Mean-periodic Solutions of Euler Differential Equations

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Abstract

Let \( \Phi : C(\mathbb{R}_+) \to \mathbb{C} \) be a given nonzero linear functional. We are looking for mean-periodic solutions for the Euler operator \( \delta = t \frac{d}{dt} \) and the functional \( \Phi \) of equations of the form \( P(\delta)y(t) = f(t) \) with a polynomial \( P \). A function \( f \) is called mean-periodic with respect to \( \Phi \) iff \( \Phi \{ f(t+\tau) \} = 0 \). A necessary condition for existence of such a solution is the requirement the right hand side \( f \) to be mean-periodic. Then, the problem is equivalent to the following nonlocal Cauchy problem: \( P(\delta)y(t) = f(t), \Phi \{ \delta^k y \} = 0, k = 0, 1, \ldots, \deg P - 1 \). The solution of the last problem has the following Duhamel-type form \( y = \delta(G \ast f) \), where \( G \) is the solution of the nonlocal Cauchy problem for \( f(t) \equiv 1 \) and \( \ast \) denotes the convolution product in \( C(\mathbb{R}_+) \)

\[
(f \ast g)(t) = \Phi \left\{ \int_\tau^t f \left( \frac{t\tau}{\sigma} \right) g(\sigma) \frac{d\sigma}{\sigma} \right\} .
\]

1. Mean-periodic functions for the Euler operator with respect to a functional

As it is well-known, the notion of mean-periodic function for the differentiation operator \( \frac{d}{dt} \) with respect to a linear functional \( \Phi : C(\mathbb{R}) \to \mathbb{C} \) is introduced in 1935 by J. Delsarte [1]. An extensive study of it is proposed in 1947 in the L. Schwartz’ memoir [4]. Let us remind this basic definition.

**Definition 1** A function \( f \in C(\mathbb{R}) \) is said to be mean-periodic with respect to a linear functional \( \Phi : C(\mathbb{R}) \to \mathbb{C} \) iff \( \Phi \{ f(t+\tau) \} = 0 \) identically.
In Dimovski and Skórnik [3] an operational method is considered for solving linear ordinary differential equations with constant coefficients in mean-periodic functions with respect to a given functional Φ.

Here we extend this approach to Euler equations.

**Definition 2** A function \( f \in C(\mathbb{R}_+) \to \mathbb{C} \), where \( \mathbb{R}_+ = (0, \infty) \), is said to be mean-periodic for the Euler operator \( \delta = t \frac{d}{dt} \) with respect to a linear functional \( \Phi : C(\mathbb{R}_+) \to \mathbb{C} \) iff \( \Phi_\tau \{ f(t\tau) \} = 0 \) identically.

In the sequel a basic role is played by the following convolution product introduced in Dimovski and Skórnik [3] and Dimovski and Hristov [2]:

**Theorem 1** ([3]). The operation

\[
(f \ast g)(t) = \Phi_\tau \left\{ \int_{t\tau}^t f\left(\frac{\sigma t}{t}\right) g(\sigma) \frac{d\sigma}{\sigma} \right\}
\]

converts \( C(\mathbb{R}_+) \) into a commutative and associative algebra.

For the sake of completeness we supply a sketch of the proof. The commutativity is almost obvious. Let us verify the associativity. It is possible to do this by a direct check, but an easier way is to verify it at first for polynomials and then to use approximation argument.

