Потоци в мрежи

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6.4 MAXIMUM FLOW

- *introduction*
- *Ford-Fulkerson algorithm*
- *maxflow-mincut theorem*
- *running time analysis*
- *Java implementation*
- *applications*

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6.4 Maximum Flow

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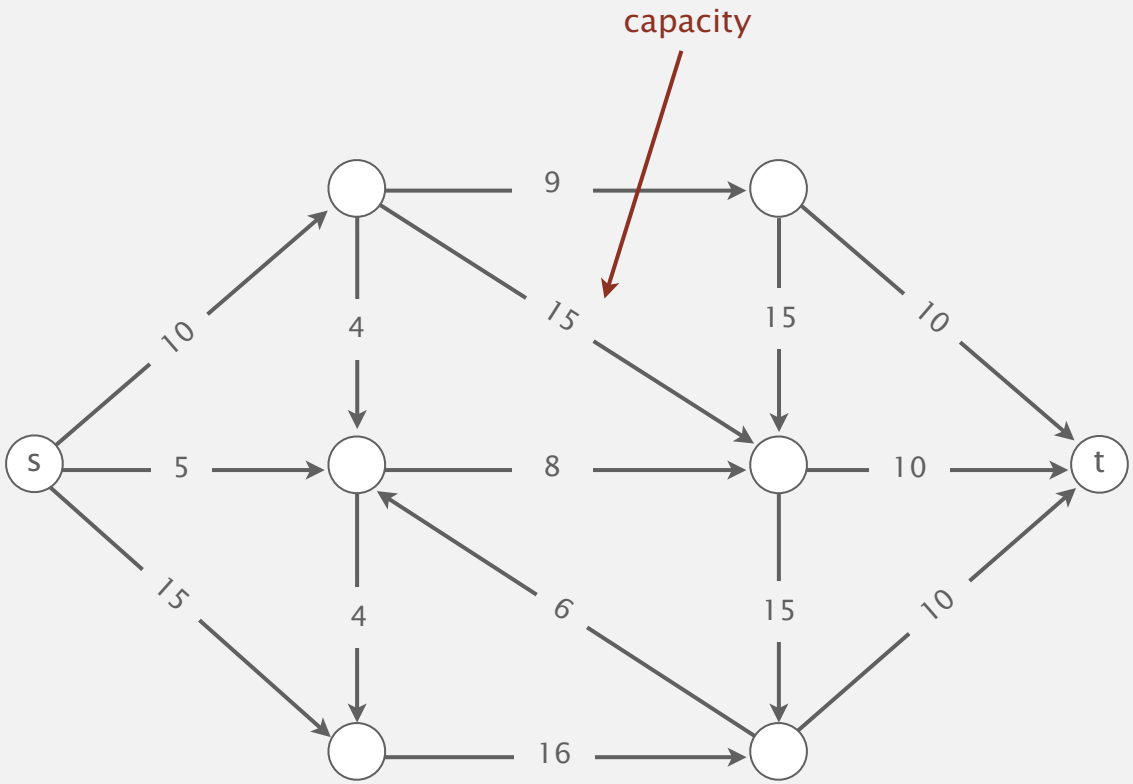
Maxflow problem

Input. An edge-weighted digraph, source vertex s, and target vertex t.

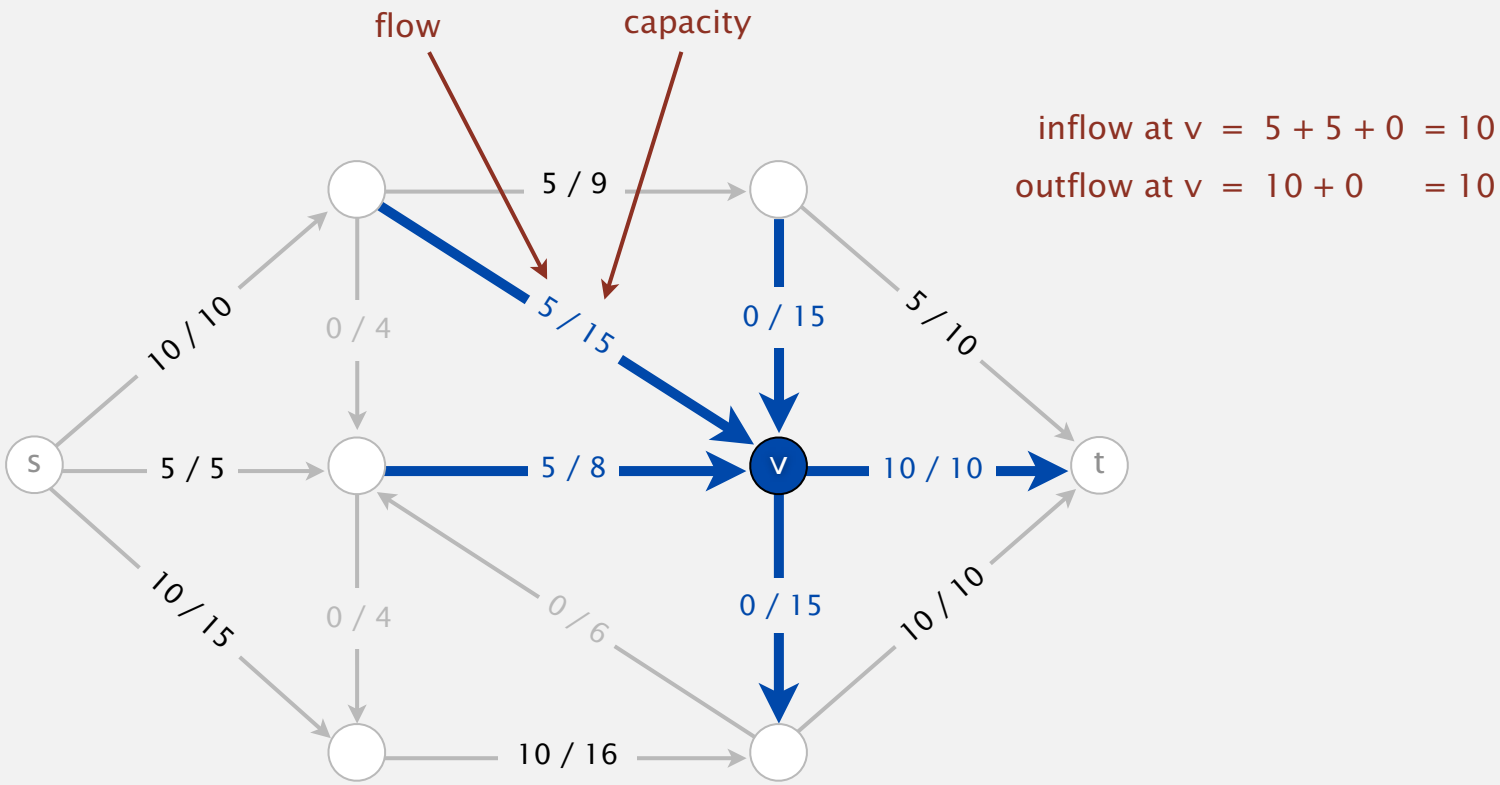
each edge has a positive capacity



Maxflow problem

Def. An *st*-flow (flow) is an assignment of values to the edges such that:

- Capacity constraint: 0 ≤ edge's flow ≤ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except *s* and *t*).



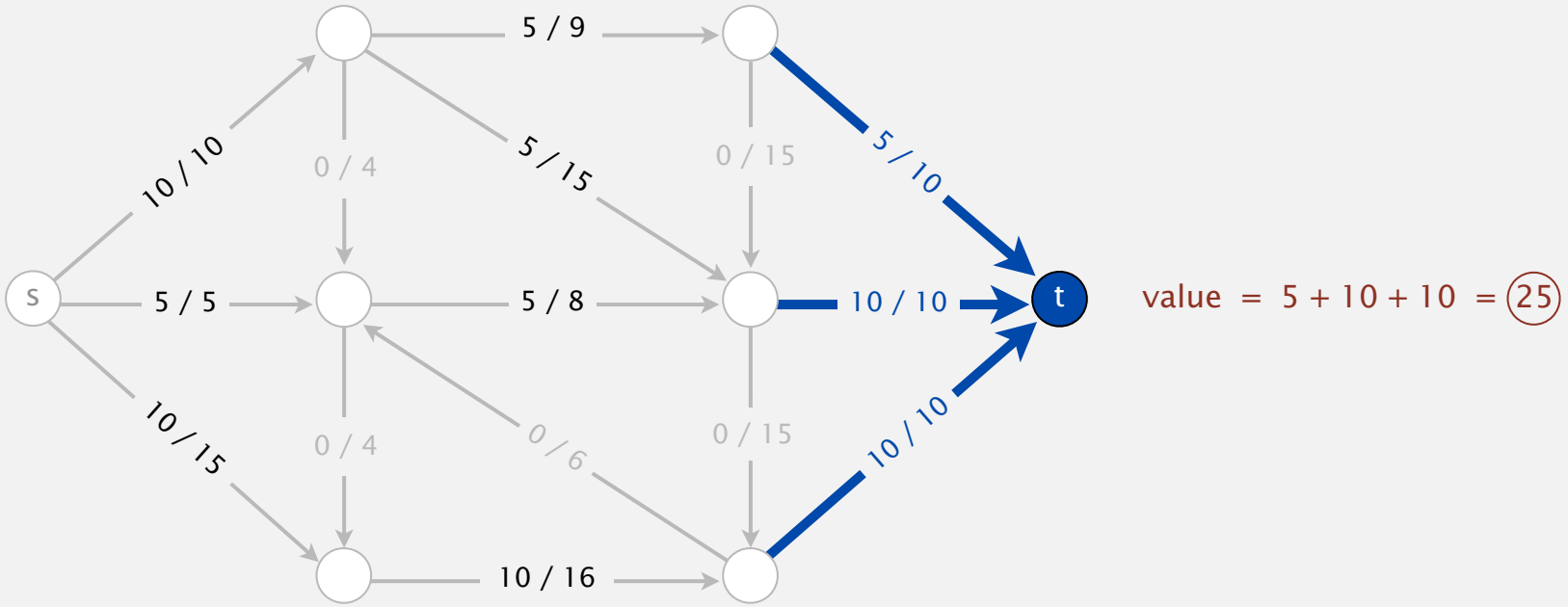
Maxflow problem

Def. An *st*-flow (flow) is an assignment of values to the edges such that:

- Capacity constraint: $0 \le$ edge's flow $\le$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except s and t).

Def. The value of a flow is the inflow at t.

we assume no edge points to s or from t



value = 5 + 10 + 10 = 25

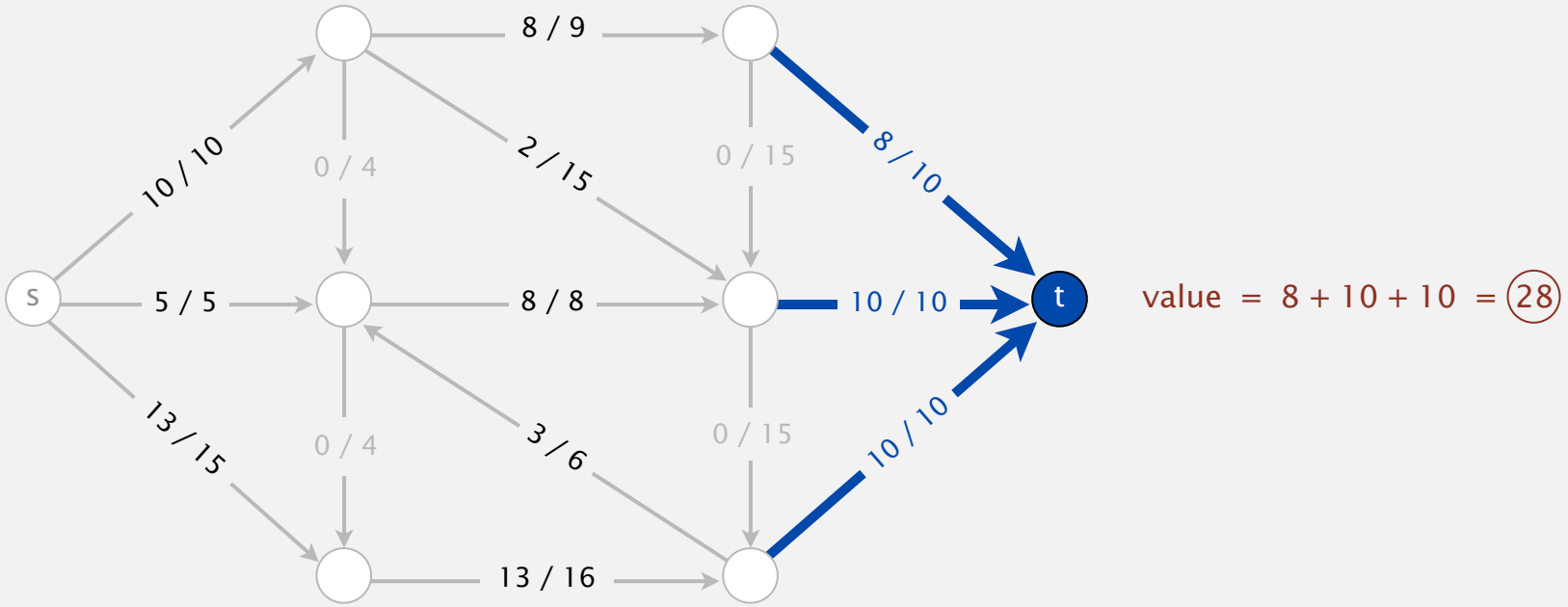
Maxflow problem

Def. An *st*-flow (flow) is an assignment of values to the edges such that:

- Capacity constraint: $0 \le$ edge's flow $\le$ edge's capacity.
- Local equilibrium: inflow = outflow at every vertex (except s and t).

Def. The value of a flow is the inflow at t.

