm{d}\sigma}{\sigma}\right\}. \tag{25}$$

Indeed, the general solution of the linear differential equation $t(\mathrm{d}y/\mathrm{d}t) - \lambda y = f(t)$ is $y = t^\lambda\left(c + \int_1^t f(\tau)/\tau^{\lambda+1}\,\mathrm{d}\tau\right)$ with an arbitrary constant c. Then, using the condition $\Phi\{y\} = 0$, one obtains the value

$$c = -\frac{1}{E(\lambda)}\Phi_\tau\left\{\int_1^\tau \left(\frac{\tau}{\sigma}\right)^\lambda f(\sigma)\frac{\mathrm{d}\sigma}{\sigma}\right\}.$$

Now suppose that $ML_\lambda = L_\lambda M$ in $C(\mathbb{R}_+)$ and $f \in C^1_\Phi(\mathbb{R}_+)$. To prove that

$$h = (M\delta_\lambda - \delta_\lambda M)f = 0,$$

consider

$$L_\lambda h = L_\lambda M\delta_\lambda f - L_\lambda\delta_\lambda Mf = M(L_\lambda\delta_\lambda)f - (L_\lambda\delta_\lambda)Mf) = Mf - Mf = 0.$$

But $L_\lambda h = 0$ implies $\delta_\lambda L_\lambda h = 0$, *i.e.* $h = 0$. Hence $M\delta_\lambda f = \delta_\lambda Mf$.

Conversely, let $M\delta_\lambda f = \delta_\lambda Mf$ for every $f \in C^1_\Phi(\mathbb{R}_+)$. If $g \in C(\mathbb{R}_+)$, then there is a function $f \in C^1_\Phi(\mathbb{R}_+)$, for which $f = L_\lambda g$. After the substitution $f = L_\lambda g$ in $\delta_\lambda Mf = M\delta_\lambda f$, one gets

$$\delta_\lambda(ML_\lambda g) = M\delta_\lambda L_\lambda g = Mg.$$

Since $\Phi\{L_\lambda g\} = 0$, then $\Phi\{ML_\lambda g\} = 0$. But the solution of the equation $\delta_\lambda y = Mg$ with the condition $\Phi\{y\} = 0$ by definition is $y = L_\lambda(Mg)$, which implies

$$ML_\lambda g = L_\lambda Mg$$

in $C(\mathbb{R}_+)$, which completes the proof. ∎

LEMMA 4.2 *The operator* L_λ *given by* (25) *is a convolution operator of the form*

$$L_\lambda f = \varphi_\lambda * f = \left\{\frac{t^\lambda}{E(\lambda)}\right\} * f. \tag{26}$$

Proof The equality (26) can be checked directly using (21) and the representation $\int_\tau^t = \int_1^t - \int_1^\tau$. ∎

THEOREM 4.3 *The commutant of* δ *in the invariant hyperplane* C_Φ *coincides with the commutant of any of the operators* L_λ *in* $C(\mathbb{R}_+)$.

Proof Let $M\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be a linear operator commuting with L_λ for some $\lambda \in \mathbb{C}$, *i.e.* $ML_\lambda = L_\lambda M$. First, it will be proved that C_Φ is an invariant hyperplane for M. Indeed, let g be a function from $C(\mathbb{R}_+)$ and f be the solution of the problem

$$\delta f - \lambda f = g, \quad \Phi\{f\} = 0. \tag{27}$$

Then

$$L_\lambda M g = ML_\lambda g = Mf \tag{28}$$

and hence

$$Mg = (\delta - \lambda)Mf.$$

Using (27) this can be written as

$$M(\delta - \lambda) f = (\delta - \lambda)Mf$$

or, equivalently,

$$(M\delta) f = (\delta M) f.$$

It remains to show that $\Phi\{Mf\} = 0$. This follows using (28) and the representation (26) of L_λ as a convolutional operator, along with the property $\Phi\{p * q\} = 0$ for arbitrary $p, q \in C(\mathbb{R}_+)$ of the convolution (20).

Conversely, let $M\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ have the hyperplane C_Φ as an invariant subspace and let $M\delta = \delta M$ in C^1_Φ. One has to prove that $ML_\lambda = L_\lambda M$ for $\lambda \in \mathbb{C}$ with $E(\lambda) \neq 0$.