Let \( f(t) = t^\mu \) and \( g(t) = t^\nu \). Then

\[
\{t^\mu\} \ast \{t^\nu\} = \Phi_\tau \left\{ \int_{t\tau}^t \frac{(t\tau)^\mu}{\sigma^\mu} \frac{d\sigma}{\sigma} \right\} = t^\mu \Phi_\tau \left\{ \tau^\mu \int_{t\tau}^t \sigma^{\nu-\mu-1} d\sigma \right\} =
\]

\[
t^\mu \Phi_\tau \left\{ \tau^\mu \frac{t^{\nu-\mu} - t^{\nu-\mu}}{\nu - \mu} \right\} = E(\mu)t^\nu - E(\nu)t^\mu
\]

with \( E(\lambda) = \Phi_\tau \{ \tau^\lambda \} \). Using this expression, it follows that

\[
(\{t^\mu\} \ast \{t^\nu\}) \ast \{t^\kappa\} = \{t^\mu\} \ast (\{t^\nu\} \ast \{t^\kappa\})
\]

(2)

since both sides of (2) have one and the same symmetric form

\[
t^\mu \frac{E(\nu)E(\kappa)}{(\mu - \nu)(\mu - \kappa)} + t^\nu \frac{E(\kappa)E(\mu)}{(\nu - \kappa)(\nu - \mu)} + t^\kappa \frac{E(\mu)E(\nu)}{(\kappa - \mu)(\kappa - \nu)},
\]

with respect to \( \mu, \nu, \) and \( \kappa \). Then, (2), differentiated \( m, n, \) and \( k \) times with respect to \( \mu, \nu, \) and \( \kappa \) correspondingly, gives

\[
(\{t^\mu(\ln t)^m\} \ast \{t^\nu(\ln t)^n\}) \ast \{t^\kappa(\ln t)^k\} = \{t^\mu(\ln t)^m\} \ast (\{t^\nu(\ln t)^n\} \ast \{t^\kappa(\ln t)^k\}).
\]
Next, passing to the limits \( \mu \to +0, \nu \to +0 \) and \( \kappa \to +0 \), one gets
\[
\left( \{ \ln(t)^m \} \ast \{ \ln(t)^n \} \right) \ast \{ \ln(t)^k \} = \{ \ln(t)^m \} \ast \left( \left\{ \{ \ln(t)^n \} \ast \{ \ln(t)^k \} \right\} \right).
\]
But the bilinearity of (1) implies for arbitrary polynomials \( P, Q \) and \( R \)
\[
\left( \{ P(\ln(t)) \} \ast \{ Q(\ln(t)) \} \right) \ast \{ R(\ln(t)) \} = \{ P(\ln(t)) \} \ast \left( \{ Q(\ln(t)) \} \ast \{ R(\ln(t)) \} \right).
\]
To finish the proof, note that if \( t \in \mathbb{R}_+ \), then \( \ln t \) covers the whole real line \( \mathbb{R} \).
Then Weierstrass’ theorem allows any function in \( C(\mathbb{R}_+) \) to be approximated
almost uniformly by polynomials of \( \ln t, t > 0 \), i.e. by a sequence uniformly convergent to the function on each segment \([a, b] \subset \mathbb{R}_+ \). Due to the continuity of the functional \( \Phi \) the desired identity \((f \ast g) \ast h = f \ast (g \ast h)\) holds for every \( f, g, h \in C(\mathbb{R}_+) \).

Further, we restrict the functional \( \Phi \) by \( \Phi\{1\} \neq 0 \) and, without essential loss of generality, we may assume \( \Phi\{1\} = 1 \).

Let \( L : C(\mathbb{R}_+) \to C(\mathbb{R}_+) \) be the right inverse operator of \( \delta = t \frac{d}{dt} \), defined by the boundary value condition \( \Phi(Lf) = 0 \). It is easy to find \( Lf(t) \) explicitly:
\[
Lf(t) = \int_1^t \frac{f(\tau)}{\tau} d\tau - \Phi\left\{ \int_1^\sigma \frac{f(\tau)}{\tau} d\tau \right\}. \tag{3}
\]
Moreover, \( Lf \) has the convolution representation
\[
Lf = \{1\} \ast f
\]
and \( L^n f = \{Q_n(\ln t)\} \ast f \), where \( Q_n \) is a polynomial of degree exactly \( n \).