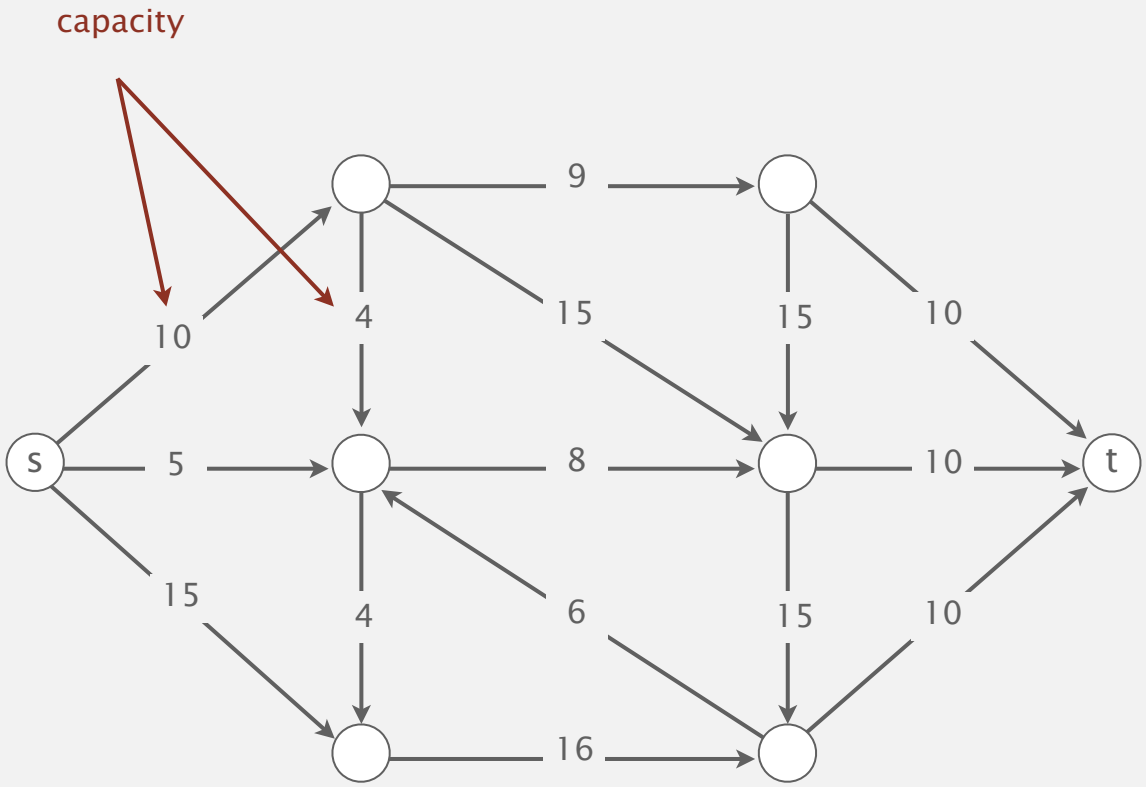
Maximum st-flow (maxflow) problem. Find a flow of maximum value.



Mincut problem

Input. An edge-weighted digraph, source vertex s, and target vertex t.

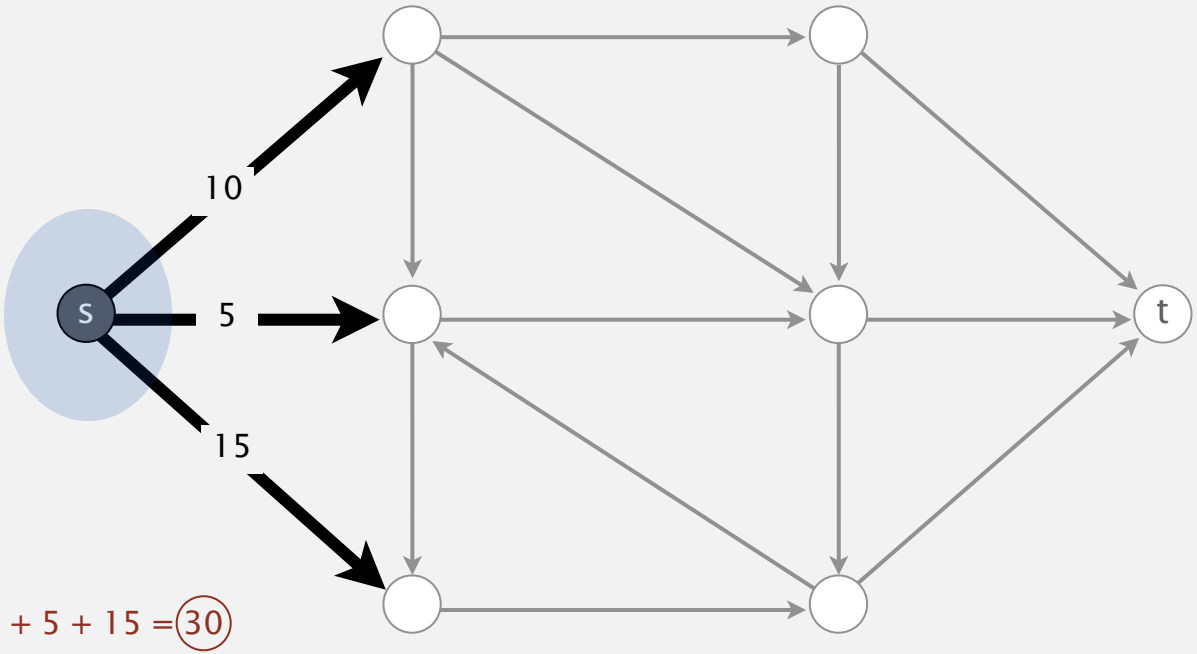
each edge has a positive capacity



Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.

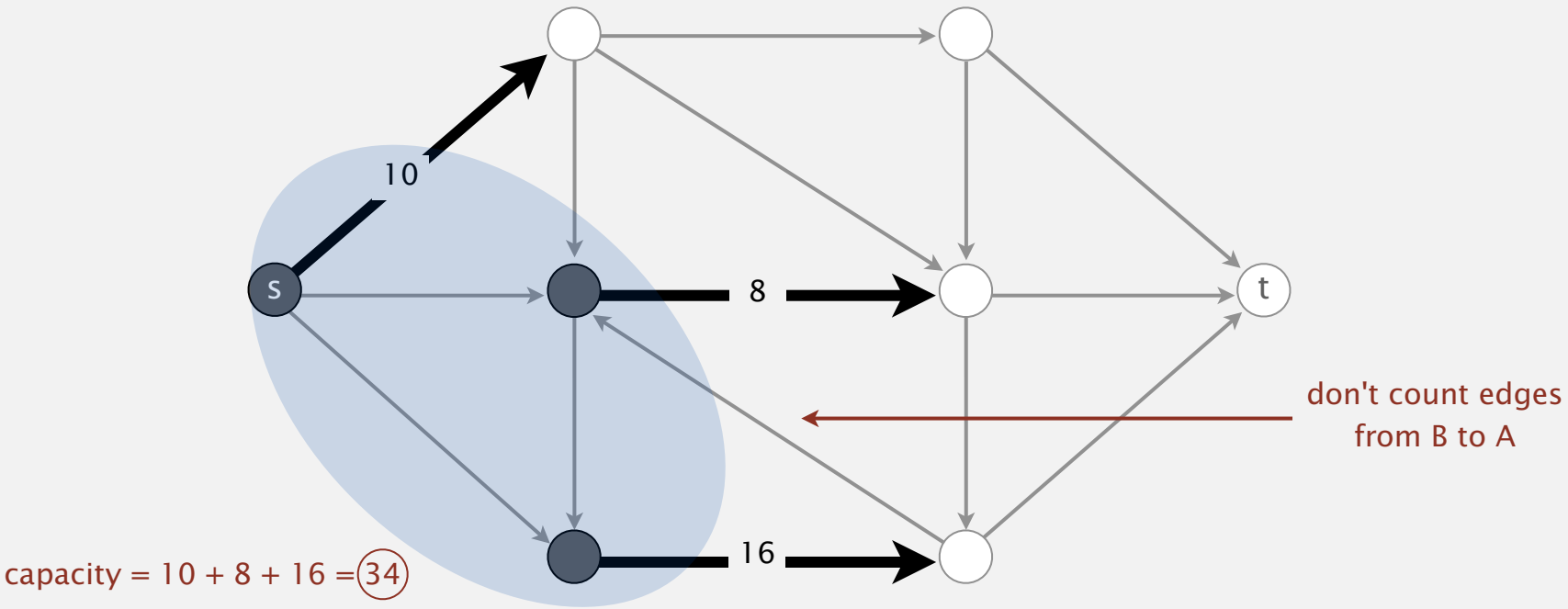


capacity = 10 + 5 + 15 = 30

Mincut problem

Def. A *st*-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.

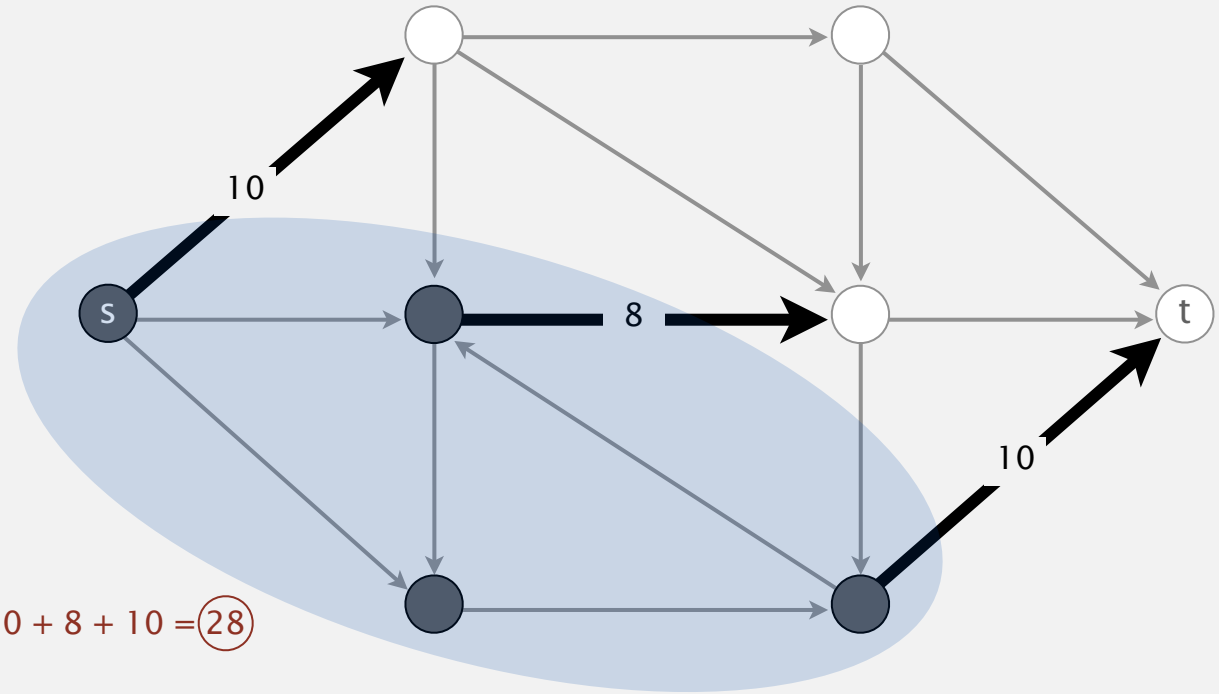


Mincut problem

Def. A st-cut (cut) is a partition of the vertices into two disjoint sets, with s in one set A and t in the other set B.

Def. Its capacity is the sum of the capacities of the edges from A to B.

Minimum st-cut (mincut) problem. Find a cut of minimum capacity.



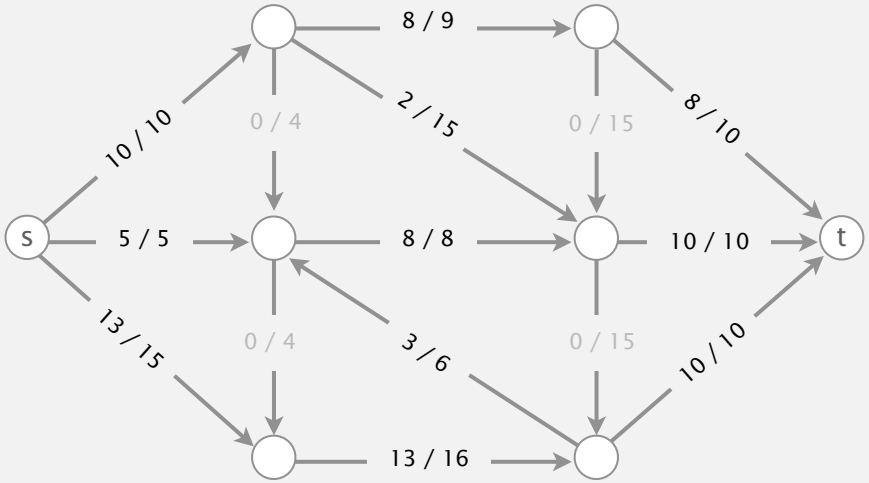
capacity = 10 + 8 + 10 = 28

Summary

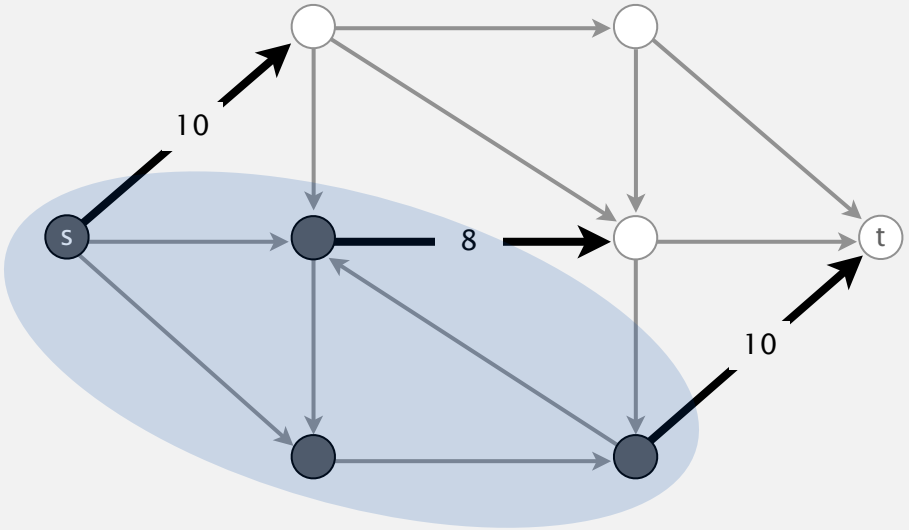
Input. A weighted digraph, source vertex s, and target vertex t.

Mincut problem. Find a cut of minimum capacity.