Let $f \in C(\mathbb{R}_+)$ be arbitrary and denote $h = (ML_\lambda - L_\lambda M) f$. Then

$$(\delta - \lambda)h = (\delta - \lambda)ML_\lambda f - Mf = M(\delta - \lambda)L_\lambda f - Mf = 0$$

and also

$$\Phi\{h\} = \Phi\{ML_\lambda f\} - \Phi\{L_\lambda Mf\} = 0,$$

according to our assumptions. Since λ is not an eigenvalue, *i.e.* $E(\lambda) \neq 0$, then $h = 0$, or

$$ML_\lambda f = L_\lambda Mf.$$

The proof is completed. ∎

Definition 4.4 *A linear operator $M\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is said to be a multiplier of the convolutional algebra $(C(\mathbb{R}_+), *)$ when for arbitrary $f, g \in C(\mathbb{R}_+)$ it holds*

$$M(f * g) = (Mf) * g.$$

Theorem 4.5 *A linear operator $M\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ with $M\colon C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$ is a multiplier of the convolution algebra $(C(\mathbb{R}_+), *)$ iff it has a representation of the form*

$$Mf(t) = \mu f(t) + (m * f)(t), \tag{29}$$

where $\mu = \text{const}$ and $m \in C(\mathbb{R}_+)$.

Proof The sufficiency is obvious. In order to prove the necessity, the notations from (18) and (19) will be used for convenience.

Let $M: C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ be an arbitrary multiplier of $(C(\mathbb{R}_+), *)$. Applying (26), one has

$$ML_\lambda f = M(\varphi_\lambda * f) = (M\varphi_\lambda) * f = \varphi_\lambda * Mf = L_\lambda Mf, \tag{30}$$

i.e. $ML_\lambda f = L_\lambda Mf$. Also, denoting $e_\lambda = M\varphi_\lambda$, one has $e_\lambda \in C^1(\mathbb{R}_+)$, and (30) gives

$$L_\lambda Mf = e_\lambda * f.$$

It remains to apply the operator $\delta_\lambda = \delta - \lambda$ and the definition of L_λ as the right inverse operator of δ_λ to obtain

$$Mf = \delta_\lambda(e_\lambda * f).$$

The right hand side can be represented in a different way using the identity

$$\delta_\lambda(u * v) = (\delta_\lambda u) * v + \Phi(u)v, \tag{31}$$

which can be checked directly. Then

$$(Mf)(t) = [(\delta_\lambda e_\lambda) * f](t) + \Phi(e_\lambda) f(t),$$

which is the representation (29) with $\mu = \Phi(e_\lambda) = \Phi\{M\varphi_\lambda\}$ and $m(t) = (\delta_\lambda e_\lambda)(t) = [\delta_\lambda M\varphi_\lambda](t)$. Thus, the necessity is proved. ■

THEOREM 4.6 *The function $\varphi_\lambda(t) = t^\lambda/E(\lambda)$ is a cyclic element of the operator L_λ.*

Proof Let $f \in C(\mathbb{R}_+)$ be arbitrarily chosen. It is needed to prove that there is a sequence of functions of the form

$$f_n(t) = \sum_{k=0}^{n} c_{nk} L_\lambda^k \varphi_\lambda(t), \quad n = 1, 2, \ldots$$

converging to $f(t)$ uniformly on any segment $[a, b]$ of $\mathbb{R}_+$.

First, it is easy to show by induction that

$$L_\lambda^k \varphi_\lambda(t) = t^\lambda p_k(\ln t), \tag{32}$$

where p_k is a polynomial of degree k, *i.e.* $p_k(\ln t) = \sum_{s=0}^{k} a_{ks}(\ln t)^s$.

Indeed, if $k = 1$, then by (26) and (20)

$$\begin{aligned} L_\lambda\varphi_\lambda(t) &= \left\{\frac{t^\lambda}{E(\lambda)}\right\} * \left\{\frac{t^\lambda}{E(\lambda)}\right\} = \frac{1}{E^2(\lambda)}\Phi_\tau\left\{\int_\tau^t \left(\frac{t\tau}{\sigma}\right)^\lambda \sigma^\lambda \frac{\mathrm{d}\sigma}{\sigma}\right\} \\ &= \frac{1}{E^2(\lambda)} t^\lambda \Phi_\tau\left\{\tau^\lambda \int_\tau^t \frac{\mathrm{d}\sigma}{\sigma}\right\} = t^\lambda\left[\frac{\Phi_\tau\{\tau^\lambda\}}{E^2(\lambda)}\ln t - \frac{\Phi_\tau\{\tau^\lambda \ln\tau\}}{E^2(\lambda)}\right]. \end{aligned}$$