Let \( MP_\delta^\Phi \) denote the space of the mean-periodic functions for \( \delta \) with respect to \( \Phi \).

**Lemma 1** If \( f \in MP_\delta^\Phi \), then \( Lf \in MP_\delta^\Phi \).

**Proof:** Let \( f \in MP_\delta^\Phi \), i.e. \( \Phi\{f(t)\} = 0 \). Consider the function \( \varphi(t) = \Phi\{ (Lf)(t) \} \). Then
\[
\delta \varphi(t) = \delta \Phi\{ (Lf)(t) \} = \Phi\{ \delta(Lf)(t) \} = \Phi\{ (\delta L)f(t) \} = \Phi\{ f(t) \} = 0,
\]
since \( f \) is mean-periodic. Then \( t \frac{d \varphi(t)}{dt} = 0 \) and \( t > 0 \) imply \( \varphi(t) \equiv C \), where \( C \) is a constant. In order to determine \( C \), let us take \( t = 1 \). Then
\[
\varphi(1) = \Phi\{ Lf(\eta) \} = \Phi\{ \left( \int_1^\eta \frac{f(\tau)}{\tau} d\tau - \Phi \left\{ \int_1^\sigma \frac{f(\tau)}{\tau} d\tau \right\} \right) \} = \\
= \Phi\{ \left( \int_1^\eta \frac{f(\tau)}{\tau} d\tau \right) \} - \Phi\left\{ \int_1^\sigma \frac{f(\tau)}{\tau} d\tau \right\} \Phi\{1\} = 0,
\]
which means \( Lf \in MP_\delta^\Phi \).
Corollary 1 Let $P(\lambda)$ be a polynomial. If $f \in MP_\Phi^\delta$, then

$$\{P(\ln t)\} \ast f \in MP_\Phi^\delta.$$  

Indeed, if $P(\lambda) = \sum_{k=0}^{\deg P} \beta_k \lambda^k$, then $\lambda_k$ can be expressed as linear combination $\lambda_k = \sum_{j=0}^{k} \gamma_j Q_j(\lambda)$, where $Q_j(\lambda)$ are the polynomials from the proof of Theorem 1. Hence

$$\{P(\ln t)\} \ast = \left\{ \sum_{k=0}^{\deg P} \nu_k Q_k(\ln t) \right\} \ast = \sum_{k=0}^{\deg P} \nu_k L^k$$

with some constants $\nu_k$. Then the lemma implies $\{P(\ln t)\} \ast f \in MP_\Phi^\delta$ provided $f \in MP_\Phi^\delta$.

Theorem 2 $MP_\Phi^\delta$ is an ideal in $(C(\mathbb{R}_+), \ast)$.

Proof: Let $f \in MP_\Phi^\delta$ and $g \in C(\mathbb{R}_+)$. If $P(\lambda)$ is an arbitrary polynomial, it follows from Lemma 1 that $P(\ln t) \ast f \in MP_\Phi^\delta$. According to Weierstrass’ approximation theorem, we can find a polynomial sequence $\{P_n\}_{n=1}^\infty$, for which $P_n(x) \rightrightarrows g(e^x)$ on each segment $[a, b] \subset \mathbb{R} = (-\infty, \infty)$. Then $P_n(\ln t) \rightrightarrows g(t)$ on each segment $[\alpha, \beta] \subset \mathbb{R}_+ = (0, \infty)$. But from Corollary 1

$$P_n(\ln t) \ast f \in MP_\Phi^\delta, \quad \forall n \in \mathbb{N}.$$  

Since the space $MP_\Phi^\delta$ is closed with respect to the uniform convergence, the limit $g \ast f = f \ast g$ is mean-periodic, too.