Maxflow problem. Find a flow of maximum value.



value of flow = 28



capacity of cut = 28

Remarkable fact. These two problems are dual!

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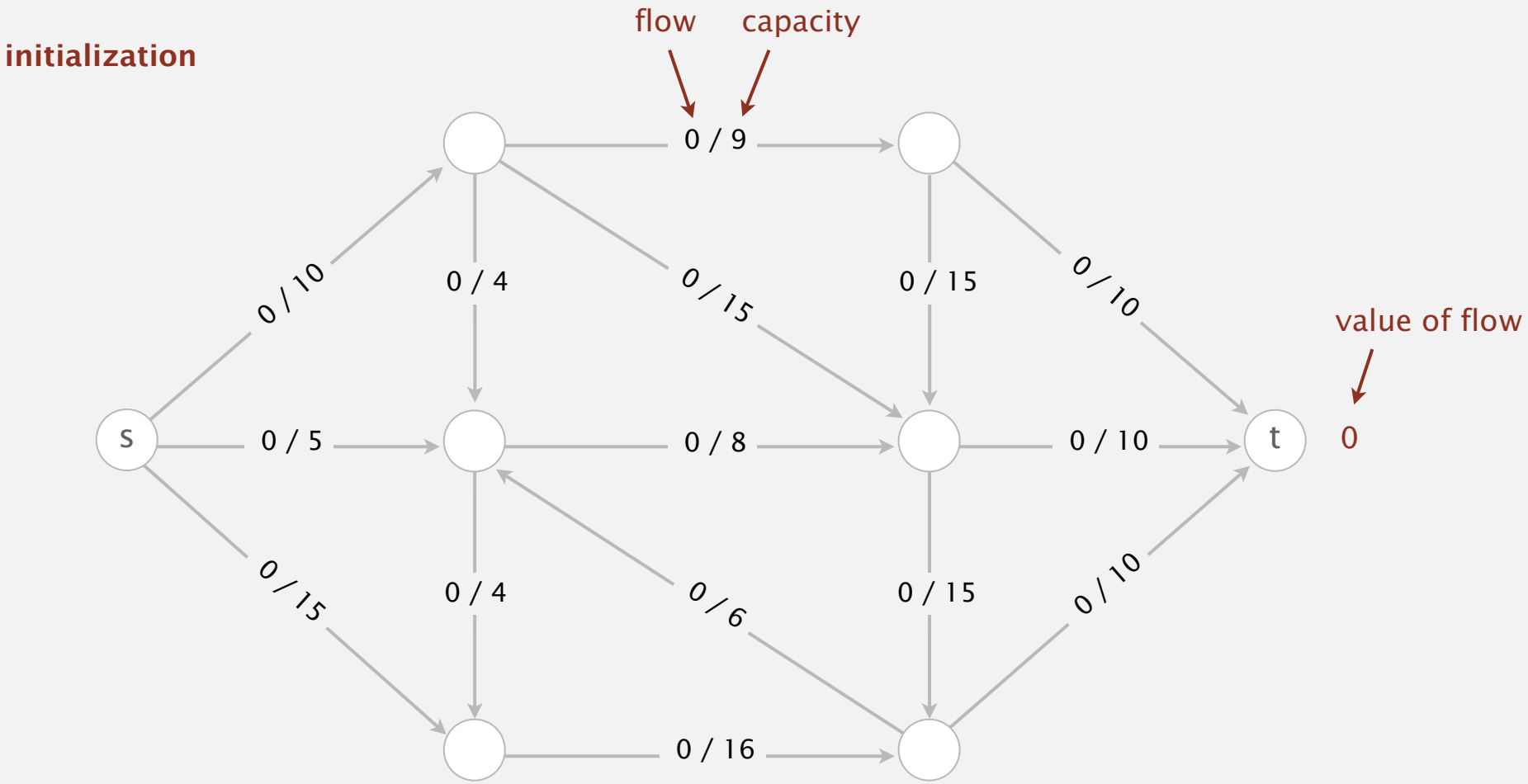
Ford-Fulkerson algorithm

Initialization. Start with 0 flow.

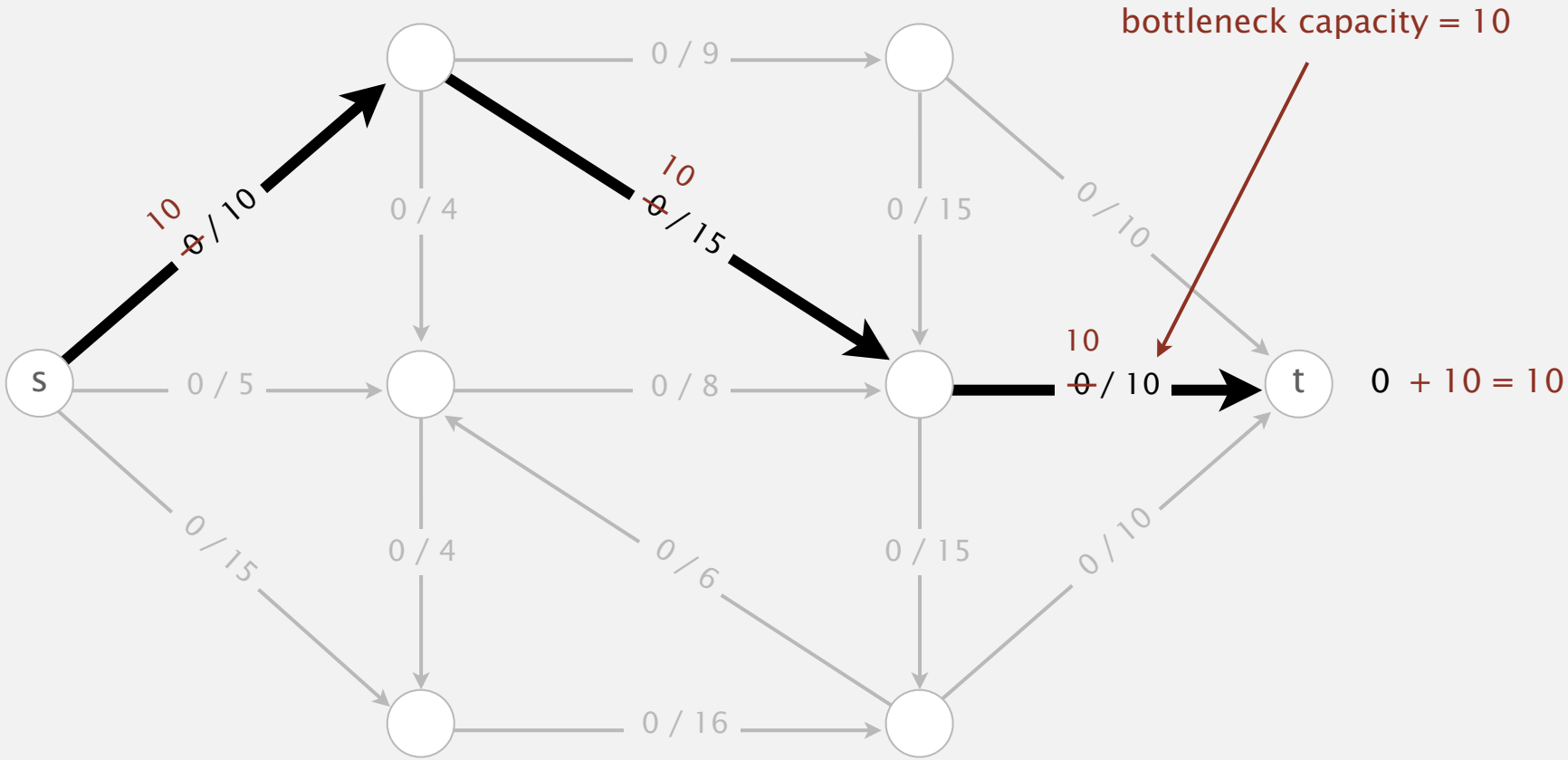


Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

1st augmenting path



Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:

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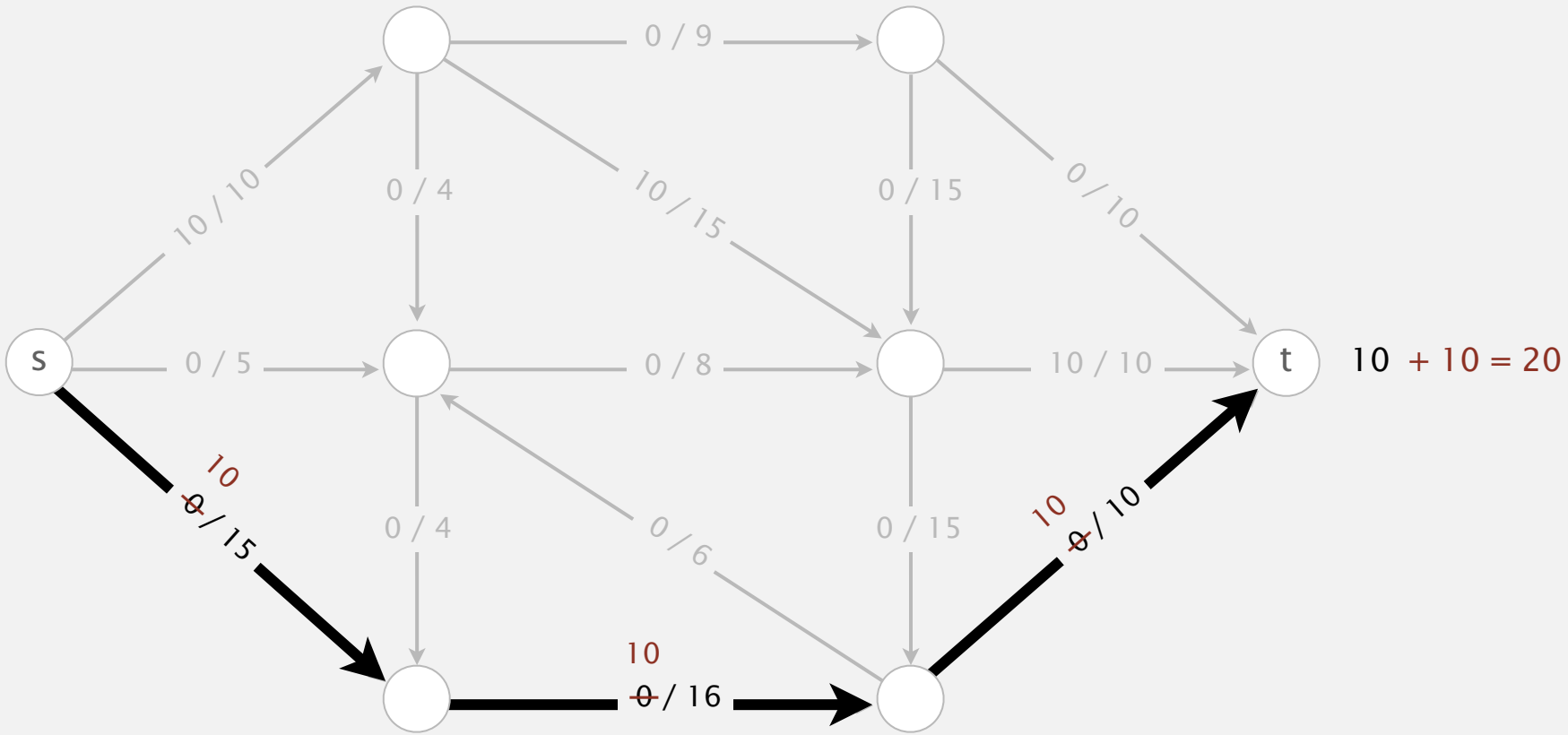
- Can decrease flow on backward edge (not empty).