Next, the inductive step will be made. Suppose that

$$L_\lambda^{k-1}\varphi_\lambda(t) = t^\lambda p_{k-1}(\ln t).$$

Then

$$\begin{aligned} L_\lambda^k\varphi_\lambda(t) &= L_\lambda(L_\lambda^{k-1}\varphi_\lambda(t)) = \left\{\frac{t^\lambda}{E(\lambda)}\right\} * L_\lambda^{k-1}\varphi_\lambda(t) \\ &= \frac{1}{E(\lambda)}\Phi_\tau\left\{\int_\tau^t \left(\frac{t\tau}{\sigma}\right)^\lambda \sigma^\lambda p_{k-1}(\ln\sigma)\frac{\mathrm{d}\sigma}{\sigma}\right\} \\ &= \frac{1}{E(\lambda)} t^\lambda \Phi_\tau\left\{\tau^\lambda \int_\tau^t p_{k-1}(\ln\sigma)\,\mathrm{d}\ln\sigma\right\}. \end{aligned}$$

The integration of p_{k-1} gives a polynomial q_k of $\ln t$ of degree k and the above chain of equalities can be continued as

$$L_\lambda^k \varphi_\lambda(t) = \frac{1}{E(\lambda)} t^\lambda \Phi_\tau \left\{\tau^\lambda [q_k(\ln t) - q_k(\ln \tau)]\right\}$$
$$= t^\lambda \left[\frac{\Phi_\tau\{\tau^\lambda\}}{E(\lambda)} q_k(\ln t) - \frac{\Phi_\tau\{\tau^\lambda q_k(\ln \tau)\}}{E(\lambda)}\right],$$

where the expression in the square brackets is obviously a polynomial p_k of $\ln t$ of degree k, as desired.

Now let $f \in C(\mathbb{R}_+)$ be arbitrarily chosen. Consider the function $\tilde{f}(t) = f(t)/t^\lambda$, which is again in $C(\mathbb{R}_+)$. Making the substitution $t = \mathrm{e}^x$, $x = \ln t$, the new function $g(x) = \tilde{f}(t)$ is in $C(-\infty, \infty)$. By Weierstrass' theorem, g can be approximated almost uniformly on $(-\infty, \infty)$ by a sequence of polynomials $\{r_n(x)\}_{n=1}^\infty$, $r_n(x) = \sum_{k=0}^n b_{nk} x^k$, i.e. the convergence is uniform on any segment $[a, b] \subset (\mathbb{R}_+)$. Returning to the old variable, $\widetilde{f}(t)$ can be approximated by the sequence of polynomials $\{r_n(\ln t) = \sum_{k=0}^n b_{nk}(\ln t)^k\}_{n=1}^\infty$. Finally, multiplying by t^λ and using (32), the desired approximation of $f(t)$ on $(\mathbb{R}_+)$ follows from the representation

$$f_n(t) = t^\lambda r_n(\ln t) = \sum_{k=0}^n b_{nk} t^\lambda (\ln t)^k = \sum_{k=0}^n c_{nk} t^\lambda p_k(\ln t) = \sum_{k=0}^n c_{nk} L_\lambda^k \varphi_\lambda(t).$$

The new coefficients c_{nk} can be calculated from the old ones b_{nk}. Thus, φ_λ is a cyclic element of L_λ in $C(\mathbb{R}_+)$. ■

THEOREM 4.7 *A linear operator M: $C(\mathbb{R}_+) \to C(\mathbb{R}_+)$, such that M: $C^1(\mathbb{R}_+) \to C^1(\mathbb{R}_+)$, and with an invariant hyperplane $C_\Phi = \{f \in C(\mathbb{R}_+) : \Phi\{f\} = 0\}$ commutes with δ in C_Φ^1 if and only if it has a representation of the form*

$$(Mf)(t) = \mu f(t) + (m * f)(t) \tag{33}$$

with a constant $\mu \in \mathbb{C}$ and $m \in C(\mathbb{R}_+)$.

