Commutants of $\frac{d^2}{dx^2}$ on the Real Half-Line

Ivan H. Dimovski, Valentin Z. Hristov

Key words and phrases: commutant, generalized translation operator, mean-periodic function, convolution algebra, ideal.

AMS subject classification: Primary: 447B37; Secondary: 47B38, 47A15

Abstract

Let $C^1_h$ denotes the space of the smooth functions $f(x)$ on the real half-line $\mathbb{R}_{\geq 0} = [0, \infty)$ satisfying the initial value condition $f'(0) - hf(0) = 0$ with fixed real $h$. We characterize the continuous linear operators $M : C^1_h \to C^1_h$ which commute with the square $D^2 = \frac{d^2}{dx^2}$ of the differentiation operator $D = \frac{d}{dx}$ on the subspace $C^2_h$ of the twice continuously differentiable functions of $C^1_h$. The explicit representation of such operators is $Mf(x) = \Phi\{Tyf(x)\}$, where

$$Tyf(x) = \frac{1}{2}\{f(x+y) + f(|x-y|)\} + \frac{h}{2}\int_{|x-y|}^{x+y} f(t)dt$$

and $\Phi$ is a linear functional on $C^1_h$.

The kernel space of this operator is denoted by $MP_\Phi$ and is called the space of the mean-periodic functions for $D^2$ determined by $\Phi$. It is proved that the space $MP_\Phi$ is invariant under the resolvent operator of $D^2$ with the boundary value conditions $y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$. A convolution structure $* : C^1_h \times C^1_h \to C^1_h$ is introduced in $C^1_h$, such that the resolvent operator is a continuous operator and $MP_\Phi$ is an ideal in the convolution algebra $(C^1_h, *)$. This result is used for effective solution in mean-periodic functions of ordinary differential equations of the form $P(D^2)y = f$ with a polynomial $P$.

A family of operators commuting with $D^2 = \frac{d^2}{dx^2}$

Let $C^1_h$ be the space of smooth functions $f$ on $\mathbb{R}_{\geq 0} = [0, \infty)$ satisfying the boundary value condition

$$f'(0) - hf(0) = 0$$

(1)
with a fixed $h \in \mathbb{R}$. By $C^2_h$ we denote the subspace of twice continuously differentiable functions of $C^1_h$.

**Lemma 1** The operators

$$T^y f(x) = \frac{1}{2} \{ f(x + y) + f(|x - y|) \} + \frac{h}{2} \int_{|x-y|}^{x+y} f(t) dt \quad (2)$$

map $C^1_h$ onto $C^1_h$ and have the following properties:

(i) $T^y f(x) = T^x f(y)$;

(ii) $T^0 f(x) = f(x)$;

(iii) $D^2 T^y = T^y D^2$ on $C^2_h$;

(iv) $T^y T^z = T^z T^y$.

**Proof:** It is seen directly that $T^y f(0) - h(T^y f)(0) = 0$ for arbitrary $f \in C^1(\mathbb{R}_{\geq 0})$ and hence $T^y : C^1_h \to C^1_h$.

The properties (i) and (ii) are obvious.

In order to prove (iii), we verify it first for $y \leq x$ and then for $x < y$. If $y \leq x$, then

$$T^y f(x) = \frac{1}{2} \{ f(x + y) + f(x - y) \} + \frac{h}{2} \int_{x-y}^{x+y} f(t) dt$$

and

$$\frac{d^2}{dx^2} T^y f(x) = \frac{1}{2} [f''(x + y) + f''(x - y)] + \frac{h}{2} [f'(x + y) - f'(x - y)] = T^y f''(x).$$

If $x < y$, then the verification of $\frac{d^2}{dx^2} T^y f(x) = T^y f''(x)$ goes in the same way.

For the proof of (iv), one may verify it first for even powers of $x$, i.e. for $f(x) = x^{2n}$, and then to proceed by approximation of an arbitrary function $f \in C^1_h$ by polynomials of the form $P(x^2)$.