2nd augmenting path

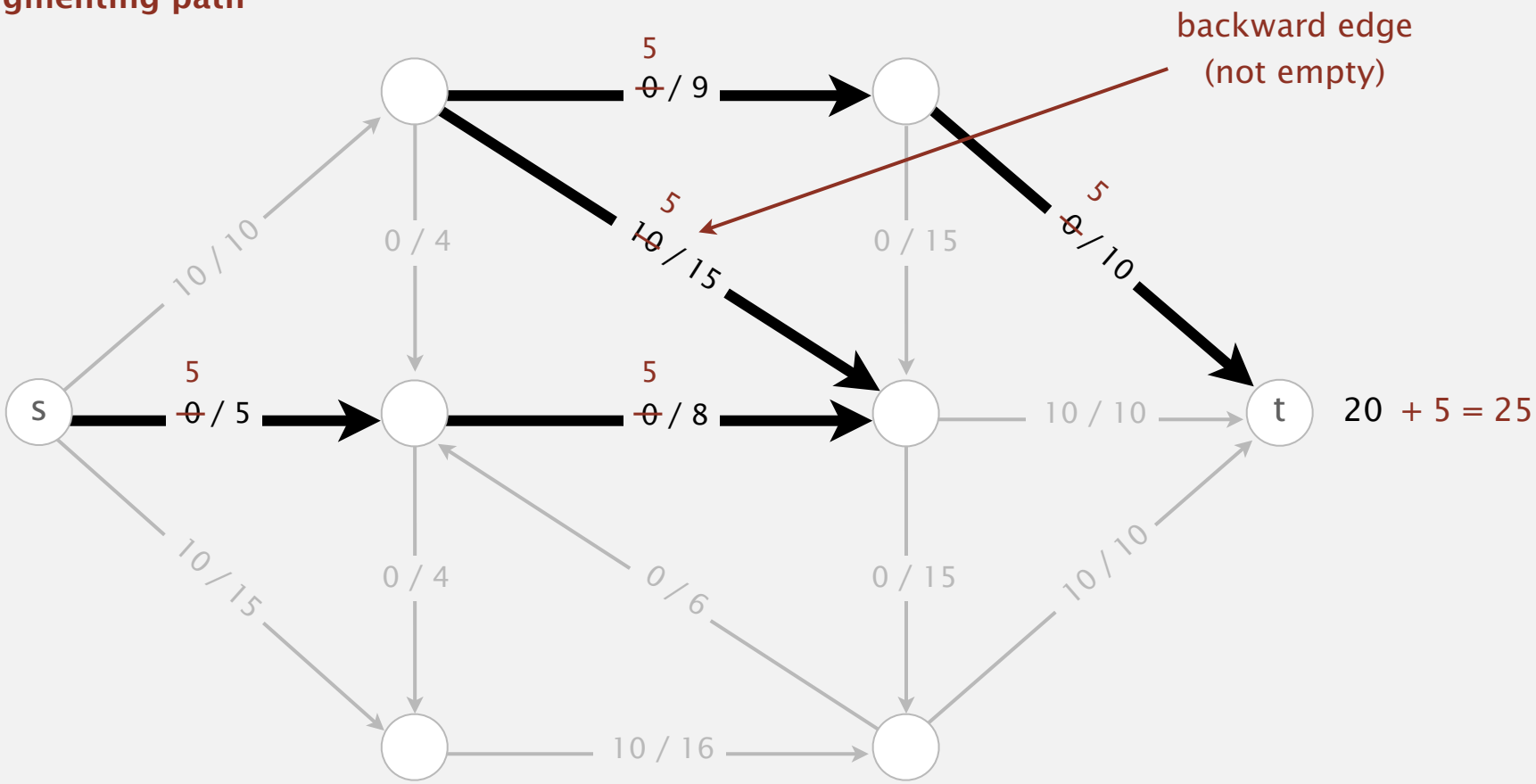


Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

3rd augmenting path

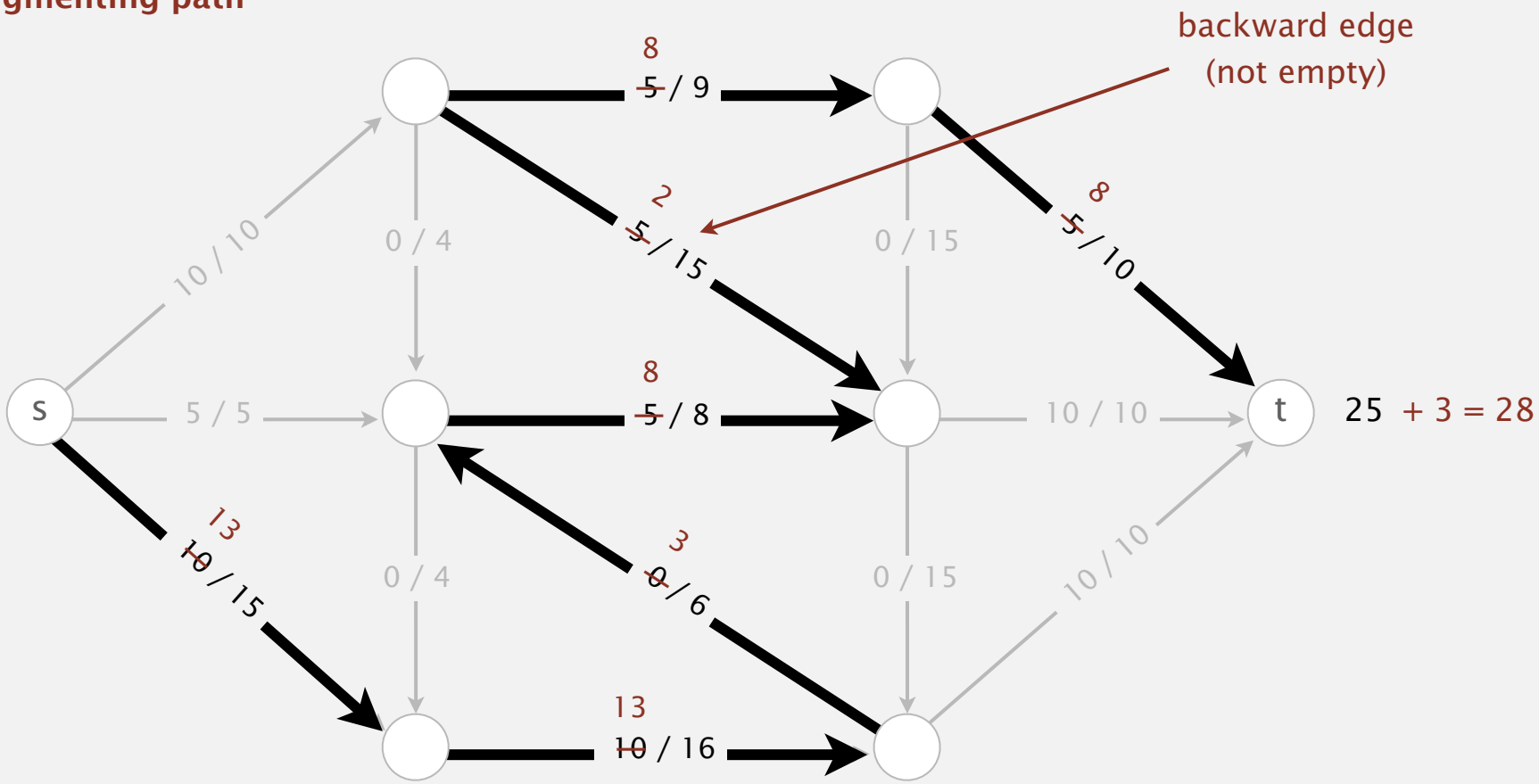


Idea: increase flow along augmenting paths

Augmenting path. Find an undirected path from s to t such that:

- Can increase flow on forward edges (not full).
- Can decrease flow on backward edge (not empty).

4th augmenting path

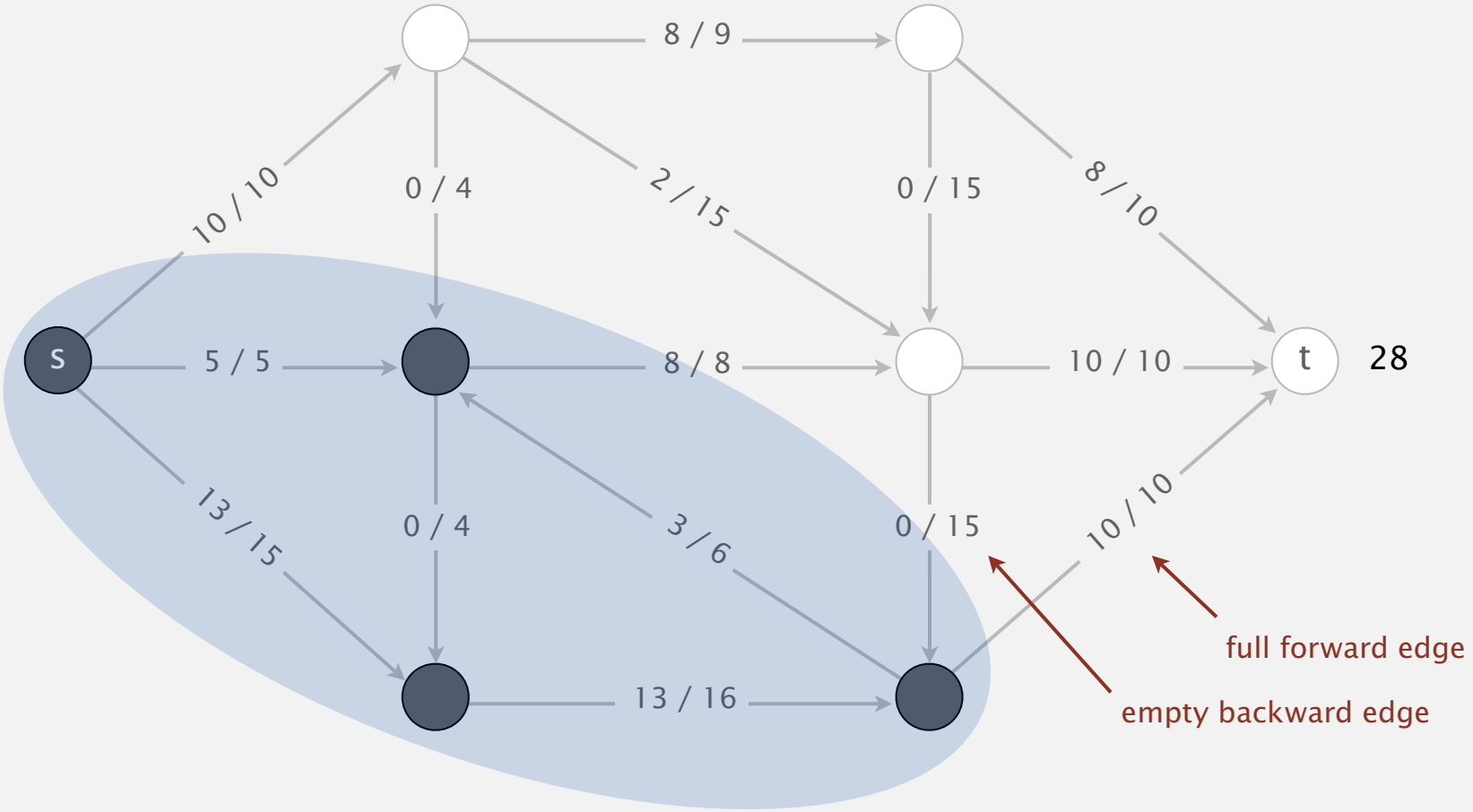


Idea: increase flow along augmenting paths

Termination. All paths from s to t are blocked by either a

- Full forward edge.
- Empty backward edge.

no more augmenting paths



Ford-Fulkerson algorithm

Ford-Fulkerson algorithm

Start with 0 flow.

While there exists an augmenting path:

- **find an augmenting path**
- **compute bottleneck capacity**
- **increase flow on that path by bottleneck capacity**

Questions.

- How to compute a mincut?
- How to find an augmenting path?
- If FF terminates, does it always compute a maxflow?
- Does FF always terminate? If so, after how many augmentations?

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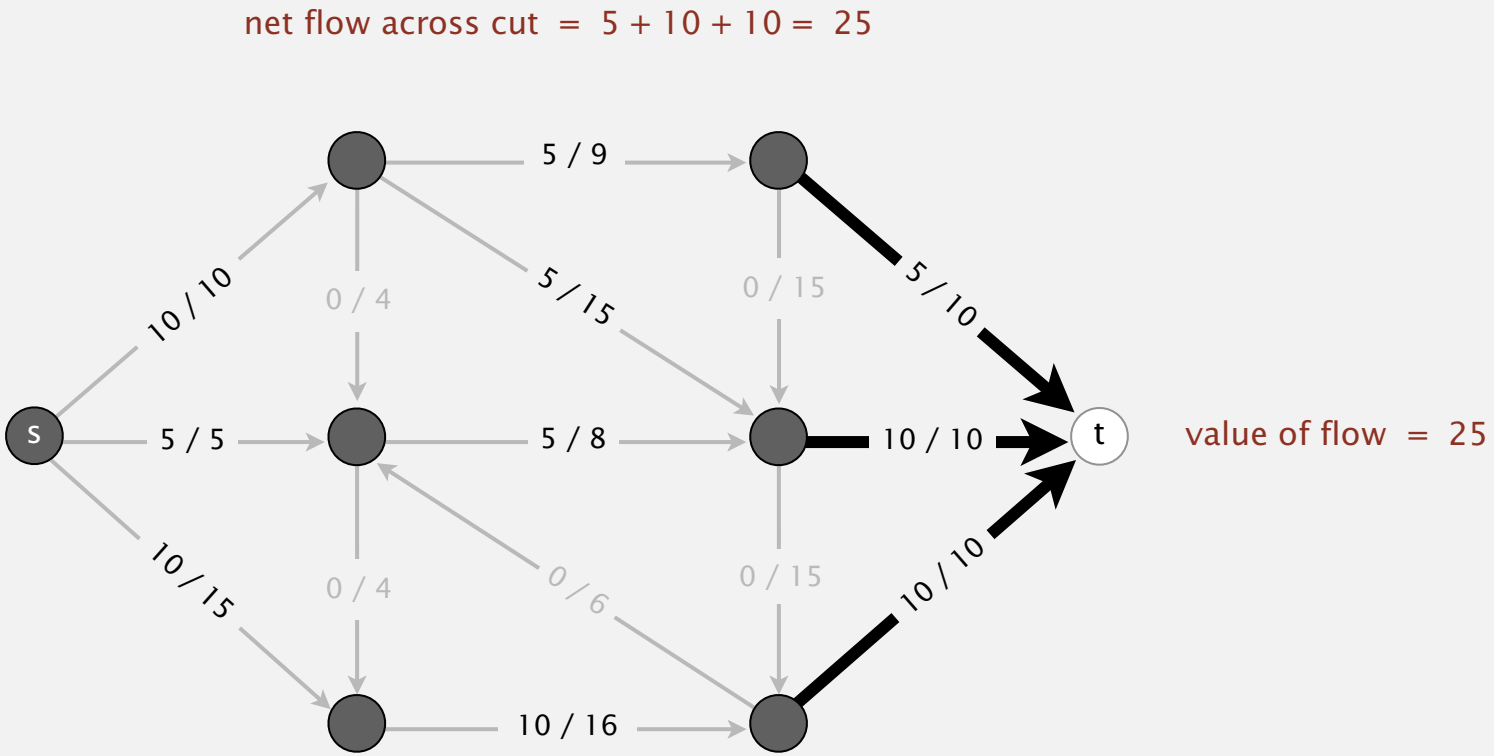
Relationship between flows and cuts

Def. The net flow across a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from from B to A.

Flow-value lemma. Let f be any flow and let (A, B) be any cut. Then, the net flow across (A, B) equals the value of f.