Proof Since $\Phi\{f * g\} = 0$ for $f, g \in C(\mathbb{R}_+)$ (see (10)), then each operator of the form (33) has C_Φ as an invariant subspace. It commutes with δ in C_Φ^1. Indeed, if $f \in C_\Phi^1$, then (31) gives

$$\delta(m * f) = m * \delta f + \Phi\{f\} m$$

and, using (33),

$$\delta M f = \mu \delta f + m * (\delta f) + \Phi\{f\} m = \mu \delta f + m * (\delta f) = M \delta f.$$

The sufficiency is proved.

In order to prove the necessity of (33), according to Lemma 4.1, $M L_\lambda = L_\lambda M$ for $\lambda \in \mathbb{C}$ with $E(\lambda) \neq 0$. As it is shown in [13] (Theorem 1.3.11, p. 33), the commutant of L_λ coincides with the ring of the multipliers of the convolution algebra $(C(\mathbb{R}_+), *)$ since L_λ has a cyclic element. By Theorem 4.6 such a cyclic element is the function $\varphi_\lambda(t) = t^\lambda/E(\lambda)$ for which $L_\lambda f = \varphi_\lambda * f$. The proof is completed. ■

Remark The constant μ and the function $m \in C(\mathbb{R}_+)$ in (29) are uniquely determined. Indeed, assume that $\mu f + m * f = \mu_1 f + m_1 * f$. Take f such that $\Phi(f) \neq 0$. Then, (23) implies $\mu \Phi(f) = \mu_1 \Phi(f)$, and hence $\mu = \mu_1$. From $m * f = m_1 * f$ for arbitrary $f \in C(\mathbb{R}_+)$ it follows that $(m - m_1) * f = 0$, and hence $m = m_1$.

5. Mean-periodic functions for the Euler operator

DEFINITION 5.1 *A function $f \in C(\mathbb{R}_+)$ is said to be mean-periodic for the Euler operator with respect to the linear functional Φ if*

$$\Phi_\tau\{f(t\tau)\} = 0$$

identically in $\mathbb{R}_+$.

It is clear that the mean-periodic functions with respect to Φ form the kernel space of the operator

$$Mf(t) = \Phi_\tau\{f(t\tau)\}$$

commuting with the Euler operator δ in $C(\mathbb{R}_+)$.

Now a connection between the mean-periodic functions and the convolutional algebra $(C(\mathbb{R}_+), *)$ will be shown.

THEOREM 5.2 *The mean-periodic functions for the Euler operator δ with respect to any non-zero functional Φ: $C(\mathbb{R}_+) \to \mathbb{C}$ form an ideal in the convolutional algebra $(C(\mathbb{R}_+), *)$.*

Proof One need prove only that the convolutional product $(f * g)(t)$ of a mean-periodic function f and an arbitrary function $g \in C(\mathbb{R}_+)$ is a mean-periodic function, too, *i.e.* it is given that $\Phi_\tau\{f(t\tau)\} = 0$ and then $\Phi_\tau\{(f * g)(t\tau)\} = 0$ is to be shown. By (20)

$$(f * g)(t\tau) = \Phi_\sigma\left\{\int_\sigma^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{\mathrm{d}\eta}{\eta}\right\}$$

and

$$\begin{aligned}
\Phi_\tau\{(f * g)(t\tau)\} &= \Phi_\tau\Phi_\sigma\left\{\int_\sigma^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{\mathrm{d}\eta}{\eta}\right\} \\
&= \Phi_\tau\Phi_\sigma\left\{\int_\sigma^{\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{\mathrm{d}\eta}{\eta}\right\} \\
&\quad + \Phi_\tau\Phi_\sigma\left\{\int_\tau^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{\mathrm{d}\eta}{\eta}\right\}. \qquad (34)
\end{aligned}$$

Interchanging τ and σ in the first term of (34) and using the Fubini commutational property of the functionals yields

$$\begin{aligned}
\Phi_\tau\Phi_\sigma\left\{\int_\sigma^{\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{\mathrm{d}\eta}{\eta}\right\} &= \Phi_\sigma\Phi_\tau\left\{\int_\tau^{\sigma} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{\mathrm{d}\eta}{\eta}\right\} \\
&= \Phi_\sigma\Phi_\tau\left\{-\int_\sigma^{\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{\mathrm{d}\eta}{\eta}\right\} \\
&= -\Phi_\tau\Phi_\sigma\left\{\int_\sigma^{\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{\mathrm{d}\eta}{\eta}\right\},
\end{aligned}$$

thus obtaining

$$\Phi_\tau\Phi_\sigma\left\{\int_\sigma^{\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta)\frac{\mathrm{d}\eta}{\eta}\right\} = 0. \qquad (35)$$