Since the operators (2) are a very special case of the generalized translation operators of B. M. Levitan (see [3]), one may rely also on a general proof in this book. □

**Theorem 1** Let $M : C^1_h \to C^1_h$ be a continuous linear operator, such that $M : C^2_h \to C^2_h$. Then the following assertions are equivalent:
(i) $MD^2 = D^2 M$ in $C^2_h$;

(ii) $MT^y = T^y M$ for each $y \geq 0$;

(iii) $M$ has the explicit representation

\[ Mf(x) = \Phi_y\{T^y f(x)\} = \Phi_y \left\{ \frac{f(x + y) + f(|x - y|)}{2} + \frac{h}{2} \int_{|x-y|}^{x+y} f(t) dt \right\} \]

with a linear functional $\Phi$ in $C^1_h$.

Proof:

(i) $\Rightarrow$ (ii)

Let $f(x)$ be an even polynomial. Then the Maclaurin expansion

\[ f(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} D^{2n} f(0) \]

gives the following representation of the translated function:

\[ T^y f(x) = T^x f(y) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} T^x D^{2n} f(0) \]

\[ = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} T^0 D^{2n} f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} D^{2n} f(x). \]

Now (ii) will follow if we apply $M$ to both sides and use $MD^{2n} f(x) = D^{2n} Mf(x)$ which follows immediately from (i) for each $n \in \mathbb{N}$:

\[ MT^y f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} MD^{2n} f(x) = \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} D^{2n} Mf(x) = T^y Mf(x). \]

(ii) $\Rightarrow$ (iii)

Let us define a continuous linear functional $\Phi$ in $C^1_h$ by $\Phi\{f\} = (Mf)(0)$. Substituting $y = 0$ in

\[ T^y Mf(x) = MT^y f(x) = MT^x f(y), \]

we obtain

\[ T^0 Mf(x) = MT^x f(0). \]

The left hand side is $Mf(x)$ and the right hand side is the value of the functional $\Phi$ for the function $T^x f$. Hence

\[ Mf(x) = \Phi_y\{T^x f(y)\} = \Phi_y\{T^y f(x)\}. \]
Thus the implication is proved using $y$ as the "dumb" variable of the functional.

(iii)$\Rightarrow$(i)

Let $Mf(x) = \Phi_y\{Tyf(x)\}$. Then $D^2Mf(x) = \Phi_y\{D^2Tyf(x)\}$. Using $D^2Ty = TyD^2$ from Lemma 1, we have

$$D^2Mf(x) = \Phi_y\{TyD^2f(x)\} = MD^2f(x).$$

Hence (iii)$\Rightarrow$(i). \hfill $\Box$

**Theorem 2** The commutant of $D^2 = \frac{d^2}{dx^2}$ in $C^1_h$ is a commutative ring.

**Proof:** Let $M : C^1_h \to C^1_h$ and $N : C^1_h \to C^1_h$ commute with $D^2 = \frac{d^2}{dx^2}$ in $C^2_h$.

According to (iii) from Theorem 1, there are linear functionals $\Phi$ and $\Psi$ in $C^1_h$, such that

$$Mf(x) = \Phi_y\{Tyf(x)\} \quad \text{and} \quad Nf(x) = \Psi_z\{Tzf(x)\}.\]

Then

$$MNf(x) = \Phi_y\Psi_z\{TyTzf(x)\} \quad \text{and} \quad NMf(x) = \Psi_z\Phi_y\{T^zTyf(x)\}.\]

By (iv) from Lemma 1, $T^zTy = TyT^z$, and hence

$$NMf(x) = \Psi_z\Phi_y\{T^zTyf(x)\} = \Psi_z\Phi_y\{TyTzf(x)\}.\]

It remains to use the Fubini property $\Psi_y\Phi_yg(y, z) = \Phi_y\Psi_zg(y, z)$ for functions $g(y, z) \in C^1(\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0})$ in order to assert that $MN = NM$. \hfill $\Box$

**Mean-periodic functions for $D^2 = \frac{d^2}{dx^2}$ in $C^1_h$**

**Definition 1** The kernel space $\ker M$ of an operator of the form $Mf(x) = \Phi_y\{Tyf(x)\}$ is called the space of the mean-periodic functions for $D^2 = \frac{d^2}{dx^2}$, associated with the linear functional $\Phi$.