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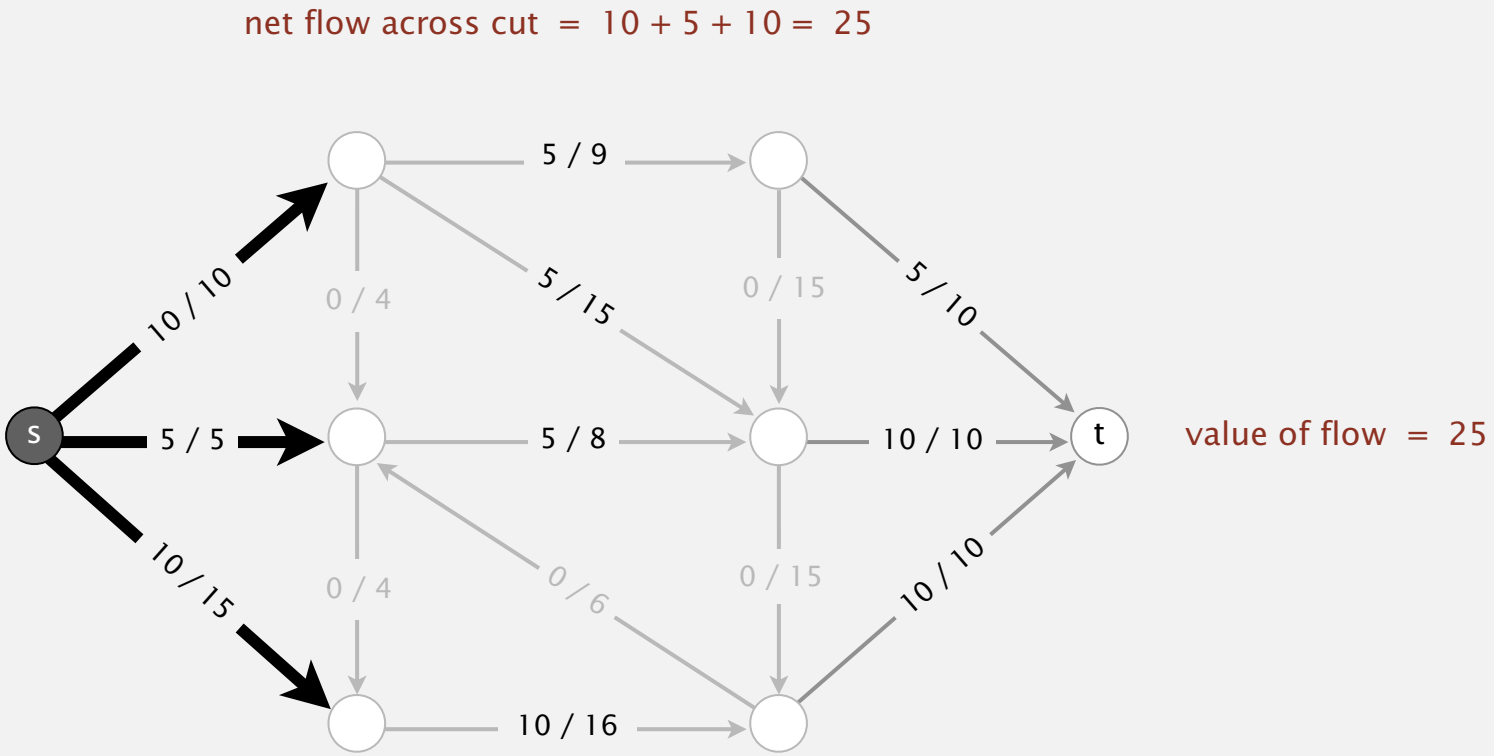

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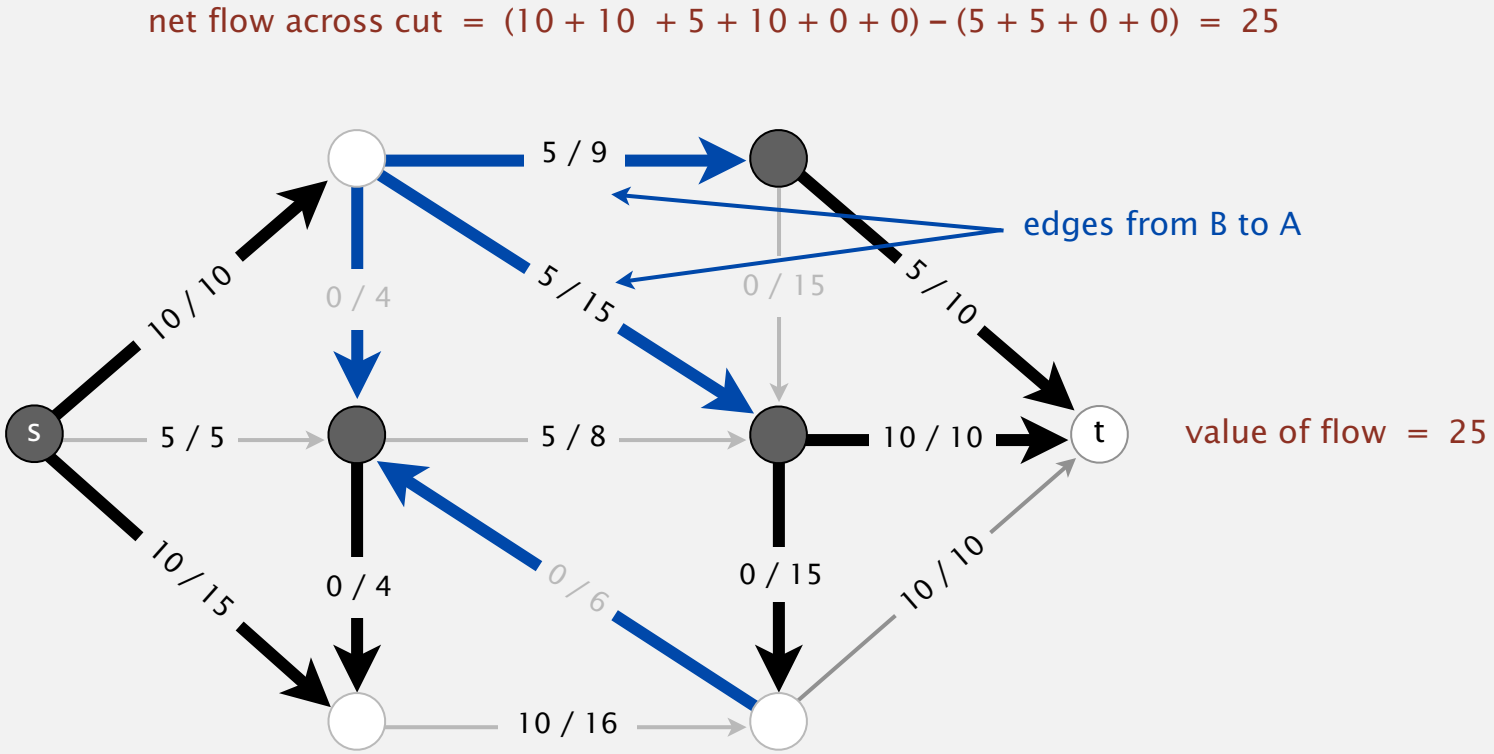


Relationship between flows and cuts

Def. The net flow across a cut (A, B) is the sum of the flows on its edges from A to B minus the sum of the flows on its edges from from B to A.

Flow-value lemma. Let f be any flow and let (A, B) be any cut. Then, the net flow across (A, B) equals the value of f.

Pf. By induction on the size of B.

- Base case: $B = \{ t \}$.
- Induction step: remains true by local equilibrium when moving any vertex from A to B.

Corollary. Outflow from s = inflow to t = value of flow.

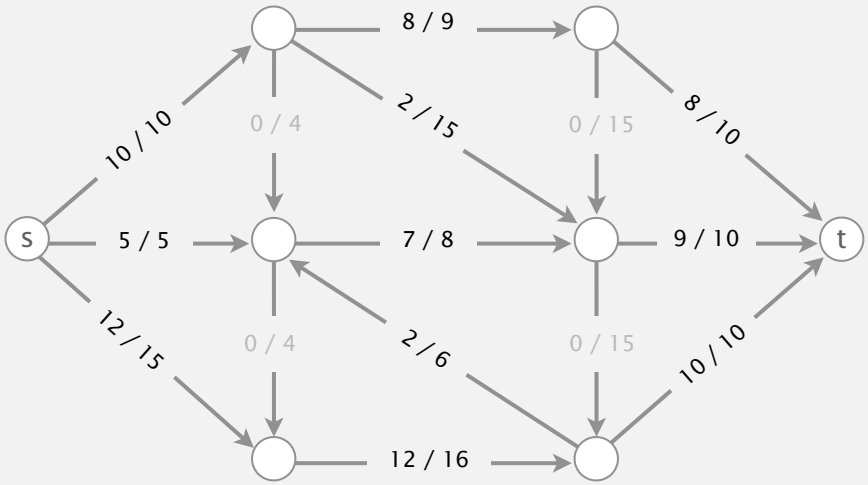
Relationship between flows and cuts

Weak duality. Let f be any flow and let (A, B) be any cut.
Then, the value of the flow ≤ the capacity of the cut.

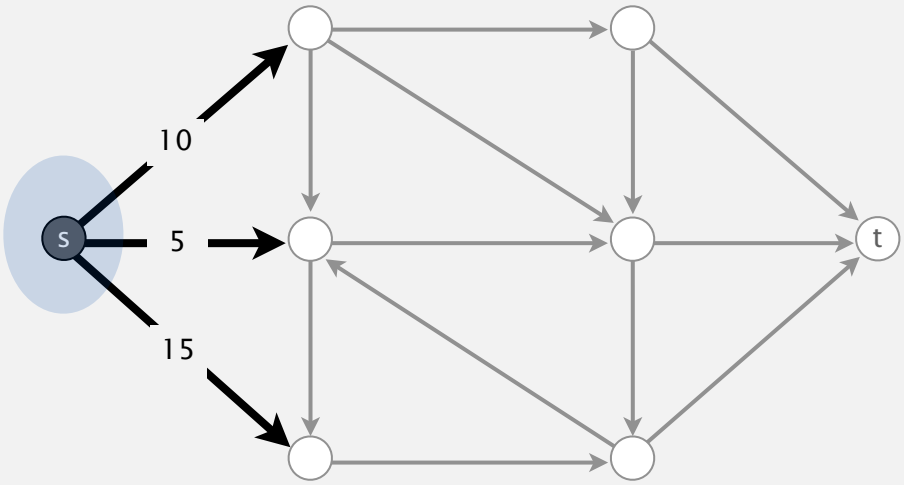
Pf. Value of flow f = net flow across cut (A, B) ≤ capacity of cut (A, B).

flow-value lemma

flow bounded by capacity



value of flow = 27



capacity of cut = 30

Maxflow-mincut theorem

Augmenting path theorem. A flow f is a maxflow iff no augmenting paths.

Maxflow-mincut theorem. Value of the maxflow = capacity of mincut.

Pf. The following three conditions are equivalent for any flow f:

i. There exists a cut whose capacity equals the value of the flow f.

ii. f is a maxflow.

iii. There is no augmenting path with respect to f.

[i $\Rightarrow$ ii]

- Suppose that (A, B) is a cut with capacity equal to the value of f.
- Then, the value of any flow f' $\leq$ capacity of (A, B) $=$ value of f.
- Thus, f is a maxflow.

weak duality

by assumption

Maxflow-mincut theorem

Augmenting path theorem. A flow f is a maxflow iff no augmenting paths.

Maxflow-mincut theorem. Value of the maxflow = capacity of mincut.

Pf. The following three conditions are equivalent for any flow f:

i. There exists a cut whose capacity equals the value of the flow f.

ii. f is a maxflow.

iii. There is no augmenting path with respect to f.

[ii ⇒ iii] We prove contrapositive: ~iii ⇒ ~ii.

- Suppose that there is an augmenting path with respect to f.
- Can improve flow f by sending flow along this path.
- Thus, f is not a maxflow.

Maxflow-mincut theorem

Augmenting path theorem. A flow f is a maxflow iff no augmenting paths.

Maxflow-mincut theorem. Value of the maxflow = capacity of mincut.

Pf. The following three conditions are equivalent for any flow f:

i. There exists a cut whose capacity equals the value of the flow f.

ii. f is a maxflow.

iii. There is no augmenting path with respect to f.

[iii ⇒ i]

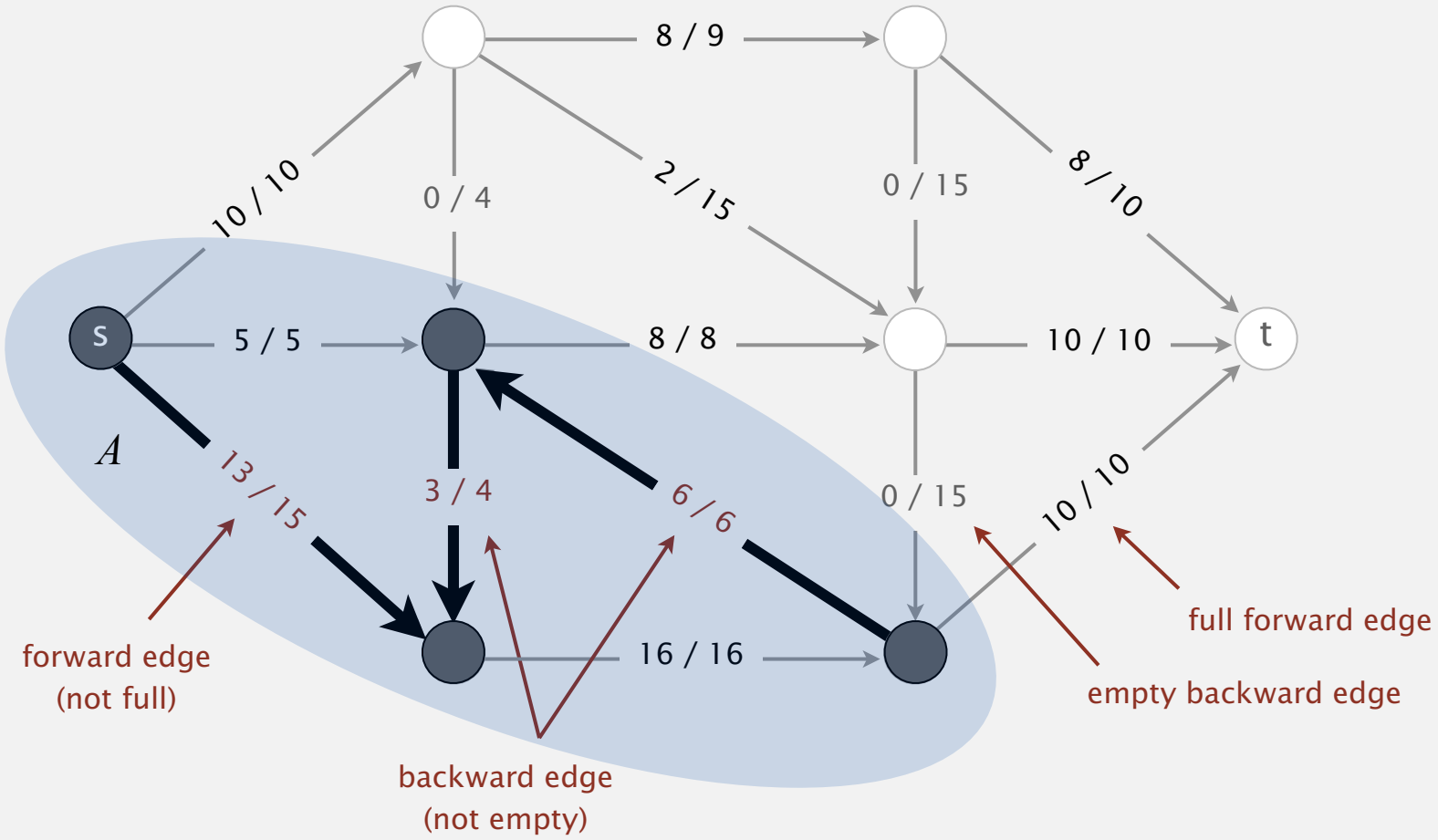
Suppose that there is no augmenting path with respect to f.