2. Nonlocal Cauchy problems for Euler equations

Let $\Phi : C(\mathbb{R}_+) \rightarrow \mathbb{C}$ be a linear functional. According to Riesz-Markov theorem $\Phi$ has a representation of the form $\Phi\{f\} = \int_{\alpha}^{\beta} f(t)d\gamma(t)$, where $0 < \alpha < \beta < +\infty$ and $\gamma$ is a function with bounded variation.

Definition 3 $P(\lambda)$ be a polynomial with $\deg P \geq 1$. The boundary value problem

$$P(\delta)y = f, \quad (4)$$

$$\Phi\{\delta^k y\} = \alpha_k, \quad k = 0, 1, 2, \ldots, \deg P - 1 \quad (5)$$

with given $\alpha_k \in \mathbb{C}$ is said to be a nonlocal Cauchy problem for the Euler equation (4).
In [3] an operational method for solution of such nonlocal boundary value problems is developed. Here we reproduce the basic elements of this approach.

First, a Mikusiński-type operational calculus for the right inverse operator $L$, defined by (3), is developed. Without any loss of generality we may assume that $\Phi\{1\} = 1$. Then (3) becomes

$$Lf(t) = \int_1^t \frac{f(\tau)}{\tau} d\tau - \Phi\left\{ \int_1^\sigma \frac{f(\tau)}{\tau} d\tau \right\}. \tag{6}$$

In fact, $L$ is the convolution operator $L = \{1\} \ast$, i.e. $Lf = \{1\} \ast f$, in the convolution algebra $(C(\mathbb{R}_+), \ast)$ with the multiplication (1).

Let $\mathfrak{M}$ be the ring of convolution fractions of the form $\frac{f}{g}$ where $f \in C(\mathbb{R}_+)$ and $g \in C(\mathbb{R}_+)$ but $g$ being a non-divisor of zero in $(C(\mathbb{R}_+), \ast)$.

Then the operator $L$ can be identified with the constant function $\{1\}$, i.e. $L = \{1\}$. By 1 we will denote the unit element of $\mathfrak{M}$ and hence $1 \neq \{1\}$.

The basic element of the operational calculus we are to develop, is played by the element $S = \frac{1}{L}$ which may be called the algebraic Euler operator.

**Lemma 2** If $f \in C^1(\mathbb{R}_+)$, then

$$\delta f = Sf - \Phi\{ f \}, \tag{7}$$

where $Sf$ is the product $S.f$ in $\mathfrak{M}$ and $\Phi\{ f \}$ is to be understood as a “numerical” operator, i.e. as the convolution fraction $\frac{\Phi\{ f \}}{\{1\}}$.

**Proof:** By an immediate check it is seen that

$$L(\delta f)(t) = f(t) - f(1) - \Phi\{ f(t) - f(1) \} = f(t) - \Phi\{ f \}.$$  

This identity can be written as

$$L(\delta f) = f - \Phi\{ f \}.L.$$  

Applying $\delta$ to both sides, we obtain (7).

**Corollary 2** For arbitrary $k \in \mathbb{N}$ and $f \in C^k(\mathbb{R}_+)$ we have

$$\delta^k f = S^k f - \Phi\{ f \} S^{k-1} - \Phi\{ \delta f \} S^{k-2} - \ldots - \Phi\{ \delta^{k-1} f \}. \tag{8}$$  

The proof proceeds by induction using (7).
Theorem 3 Let $\lambda \in \mathbb{C}$ be such that $E\{\lambda\} = \Phi\{\tau^\lambda\} \neq 0$. Then

$$
\frac{1}{S - \lambda} = \left\{ \frac{t^\lambda}{E(\lambda)} \right\}
$$

and

$$
\frac{1}{(S - \lambda)^k} = \left\{ \frac{1}{(k - 1)!} \frac{d^{k-1}}{d\lambda^{k-1}} \left( \frac{t^\lambda}{E(\lambda)} \right) \right\}.
$$