The second term in (34) also vanishes

$$\Phi_\tau \Phi_\sigma \left\{ \int_\tau^{t\tau} f\left(\frac{t\tau\sigma}{\eta}\right) g(\eta) \frac{\mathrm{d}\eta}{\eta} \right\} = \Phi_\tau \left\{ \int_\tau^{t\tau} \Phi_\sigma \left\{ f\left(\frac{t\tau\sigma}{\eta}\right)\right\} g(\eta) \frac{\mathrm{d}\eta}{\eta} \right\} = 0 \qquad (36)$$

since f is mean-periodic and hence

$$\Phi_\sigma \left\{ f\left(\frac{t\tau\sigma}{\eta}\right)\right\} = 0.$$

Finally, equations (34)–(36) give the desired result $\Phi_\tau\{(f * g)(t\tau)\} = 0$. ■

6. Application to the Euler differential equation

Now Theorem 5.2 will be applied to find necessary and sufficient conditions in order the Euler differential equation

$$P(\delta)y(t) = f(t), \quad 0 < t < \infty, \qquad (37)$$

to have a unique mean-periodic solution with respect to a non-zero linear functional Φ in $C(\mathbb{R}_+)$. Here, $\delta = t(\mathrm{d}/\mathrm{d}t)$ is the Euler operator and $P(\mu) = a(\mu - \mu_1)(\mu - \mu_2)\cdots(\mu - \mu_k)$ is a polynomial.

THEOREM 6.1 *In order for the Euler differential equation* (37) *to have a unique mean-periodic solution with respect to a non-zero linear functional* Φ *in* $C(\mathbb{R}_+)$, *it is necessary and sufficient no roots of the equation* $P(\lambda) = 0$ *to be roots of the Euler indicatrix* $E(\lambda) = \Phi_\tau(\tau^\lambda)$.

Proof Consider the Euler differential equation (37). It is clear that in order for y to be a mean-periodic solution, the right hand side, *i.e.* the function $f(t)$, should be mean-periodic, too. Formally, let $Mf(t) = \Phi_\tau\{f(t\tau)\}$. Applying M to (37) and using the commutativity of $\delta = t(\mathrm{d}/\mathrm{d}t)$ and M yields

$$P(\delta)My(t) = Mf(t).$$

Then from $My = 0$ it follows that $Mf = 0$, *i.e.* the requirement f to be mean-periodic is a necessary condition for existing of a mean-periodic solution y. It can be shown that it is also a sufficient condition, but in general the solution may not be unique. Indeed, if a root μ of the equation $P(\lambda) = 0$ is a root of the Euler indicatrix $E(\lambda)$, then the function t^μ is a solution of the homogeneous equation $P(\delta)u = 0$, and hence the uniqueness of the solution holds no more.

Now it will be shown that if neither of the roots $\mu_1, \mu_2, \ldots, \mu_k$ of the equation $P(\lambda) = 0$ is a root of the Euler indicatrix $E(\lambda) = \Phi_\tau\{\tau^\lambda\}$, then there exists a unique mean-periodic solution of the Euler equation $P(\delta)y = f$, provided f is a mean-periodic function with respect to Φ.

Assuming that y is a mean-periodic solution of (37), an explicit expression for y will be obtained. Let P be a polynomial of degree k

$$P(\mu) = a(\mu - \mu_1)(\mu - \mu_2)\cdots(\mu - \mu_k).$$

From the assumption that y is a mean-periodic solution it follows that

$$\Phi\{y\} = \Phi\{\delta y\} = \cdots = \Phi\{\delta^{k-1} y\} = 0. \tag{38}$$

Indeed, the mean-periodicity of y means that

$$\Phi_\tau\{y(t\tau)\} = 0.$$

Applying the operator δ to this identity with respect to t, Theorem 2.1 gives

$$\Phi_\tau\{(\delta^n y)(t\tau)\} = 0, \quad n = 1, 2, \ldots, k-1.$$

It remains to put $t = 1$ in order to obtain the boundary conditions (38).