- Let (A, B) be a cut where A is the set of vertices connected to s by an undirected path with no full forward or empty backward edges.
- By definition, s is in A; since no augmenting path, t is in B.
- Capacity of cut = net flow across cut ← forward edges full; backward edges empty
  = value of flow f. ← flow-value lemma

Computing a mincut from a maxflow

To compute mincut (A, B) from maxflow f:

- By augmenting path theorem, no augmenting paths with respect to f.
- Compute A = set of vertices connected to s by an undirected path with no full forward or empty backward edges.



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Ford-Fulkerson algorithm

Ford-Fulkerson algorithm

Start with 0 flow.

While there exists an augmenting path:

- **find an augmenting path**
- **compute bottleneck capacity**
- **increase flow on that path by bottleneck capacity**

Questions.

- How to compute a mincut? Easy. ✔
- How to find an augmenting path? BFS works well.
- If FF terminates, does it always compute a maxflow? Yes. ✔
- Does FF always terminate? If so, after how many augmentations?

yes, provided edge capacities are integers (or augmenting paths are chosen carefully)

requires clever analysis

Ford-Fulkerson algorithm with integer capacities

Important special case. Edge capacities are integers between 1 and U.

flow on each edge is an integer

Invariant. The flow is integer-valued throughout Ford-Fulkerson.

Pf. [by induction]

- Bottleneck capacity is an integer.
- Flow on an edge increases/decreases by bottleneck capacity.

Proposition. Number of augmentations ≤ the value of the maxflow.

Pf. Each augmentation increases the value by at least 1.

important for some applications (stay tuned)

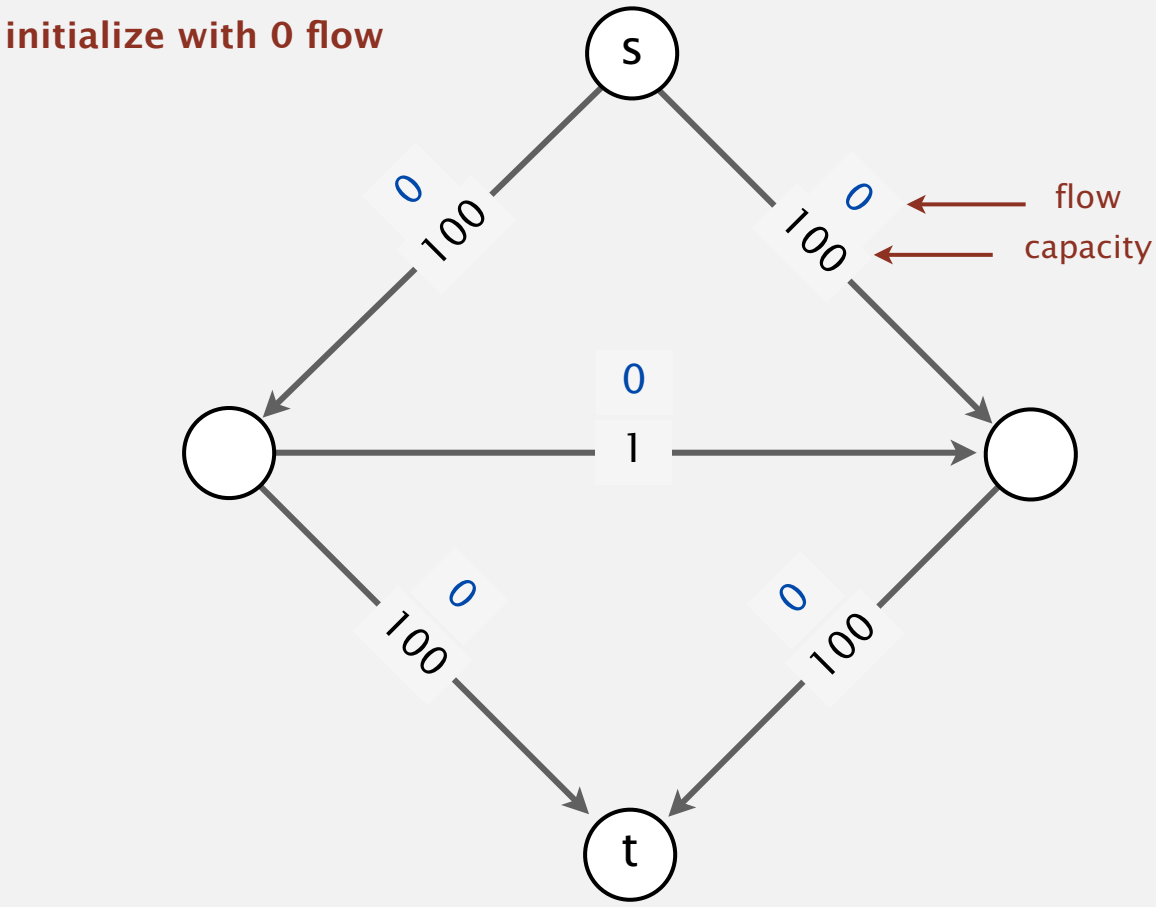
and FF finds one!

Integrality theorem. There exists an integer-valued maxflow.

Pf. Ford-Fulkerson terminates and maxflow that it finds is integer-valued.

Bad case for Ford-Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.



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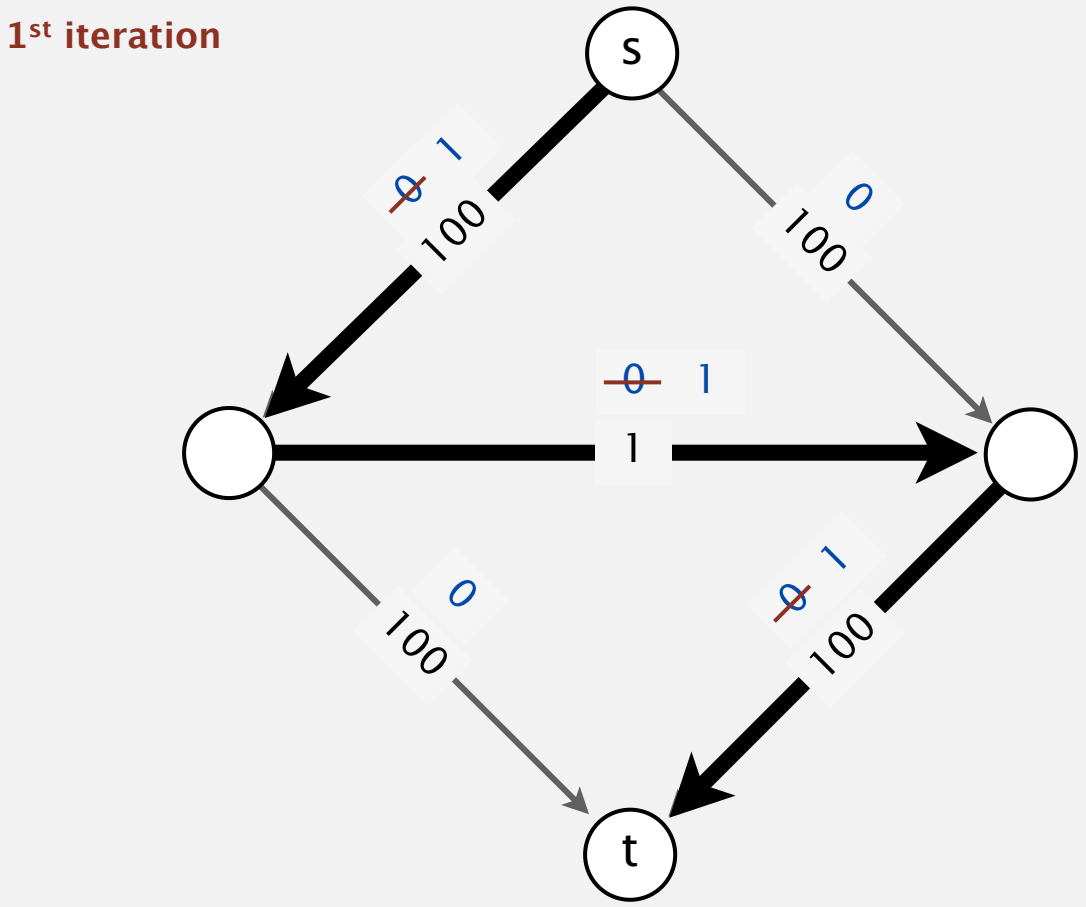

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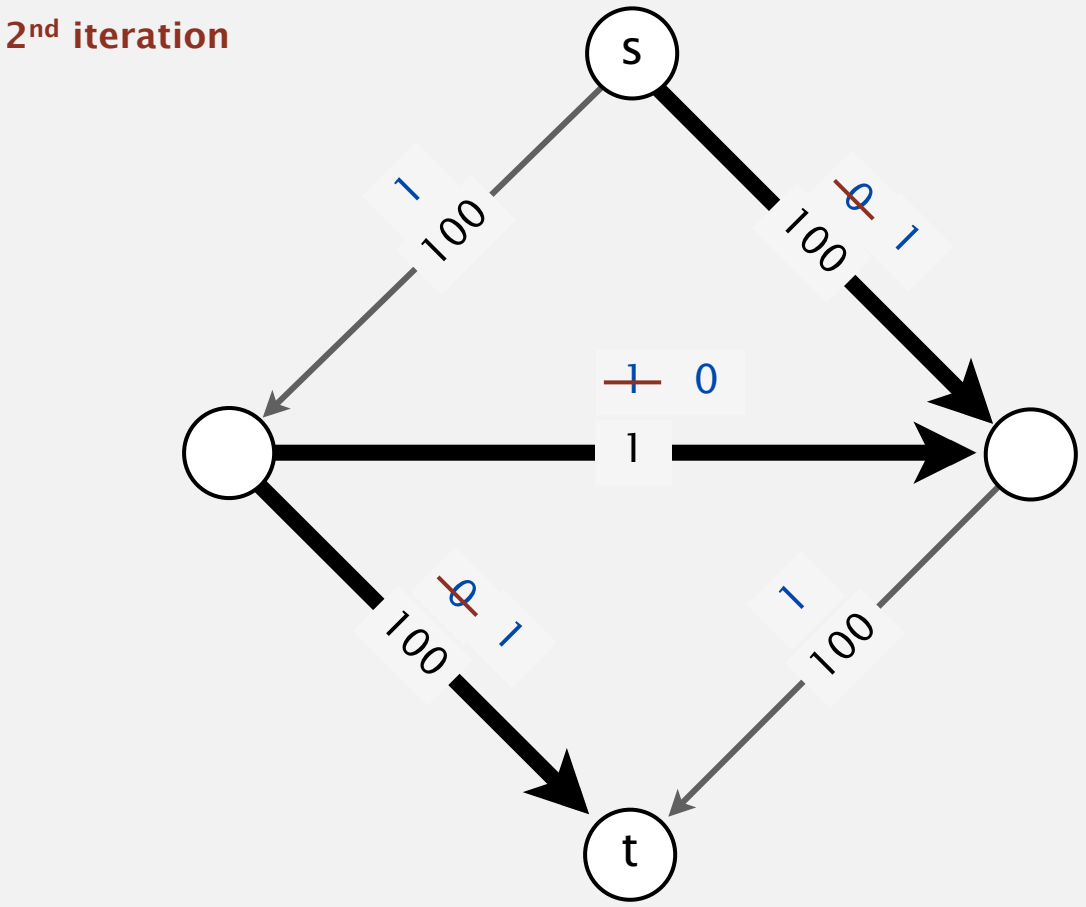

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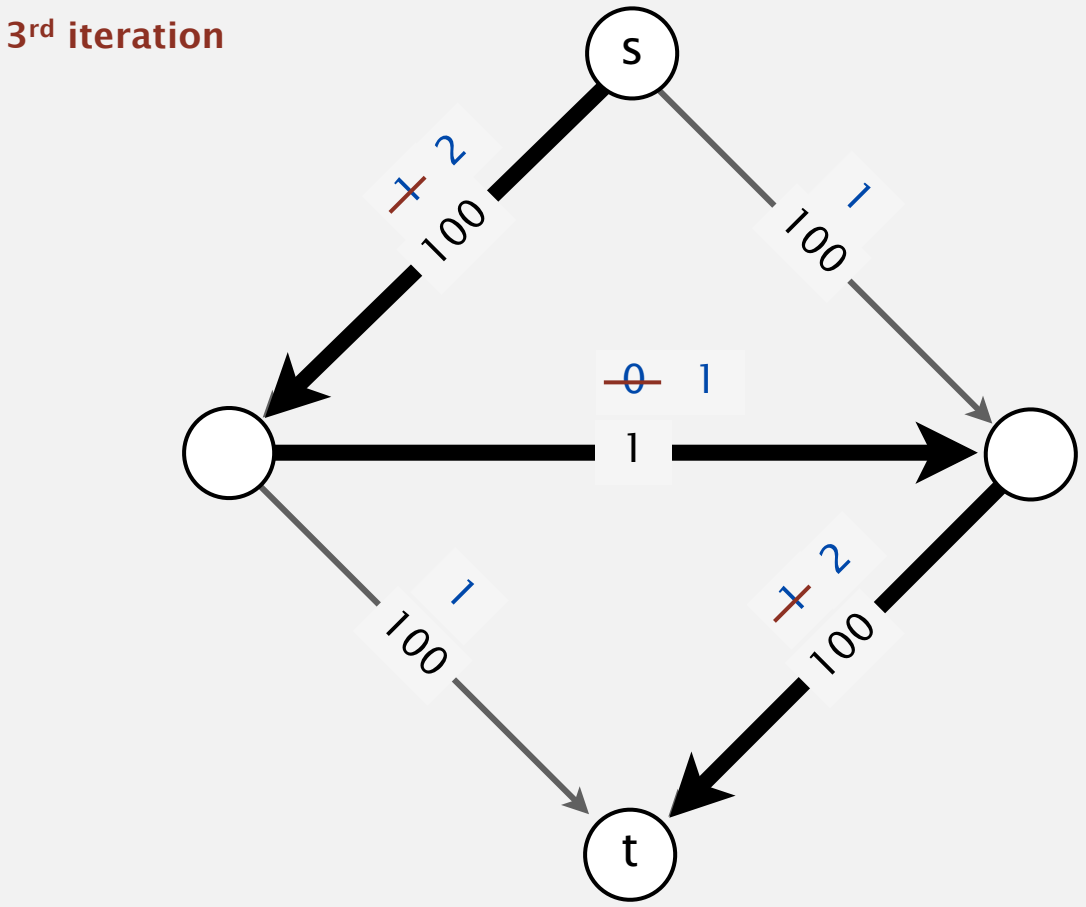