The proof is given in [3]. In fact

$$
\frac{1}{S - \lambda} = L_\lambda,
$$

where $L_\lambda$ is the resolvent operator, for which $L_\lambda f(x) = y$ is the solution of the boundary value problem

$$
\delta y - \lambda y = f, \quad \Phi\{y\} = 0.
$$

It has the form

$$
L_\lambda f(t) = \int_1^t \left( \frac{t}{\tau} \right)^\lambda f(\tau) \frac{d\tau}{\tau} - \frac{t^\lambda}{E(\lambda)} \Phi \left\{ \int_1^\sigma \left( \frac{\sigma}{\tau} \right)^\lambda f(\tau) \frac{d\tau}{\tau} \right\}.
$$

$L_\lambda$ can be represented as the convolution operator

$$
L_\lambda f = \left\{ \frac{t^\lambda}{E(\lambda)} \right\} * f.
$$

Theorem 4 Let $P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$ be a polynomial of degree $n$. Then the boundary value problem (4)-(5) is equivalent in $M$ to the linear algebraic equation

$$
P(S)y = f + Q(S),
$$

where

$$
Q(S) = \sum_{k=0}^{n-1} \sum_{m=0}^{n-k-1} a_k \alpha_m S^{n-k-m-1} = \sum_{\mu=0}^{n-\mu-1} \left( \sum_{\nu=0}^{n-\mu-1} a_p \alpha_{n-\mu-\nu-1} \right) S^\mu.
$$

Proof: First, let $y \in C^n(\mathbb{R}_+)$ be a solution of (4)-(5). Then using (8) we obtain (11).
Conversely, let \( y \in C^n(\mathbb{R}_+) \) satisfies (11). Let us involve the right inverse \( L \) of \( \delta \) substituting \( S = \frac{1}{L} \). Then, \( L^n \), applied to both sides of (11), gives

\[
L^n P \left( \frac{1}{L} \right) y = L^n f + L^n Q \left( \frac{1}{L} \right).
\]

This can be written as

\[
\tilde{P}(L)y - \tilde{Q}(L) = L^n f. \tag{12}
\]

where \( \tilde{P}(\lambda) = \lambda^n P \left( \frac{1}{\lambda} \right) \) and \( \tilde{Q}(\lambda) = \lambda^n Q \left( \frac{1}{\lambda} \right) \) are the reciprocal polynomials of \( P \) and \( Q \) respectively. It remains to apply \( \delta^n \) to both sides of (12):

\[
\delta^n \tilde{P}(L)y - \delta^n \tilde{Q}(L) = \delta^n L^n f = f.
\]

Since \( L \) is a right inverse of \( \delta \), then \( \delta^n L^k = \delta^{n-k} \delta^k L^k = \delta^{n-k} \) for \( k = 0, 1, 2, \ldots, n \). The first term \( \delta^n \tilde{P}(L)y \) becomes \( P(\delta)y \), while the second term \( \delta^n \tilde{Q}(L) \) is zero due to the fact that \( \deg Q \leq n - 1 \) and there will always be at least first power of \( \delta \) acting on the constant function \( \{1\} \). Thus \( P(\delta)y = f \) is proved.

The verification of \( \Phi \{ \delta^k y \} = \alpha_k, k = 0, 1, 2, \ldots, n - 1 \), is more complicated but again straightforward.

**Theorem 5** Let \( y \in MP_\Phi \) be a mean-periodic solution of the Euler differential equation \( P(\delta)y = f \). Then a necessary condition for existence of such a solution is \( f \in MP_\Phi \). The problem of solving this equation in \( MP_\Phi \) is equivalent to the nonlocal Cauchy boundary value problem (4)-(5) with the homogeneous initial conditions \( \Phi \{ \delta^k y \} = 0, \ k = 0, 1, 2, \ldots, \deg P - 1 \).