Next, unique solution of (37) is

$$y = \frac{1}{a} L_{\mu_k} L_{\mu_{k-1}} \cdots L_{\mu_1} f(t). \tag{39}$$

Indeed, the equation (37) can be represented as

$$(\delta - \mu_1)[(\delta - \mu_2)\cdots(\delta - \mu_k) y(t)] = \frac{1}{a} f(t).$$

Denoting the square brackets by $u_1(t)$ yields

$$\delta u_1 - \mu_1 u_1 = \frac{1}{a} f,$$

for u_1 with $\Phi\{u_1\} = 0$, as it follows from (38). This equation has the unique solution $u_1 = (1/a) L_{\mu_1} f$ with L_{μ_1} defined as in Lemma 4.1. Next solve

$$\delta u_2 - \mu_2 u_2 = u_1, \quad \Phi\{u_2\} = 0,$$

for $u_2(t) = (\delta - \mu_3)\cdots(\delta - \mu_k) y(t)$ with the unique solution $u_2 = L_{\mu_2} u_1$. Continuing in the same manner one gets the unique solution (39) of the initial equation (37). Now it is easy to verify that (39) is indeed a mean-periodic solution. It can be written in the form of convolutional product using Lemma 4.2:

$$y = \frac{1}{a} L_{\mu_k} L_{\mu_{k-1}} \cdots L_{\mu_1} f(t) = \left(\frac{1}{a} \varphi_{\mu_k} * \varphi_{\mu_{k-1}} * \cdots * \varphi_{\mu_1}\right) * f = \varphi * f \tag{40}$$

with $\varphi := (1/a) \varphi_{\mu_k} * \varphi_{\mu_{k-1}} * \cdots * \varphi_{\mu_1}$. It remains to use Theorem 5.2 to assert that the mean-periodicity of f implies the mean-periodicity of y. ■

References

[1] Levin, B.Ya., 1964, *Distribution of Zeros of Entire Functions, Translations of Mathematical Monographs*, Vol. 5 (AMS: Providence, Rhode Island).

[2] Elizarraraz, D. and Verde-Star, L., 1977, On a class of differential equations that contains the equations of Euler and Chebyshev. *Advances in Applied Mathematics*, **19**, 514–528.

[3] Bourbaki, N., 1951, *Éléments de mathématique, Livre IV, Fonctions d'une variable réelle, Théorie élémentaire* (Paris: Hermann).

[4] Edwards, R.E., 1965, *Functional Analysis. Theory and Applications* (New York: Holt, Rinehart and Winston).

[5] Delsarte, J., 1935, Les fonctions moyenne-périodiques. *Journal De Mathematiques Pures et Appliquees*, **14**, 403–453.

[6] Schwartz, L., 1947, Théorie générale des fonctions moyenne-périodiques. *Annals of Mathematics*, **48**, 857–929.

[7] Dimovski, I.H., 1978, Representation of operators which commute with differentiation in an invariant hyperplane. *Comptes rendus de l'Academie Bulgare des Sciences*, **31**(10), 1245–1248.

[8] Dimovski, I.H. and Skórnik, K., 2000, Mean-periodic operational calculi, *Algebraic Analysis and Related Topics, Banach Center Publications*, Vol. 53 (Warszawa: Institute of Mathematics, Polish Academy of Sciences) pp. 105–112.

[9] Dimovski, I.H. and Skórnik, K., 2001, Mean-periodic operational calculi. Singular cases. *Fractional Calculus & Applied Analysis*, **4**(2), 237–243.

[10] Dimovski, I.H. and Hristov, V.Z., 2005, Commutants of the Pommiez operator. *International Journal of Mathematics and Mathematical Sciences*, **2005** (8), 1239–1251.

[11] Korobeinik, Yu.F., 1983, *Shift Operators on Numerical Families* (Rostov: Rostov University) (in Russian).

[12] Kahane, C., 1969, On operators commuting with differentiation. *American Mathematical Monthly*, **76**, 171–173.

[13] Dimovski, I.H., 1990, *Convolutional calculus* (Dordrecht: Kluwer). (or 1982, Az Buki, Bulg. Math. Monographs, 2 (Sofia: Publ. House of Bulg. Acad. Sci.)).