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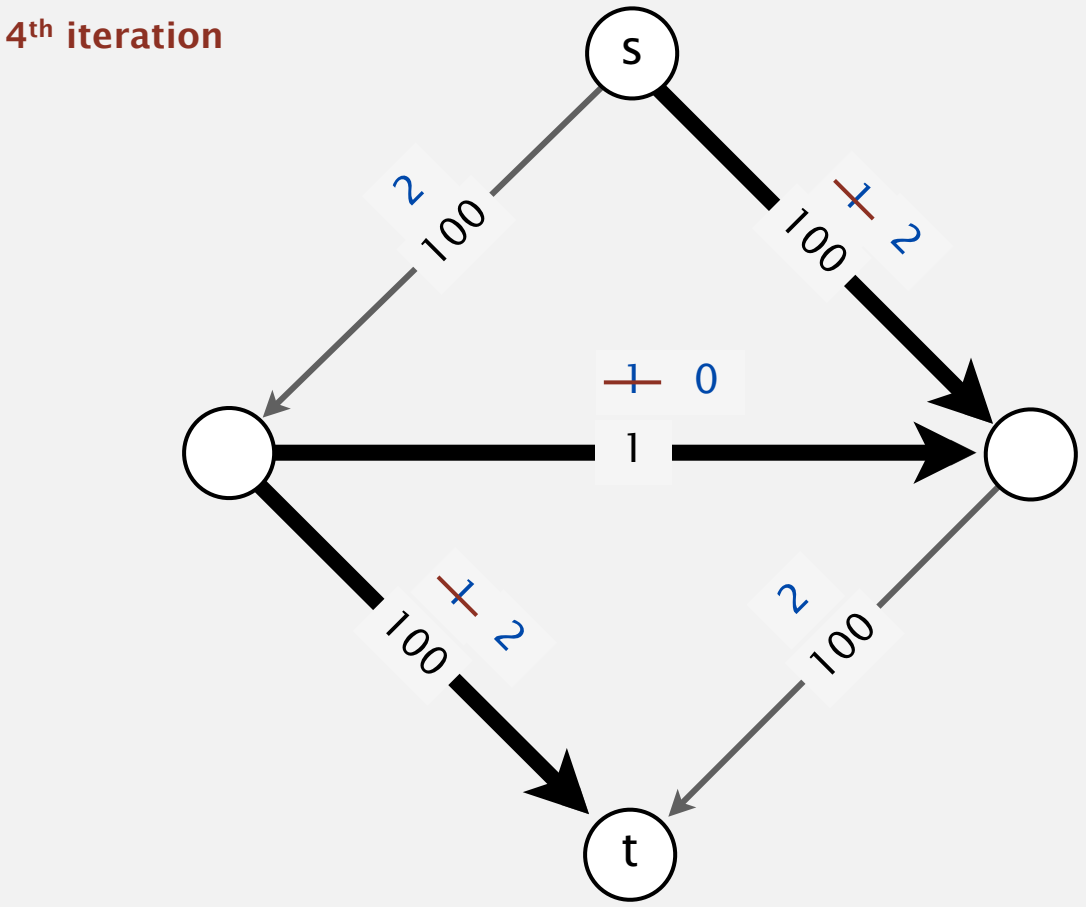

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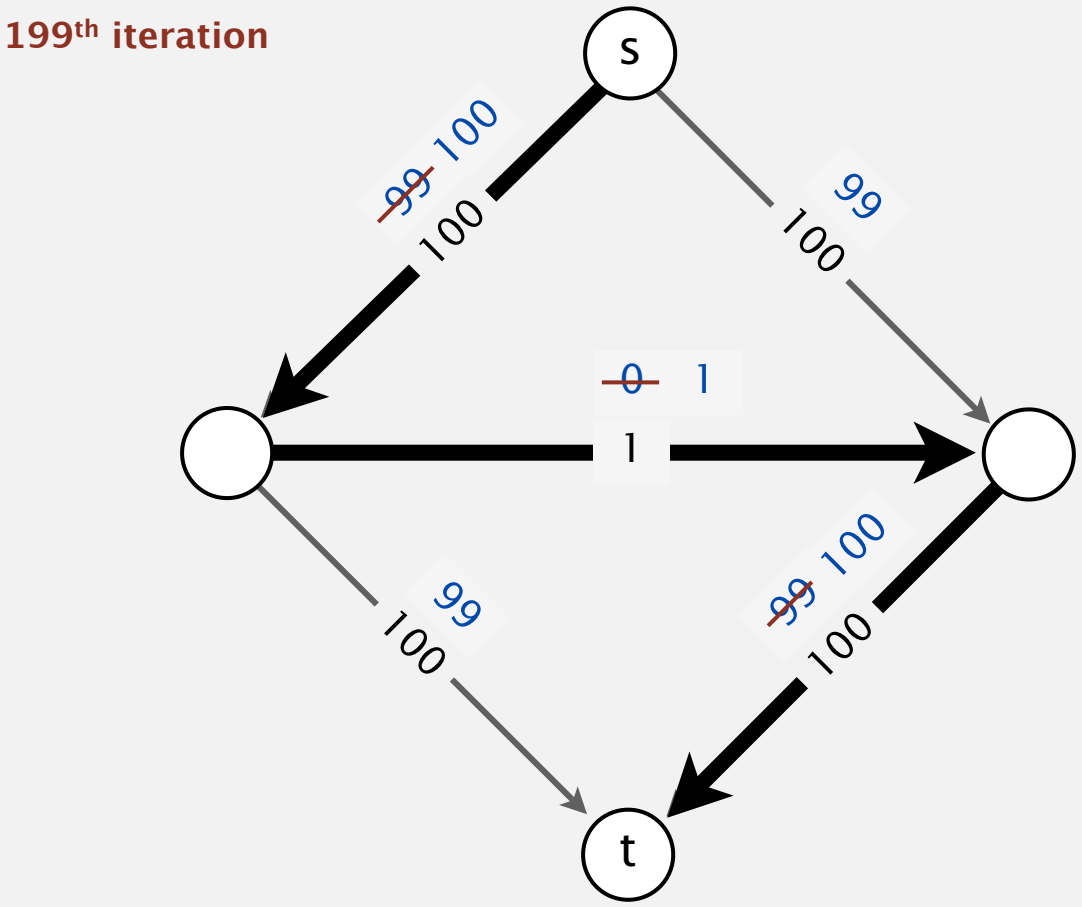

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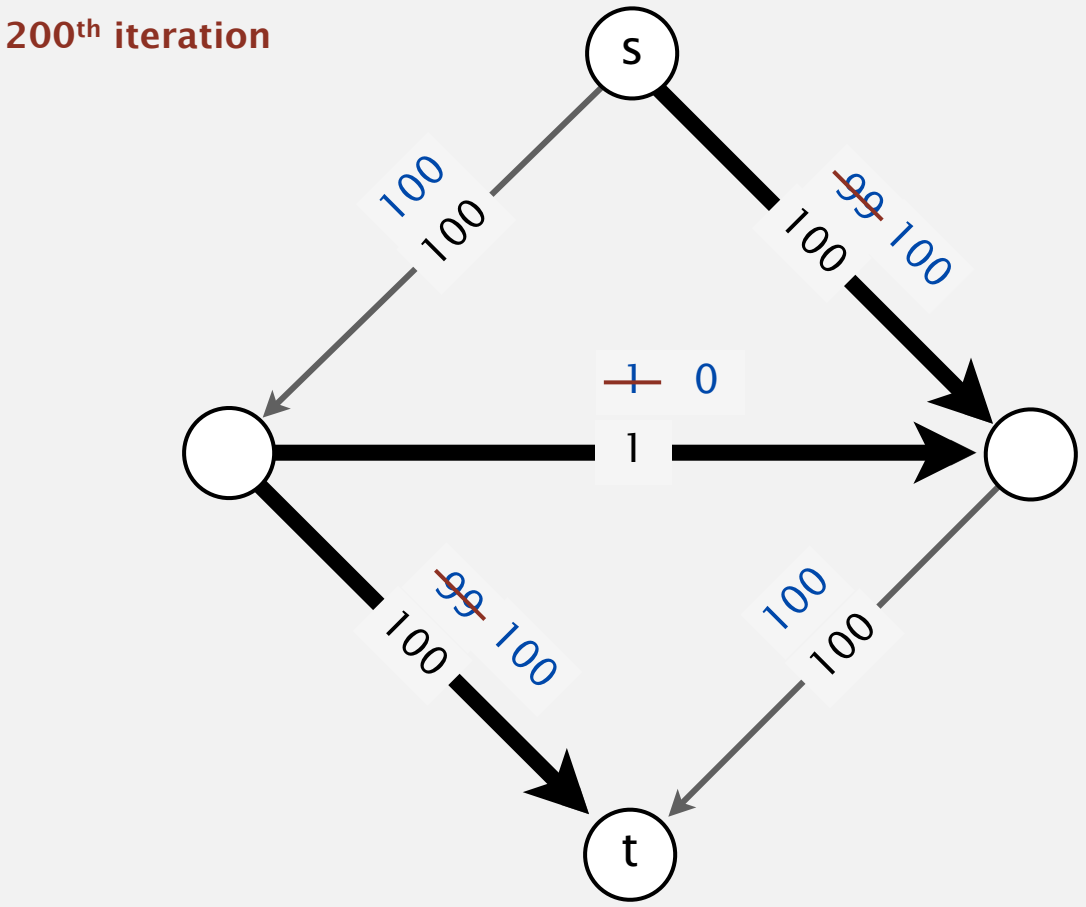
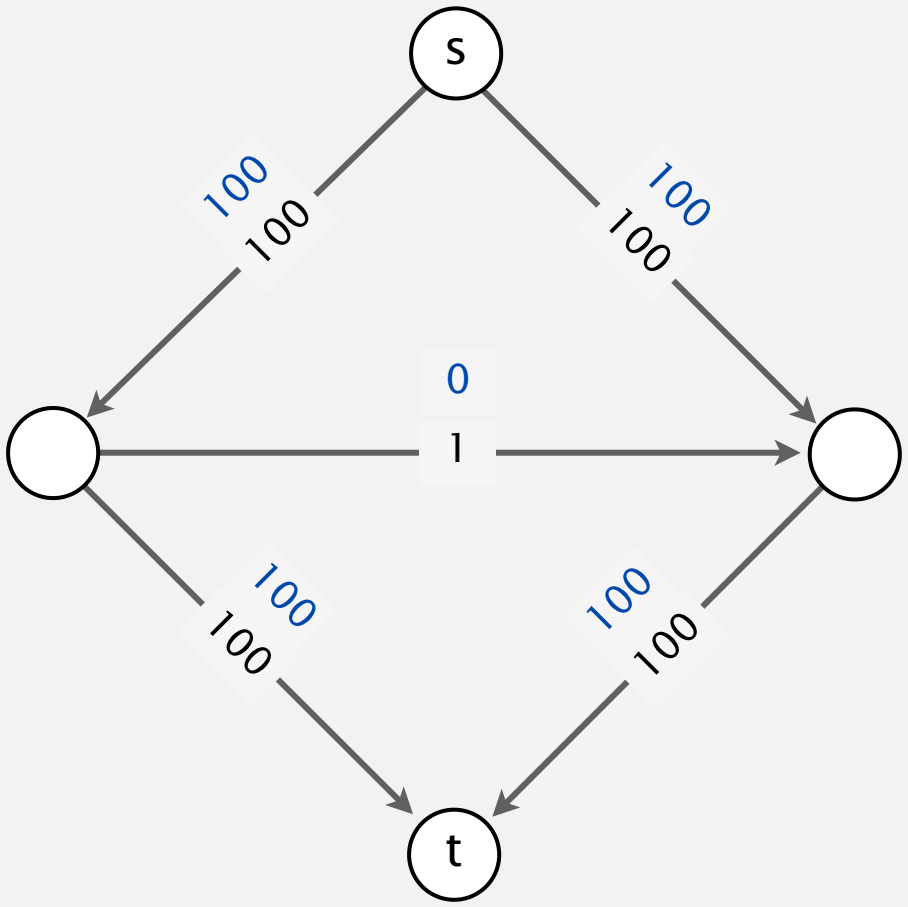


Bad case for Ford-Fulkerson

Bad news. Even when edge capacities are integers, number of augmenting paths could be equal to the value of the maxflow.

can be exponential in input size

Good news. This case is easily avoided. [use shortest/fattest path]



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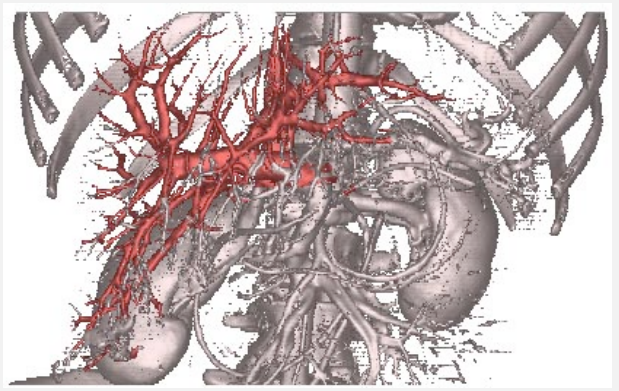
Maxflow and mincut applications

Maxflow/mincut is a widely applicable problem-solving model.