**Proof:** Let \( y \) be a mean-periodic solution of (4), i.e. \( \Phi \{ y(t\tau) \} = 0 \). According to Dimovski and Hristov [2] the operator \( M : C(\mathbb{R}_+) \to C(\mathbb{R}_+) \) given by \( Mf(t) = \Phi \{ y(t\tau) \}, \ f \in C(\mathbb{R}_+) \), belongs to the commutant of \( \delta \), i.e. \( M\delta = \delta M \). Applying \( M \) to both sides of \( P(\delta)y = f \) and using \( My(t) = 0 \), we get

\[
Mf = MP(\delta)y = P(\delta)My = 0.
\]

Hence \( f \in MP_\Phi \).

Now we continue with the proof of the equivalence.

First, let \( P(\delta)y = f, \ f \in MP_\Phi, \ \Phi \{ \delta^k y \} = 0, \ k = 0, 1, 2, \ldots, n - 1 \). Let \( M \) be the operator from the commutant of \( \delta \), which corresponds to the
functional $\Phi$ as above. Consider the function $u = My$. We need to prove that $u = 0$. One has
\[
\delta^k u(t) = \delta^k My(t) = M \delta^k y(t) = \Phi_x\{(\delta^k y)(t\tau)\}.
\]
Substituting $t = 1$ in this equality, we obtain
\[
\delta^k u(1) = \Phi_x\{(\delta^k y)(\tau)\} = 0.
\]
Thus $u$ is a solution of the ordinary Cauchy problem
\[
P(\delta) u = 0, \quad \delta^k u(1) = 0, \quad k = 0, 1, 2, \ldots, n - 1,
\]
which has the unique solution $u = 0$, i.e. $My = 0$, which means that $y$ is mean-periodic.

Conversely, let $P(\delta) y = f$ with a mean-periodic solution $y$, i.e. $My = \Phi_x\{y(t\tau)\} = 0$. Applying $\delta^k$, one has
\[
0 = \delta^k My = M \delta^k y,
\]
which means that $\delta^k y$ is mean-periodic for every $k \geq 0$. This can be written also as
\[
\Phi_x\{(\delta^k y)(t\tau)\} = 0.
\]
Finally, we substitute $t = 1$ and get
\[
\Phi_x\{(\delta^k y)(\tau)\} = \Phi\{\delta^k y\} = 0, \quad k = 0, 1, 2, \ldots, n - 1.
\]

Now we may assert that the problem of solving the Euler equations in mean-periodic functions is equivalent to solving the algebraic problem $P(S)y = f$ in $\mathfrak{M}$.

Its formal solution in $\mathfrak{M}$ is
\[
y = \frac{1}{P(S)} f
\]
provided $P(S)$ is non-divisor of zero.

**Theorem 6** $P(S)$ is a non-divisor of 0 iff $\Phi_x\{\tau^\lambda\} \neq 0$ for each zero of the polynomial $P(\lambda)$, i.e. if $P(\lambda) = 0$ implies $\Phi_x\{\tau^\lambda\} \neq 0$.

**Proof:** Let $P(\lambda) = a_0(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n)$, $a_0 \neq 0$. $P(S)$ is a divisor of 0 if and only if at least one of the multipliers $S - \lambda_k$ is a divisor of zero. Let $S - \lambda$ be a divisor of zero, i.e. there exists an element (convolution fraction) $\frac{u}{v} \in \mathfrak{M}$, such that $(S - \lambda)\frac{u}{v} = 0$. The last equality is equivalent to
\[
(1 - \lambda L)u = 0,
\]
i.e. $\delta u - \lambda u = 0$, with the solution $u = Ct^\lambda$, $C \neq 0$ and $\Phi\{u\} = 0$, i.e. $\Phi_x\{\tau^\lambda\} = 0$. 