We use the notation $MP_\Phi = \ker M$, i.e. $MP_\Phi = \{f \in C^1_h : \Phi_y\{Tyf(x)\} = 0\}$.

In order to reveal some specific properties of $MP_\Phi$, let us introduce the resolvent operator $R_{-\lambda^2}$ of the operator $D^2 = \frac{d^2}{dx^2}$, defined by the the boundary value conditions $y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$. In other words, $y(x) = R_{-\lambda^2}f(x)$ is the solution of the differential equation

$$y'' + \lambda^2y = f(x)$$

satisfying the boundary value conditions $y'(0) - hy(0) = 0$ and $\Phi\{y\} = 0$.\]
Lemma 2  The resolvent operator $R_{-\lambda^2}$ of $D^2 = \frac{d^2}{dx^2}$ has the following explicit form

$$R_{-\lambda^2}f(x) = \frac{1}{\lambda} \int_0^x \sin \lambda (x-t)f(t)dt$$

$$-\frac{\lambda \cos \lambda x + h \sin \lambda x}{\lambda E(\lambda)} \Phi_y \left\{ \int_0^y \sin \lambda (y-t)f(t)dt \right\},$$

where $E(\lambda) = \Phi_t \left\{ \frac{\lambda \cos \lambda t + h \sin \lambda t}{\lambda} \right\}$, is an entire function of exponential type.

The proof is a matter of a direct check.

Lemma 3  $R_{-\lambda^2}$ maps $MP_{\Phi}$ into itself, i.e. $R_{-\lambda^2}(MP_{\Phi}) \subset MP_{\Phi}.$

Proof: Let $f \in MP_{\Phi}$, i.e. $\Phi_y\{T^yf(x)\} = 0$. We are to prove that $\varphi(x) = \Phi_y\{T^yR_{-\lambda^2}f(x)\} \equiv 0$. Indeed, we have

$$(D^2 + \lambda^2)\varphi(x) = \Phi_y\{(D^2 + \lambda^2)T^yR_{-\lambda^2}f(x)\}$$

$$= \Phi_y\{T^y(D^2 + \lambda^2)R_{-\lambda^2}f(x)\} = \Phi_y\{T^yf(x)\} \equiv 0,$$

since $(D^2 + \lambda^2)R_{-\lambda^2}f(x) = f(x)$. Hence $\varphi(x)$ belongs to the kernel space of $D^2 + \lambda^2$, i.e. $\varphi(x) = A \cos \lambda x + B \sin \lambda x$ with constants $A$ and $B$. $\varphi$ satisfies the condition $\varphi'(0) - h\varphi(0) = 0$ and hence $B\lambda - hA = 0$. In other words, $\varphi(x)$ is a function of the form $\varphi(x) = A \left( \cos x + \frac{h \sin \lambda}{\lambda} \right)$. Using the boundary value condition $\Phi\{f\} = 0$, we obtain

$$0 = A \Phi_t \left\{ \cos x + \frac{h \sin \lambda}{\lambda} \right\} = AE(\lambda).$$

But $E(\lambda) \neq 0$ and hence $A = 0$. Thus we proved that $\varphi(x) \equiv 0$. $\square$

For the sake of simplicity, from now on we restrict our considerations to the case $h = 0$, i.e. to the space

$$C^1_0 = \{ f \in C^1(\mathbb{R}_{\geq 0}), f'(0) = 0 \}.$$

This is possible due to an explicit isomorphism between $C^1_h$ and $C^1_0$. 

5
Lemma 4 The linear operator

\[ \tau f(x) = f(x) + h \int_0^x e^{-h(x-t)} f(t) dt \] (4)

maps \( C^1_h \) onto \( C^1_0 \) and its inverse is

\[ \tau^{-1} f(x) = f(x) + h \int_0^x f(t) dt. \] (5)

If \( f \in C^2_h \), then \( \tau f \in C^2_0 \) and \( (\tau f)'' = \tau f'' \).

The proof is a matter of simple check (see Dimovski [1], p.153).