- Data mining.
- Open-pit mining.
- Bipartite matching.
- Network reliability.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Distributed computing.
- Security of statistical data.
- Egalitarian stable matching.
- Multi-camera scene reconstruction.
- Sensor placement for homeland security.
- Many, many, more.



liver and hepatic vascularization segmentation

Bipartite matching problem

N students apply for N jobs.

Each gets several offers.

Is there a way to match all students to jobs?

bipartite matching problem

1	Alice	6	Adobe
	Adobe		Alice
	Amazon		Bob
	Google		Carol
2	Bob	7	Amazon
	Adobe		Alice
	Amazon		Bob
3	Carol		Dave
	Adobe		Eliza
	Facebook	8	Facebook
	Google		Carol
4	Dave	9	Google
	Amazon		Alice
	Yahoo		Carol
5	Eliza	10	Yahoo
	Amazon		Dave
	Yahoo		Eliza

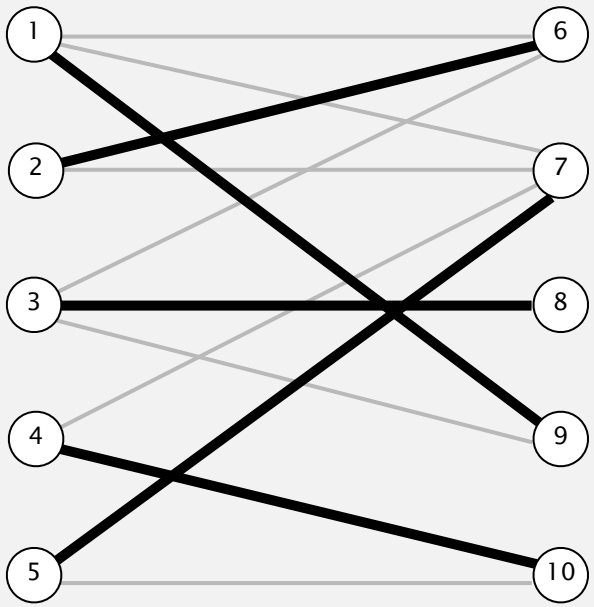
Bipartite matching problem

Given a bipartite graph, find a perfect matching.

perfect matching (solution)

Alice — Google
Bob — Adobe
Carol — Facebook
Dave — Yahoo
Eliza — Amazon

bipartite graph



bipartite matching problem

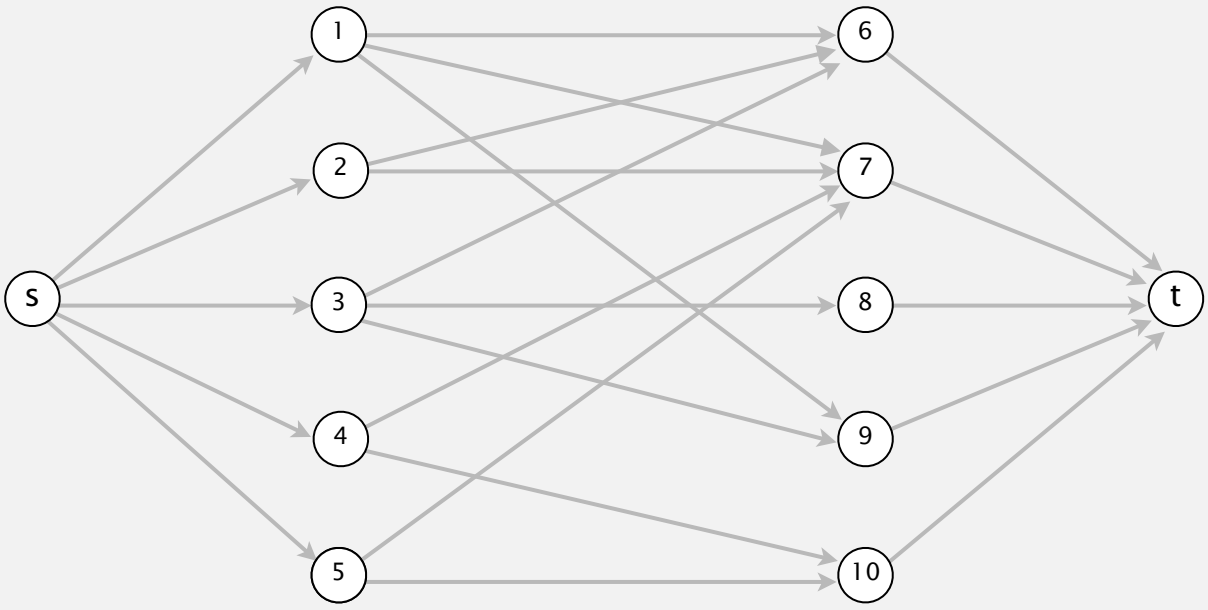
1 Alice
Adobe
Amazon
Google
2 Bob
Adobe
Amazon
3 Carol
Adobe
Facebook
Google
4 Dave
Amazon
Yahoo
5 Eliza
Amazon
Yahoo

6 Adobe
Alice
Bob
Carol
7 Amazon
Alice
Bob
Dave
Eliza
8 Facebook
Carol
9 Google
Alice
Carol
10 Yahoo
Dave
Eliza

Network flow formulation of bipartite matching

- Create s, t, one vertex for each student, and one vertex for each job.
- Add edge from s to each student (capacity 1).
- Add edge from each job to t (capacity 1).
- Add edge from student to each job offered (infinite capacity).

flow network



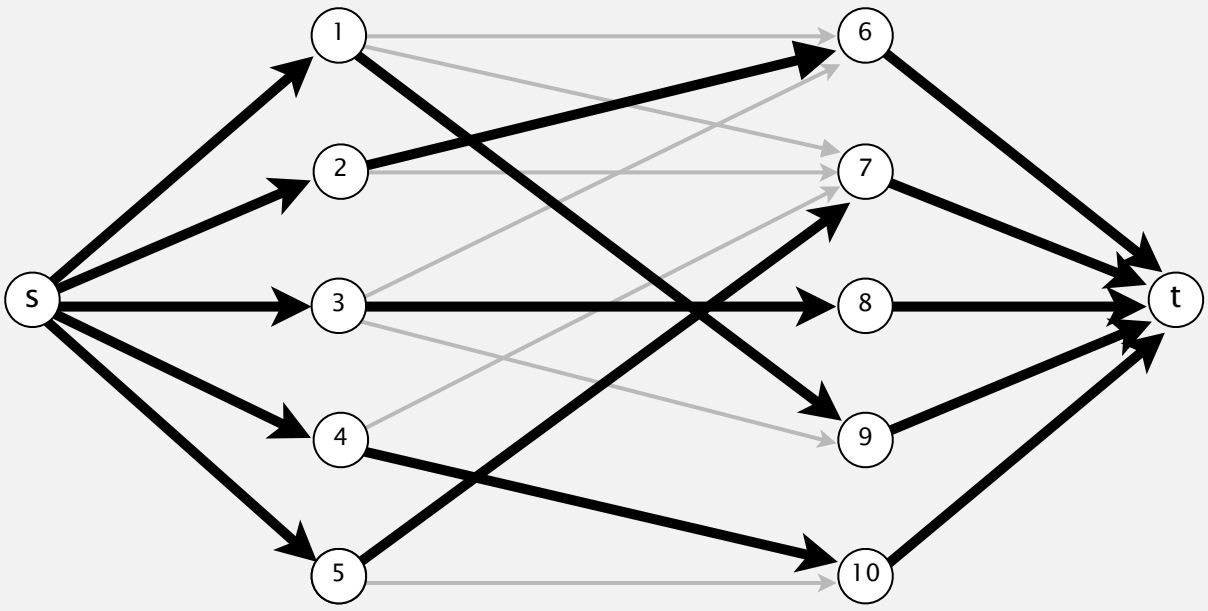
bipartite matching problem

1	Alice	6	Adobe
	Adobe		Alice
	Amazon		Bob
	Google		Carol
2	Bob	7	Amazon
	Adobe		Alice
	Amazon		Bob
3	Carol		Dave
	Adobe		Eliza
	Facebook	8	Facebook
	Google		Carol
4	Dave	9	Google
	Amazon		Alice
	Yahoo		Carol
5	Eliza	10	Yahoo
	Amazon		Dave
	Yahoo		Eliza

Network flow formulation of bipartite matching

1-1 correspondence between perfect matchings in bipartite graph and integer-valued maxflows of value N.

flow network



bipartite matching problem

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	Adobe		Alice
	Amazon		Bob
	Google		Carol
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	Google		Carol
4	Dave	9	Google
	Amazon		Alice
	Yahoo		Carol
5	Eliza	10	Yahoo
	Amazon		Dave
	Yahoo		Eliza

Maximum flow algorithms: theory

(Yet another) holy grail for theoretical computer scientists.

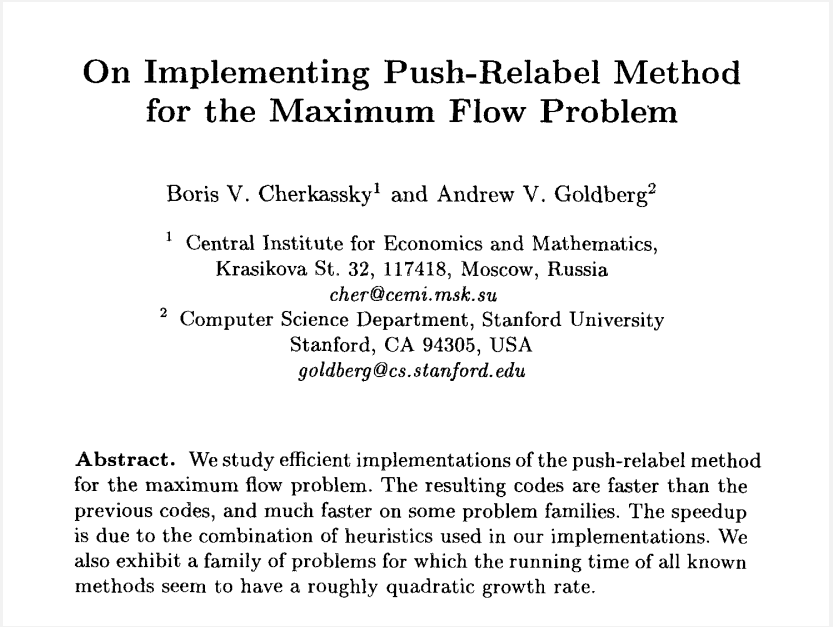
year	method	worst case	discovered by
1951	simplex	$E^3 U$	Dantzig
1955	augmenting path	$E^2 U$	Ford-Fulkerson
1970	shortest augmenting path	E^3	Dinitz, Edmonds-Karp
1970	fattest augmenting path	$E^2 \log E \log(E U)$	Dinitz, Edmonds-Karp
1977	blocking flow	$E^{5/2}$	Cherkasky
1978	blocking flow	$E^{7/3}$	Galil
1983	dynamic trees	$E^2 \log E$	Sleator-Tarjan
1985	capacity scaling	$E^2 \log U$	Gabow
1997	length function	$E^{3/2} \log E \log U$	Goldberg-Rao
2012	compact network	$E^2 / \log E$	Orlin
?	?	E	?

maxflow algorithms for sparse digraphs with E edges, integer capacities between 1 and U

Maximum flow algorithms: practice

Warning. Worst-case order-of-growth is generally not useful for predicting or comparing maxflow algorithm performance in practice.

Best in practice. Push-relabel method with gap relabeling: $E^{3/2}$.



On Implementing Push-Relabel Method for the Maximum Flow Problem

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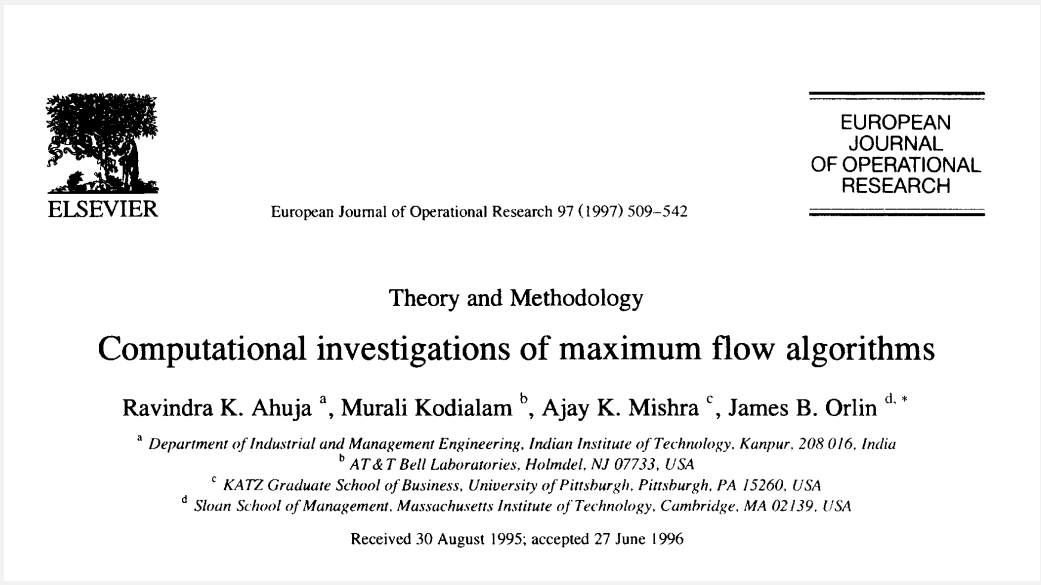
Abstract. We study efficient implementations of the push-relabel method for the maximum flow problem. The resulting codes are faster than the previous codes, and much faster on some problem families. The speedup is due to the combination of heuristics used in our implementations. We also exhibit a family of problems for which the running time of all known methods seem to have a roughly quadratic growth rate.



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Theory and Methodology

Computational investigations of maximum flow algorithms

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Summary

Mincut problem. Find an st-cut of minimum capacity.

Maxflow problem. Find an st-flow of maximum value.

Duality. Value of the maxflow = capacity of mincut.

Proven successful approaches.

- Ford-Fulkerson (various augmenting-path strategies).
- Preflow-push (various versions).

Open research challenges.

- Practice: solve real-world maxflow/mincut problems in linear time.
- Theory: prove it for worst-case inputs.
- Still much to be learned!