8
An extension of the Duhamel principle to nonlocal Cauchy problems for Euler equations

Assuming that $\frac{1}{SP(S)} = G(t) \in C^{(n)}(\mathbb{R}_+)$, we see that $G$ is the solution of the nonlocal Cauchy boundary value problem:

$$P(\delta)G = 1, \quad \Phi\{\delta^kG\} = 0, \quad k = 0, 1, 2, \ldots, n - 1.$$ 

From the equation

$$a_0\delta^nG + a_1\delta^{n-1}G + \ldots + a_nG = 1$$

we obtain $a_0\Phi\{\delta^nG\} = 1$, or

$$\Phi\{\delta^nG\} = \frac{1}{a_0} \quad (14)$$

Then the formal representation (13) becomes

$$y = \delta(G * f) = \delta(f * G) = (\delta f) * G.$$ 

This is an extension of the classical Duhamel principle to nonlocal Cauchy problems for Euler equations.

**Heaviside algorithm for interpreting $\frac{1}{P(S)}$ as a function**

Let $P(S)$ be a non-divisor of zero in $\mathfrak{M}$. First, decompose $\frac{1}{P(S)}$ in simple fractions. Factorizing $P(S)$ as

$$P(S) = a_0(S - \mu_1)^{k_1}(S - \mu_2)^{k_2}\ldots(S - \mu_l)^{k_l}, \quad k_1 + k_2 + \ldots + k_l = n,$$

we have

$$\frac{1}{P(S)} = \sum_{j=1}^{l} \sum_{m=1}^{k_j} \frac{A_{jm}}{(S - \mu_j)^m},$$

where $A_{jm}$ are constants. It remains to replace each fraction $\frac{A_{jm}}{(S - \mu_j)^m}$ by the explicit expressions given in Theorem 3.

**Example:** Let all the zeros of the polynomial $P$ be simple, i.e. $P(\mu) = a_0(\mu - \mu_1)(\mu - \mu_2)\ldots(\mu - \mu_n)$ with $\mu_\nu \neq \mu_\kappa$ for $\nu \neq \kappa$. Then

$$\frac{1}{P(S)} = \sum_{k=1}^{n} \frac{1}{P'(\mu_k)} \frac{1}{S - \mu_k} = \sum_{k=1}^{n} \frac{t^{\mu_k}}{P'(\mu_k)E(\mu_k)}.$$
As a particular case, let us take the functional
\[ \Phi \{ f \} = \frac{f(1) + f(e)}{2}. \]

Then \( E(\lambda) = \frac{1 + e^\lambda}{2} \) and \( \frac{1}{S - \lambda} = \frac{2t^\lambda}{1 + e^\lambda} \). We can write
\[ G(t) = \frac{1}{P(S)} = \sum_{k=1}^{n} \frac{2t^{\mu_k}}{P'(\mu_k)(1 + e^{\mu_k})} \]
and then the solution of the equation \( P(\delta)y(t) = f(t) \) with boundary value conditions \( \delta^k y(1) + \delta^k y(e) = 0, \ k = 0, 1, \ldots, n - 1, \) is
\[ y = G * f = \sum_{k=1}^{n} \frac{2}{P'(\mu_k)(1 + e^{\mu_k})} (t^{\mu_k} * f), \]
where
\[ t^{\mu_k} * f = \frac{1}{2} \left\{ \int_{1}^{t} \left( \frac{t}{\tau} \right)^{\mu_k} f(\tau) \frac{d\tau}{\tau} + \int_{e}^{t} \left( \frac{te}{\tau} \right)^{\mu_k} f(\tau) \frac{d\tau}{\tau} \right\}. \]

One can proceed in a similar way in the case of multiple zeros of \( P(\mu) \). The only difference is that, according to (10), any fraction of the form \( \frac{1}{(S - \mu)^l} \) should be replaced by the function
\[ \frac{1}{(l-1)!} \frac{d^{l-1}}{d\mu^{l-1}} \left( \frac{t^{\mu}}{E(\mu)} \right). \]

As for the resonance case, then additional considerations are needed. They are left for a next publication.

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