Due to Lemma 4, instead of the resolvent operator \( R_{-\lambda^2} \) of \( D^2 \) with boundary value conditions \( y'(0) - hy(0) = 0 \) and \( \Phi\{y\} = 0 \), we may consider the resolvent operator \( \tilde{R}_0 \) of \( D^2 \), defined by the boundary value conditions \( y'(0) = 0 \) and \( \tilde{\Phi}\{y\} = 0 \), where \( \tilde{\Phi} = \Phi \circ \tau^{-1} \).

From now on we will use the notation \( \Phi \) instead of \( \tilde{\Phi} \), assuming that we are all the time in the case \( h = 0 \).

For a further simplification we assume that \( \lambda = 0 \) is not an eigenvalue of the eigenvalue problem \( y'' + \lambda^2 y = 0, \ y'(0) = 0, \ \Phi\{y\} = 0 \). This means that there exists a right inverse operator \( R \) of \( D^2 \), such that \( (Rf)'(0) = 0, \ \Phi\{Rf\} = 0 \) which is possible when \( \Phi\{1\} \neq 0 \). If so, we may assume additionally that \( \Phi\{1\} = 1 \) without any loss of generality. Then the right inverse of \( D^2 \) has the form

\[ Rf(x) = \int_0^x (x-t)f(t) dt - \Phi_y \left\{ \int_0^y (y-t)f(t) dt \right\}. \]

In Dimovski [1], pp. 148-151, the following theorem is proved:

Theorem 3 The operation

\[ (f * g)(x) = \int_0^x dt \int_0^t f(t-\tau)g(\tau) d\tau + \frac{1}{2} \Phi_t \left\{ \int_0^t \psi(x,\tau) d\tau \right\}, \] (6)

where

\[ \psi(x,t) = \int_x^t f(t+x-\tau)g(\tau) d\tau + \int_{-x}^t f(|t-x-\tau|)g(|\tau|) d\tau, \]

is an inner operation in \( C^1_0 \), which is bilinear, commutative, and associative, and the operator \( R \) is the convolution operator \( R = \{1\} * \), i.e. \( Rf = \{1\} * f \).
Theorem 4 The subspace $MP_\Phi$ of mean-periodic functions for $D^2$ associated with the linear functional $\Phi$ form an ideal in the convolution algebra $(C^1_0, *)$.

Proof: By Lemma 3, if $f \in MP_\Phi$, then $Rf \in MP_\Phi$. But from Theorem 2 $Rf = \{1\} * f$ and $R^k f = \{Q_k(x^2)\} * f$, where $Q_k$ is a polynomial of degree $k$.

Choose a polynomial sequence $\{P_n(x)\}_{n=1}^\infty$ converging to $g(\sqrt{x})$ uniformly on each segment $[a, b] \subset [0, \infty)$. Then $\{P_n(x^2)\}_{n=1}^\infty$ converges to $g(x)$ in $C^1_0$.

But $P_n(x^2) = \sum_{k=0}^{n} \alpha_k Q_k(x^2)$ with some constants $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$. Then $\{P_n(x^2)\} * f \in MP_\Phi$ since $\{Q_k(x^2)\} * f \in MP_\Phi$, $k = 0, 1, 2, \ldots, n$. Obviously the limit of a sequence of mean-periodic functions is also mean-periodic.

Hence $g * f \in MP_\Phi$ for arbitrary $g \in C^1_0$ and therefore $MP_\Phi$ is an ideal in $(C^1_0, *)$.

Theorem 4 may be used to study the problem of solution of ordinary differential equations with constant coefficients of the form

$$P \left( \frac{d^2}{dx^2} \right) y = f(x)$$

in mean-periodic functions of the space $MP_\Phi$ and to extend the Heaviside algorithm for obtaining such solutions in explicit form. This will be left for a subsequent publication, but analogous considerations for the Dunkl operator $D_k$ instead of $D^2$ can be seen in Dimovski, Hristov, and Sifi [2].

References


Contact information:

1,2 Institute of Mathematics and Informatics  
Bulgarian Academy of Sciences  
"Acad. G. Bonchev" Str., Block 8  
1113 Sofia, BULGARIA  
e-mails: 1 dimovski@math.bas.bg , 2 valhrist@bas.bg