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ON SOME ITERATIVE ALGORITHMS FOR POLYNOMIAL FACTORIZATION[1]

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Abstract

In this paper some new iterative algorithms for polynomial factorization are analysed. The issues related to their speed of convergence and efficiency are discussed. New accurate estimates of the speed of convergence of the classical Tanabe method for simultaneously finding the zeroes of algebraic polynomials with weaker restrictions imposed on the system of initial approximations are received. The result is basic for the study of factorization algorithms based on Tanabe iteration. A numerical example is presented.

Key words: factorization of the polynomial $f(x)$ into quadratic factors, parallel iteration, Dvorchuk method, Tanabe method, Kyurkchiev–Zheng–Marinov method, order of convergence, efficiency index

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1. Introduction. Let

$$f(x) := x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \tag{1}$$

be the polynomial with simple roots x_i, $i = 1, 2, \ldots, n$.

Dvorchuk's method [1, 2] yields the factorization of the polynomial $f(x)$ into quadratic factors, i.e.

$$f(x) = \prod_{j=1}^{m}(x^2 + p_jx + q_j), \tag{2}$$

where n is even, i.e. $n = 2m$.

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The factorization with real coefficients $p_j,\ q_j,\ \ j=1,2,\ldots,m$ exists for real $a_i,\ \ i=1,2,\ldots,n$.

Hence for polynomials with real coefficients we can only perform the calculation by using real arithmetic.

Denote by t_i^0, t_{i+m}^0 the roots of the polynomials

$$x^2+p_i^0x+q_i^0,\quad i=1,2,\ldots,m,$$

and by t_i^k, t_{i+m}^k the roots of

$$x^2+p_i^kx+q_i^k,\quad i=1,2,\ldots,m.$$

The following algorithms are well-known and described in the literature.

Algorithm 1 (based on Dvorchuk method)**.**

$$\begin{aligned}
p_i^{k+1} &= p_i^k+\frac{f(t_i^k)}{\prod\limits_{j\neq i}^{n}(t_i^k-t_j^k)}+\frac{f(t_{i+m}^k)}{\prod\limits_{j\neq i+m}^{n}(t_{i+m}^k-t_j^k)},\\
q_i^{k+1} &= q_i^k-t_{i+m}^k\frac{f(t_i^k)}{\prod\limits_{j\neq i}^{n}(t_i^k-t_j^k)}-t_i^k\frac{f(t_{i+m}^k)}{\prod\limits_{j\neq i+m}^{n}(t_{i+m}^k-t_j^k)},
\end{aligned}\tag{3}$$

$$i=1,2,\ldots,m;\quad k=0,1,\ldots.$$

Remark 1. Dvorchuk algorithm is based on the classic WEIERSTRASS–DOCHEV iteration [3, 4] (see, also [5]).

Factorization methods of BAIRSTOW [6], HITCHKOCK [7], Dvorchuk are widely used.

Algorithm 2 (based on KYURKCHIEV–ZHENG–MARINOV method [8])**.**

$$\begin{aligned}
p_i^{k+1} &= p_i^k+\frac{f(t_i^k)}{\prod\limits_{j\neq i}^{n}(t_i^k-t_j^k-\Delta_j^{R,k})}+\frac{f(t_{i+m}^k)}{\prod\limits_{j\neq i+m}^{n}(t_{i+m}^k-t_j^k-\Delta_j^{R,k})},\\
q_i^{k+1} &= q_i^k-t_{i+m}^k\frac{f(t_i^k)}{\prod\limits_{j\neq i}^{n}(t_i^k-t_j^k-\Delta_j^{R,k})}-t_i^k\frac{f(t_{i+m}^k)}{\prod\limits_{j\neq i+m}^{n}(t_{i+m}^k-t_j^k-\Delta_j^{R,k})}\\
&\quad+\frac{f(t_i^k)}{\prod\limits_{j\neq i}^{n}(t_i^k-t_j^k-\Delta_j^{R,k})}\cdot\frac{f(t_{i+m}^k)}{\prod\limits_{j\neq i+m}^{n}(t_{i+m}^k-t_j^k-\Delta_j^{R,k})},
\end{aligned}\tag{4}$$

$$i = 1, 2, \ldots, m; \quad k = 0, 1, \ldots,$$

where

$$\Delta_s^{R,k} = -\frac{f(t_s)}{\prod\limits_{l \neq s}^{n} (t_s^k - t_l^k - \Delta_l^{R-1,k})},$$

and R is a fixed integer.

Remark 2. Algorithm 2 is based on Kyurkchiev and Andreev iterative procedure [9].

Procedure (4) has order of convergence, which is a function of the parameter R, and it is an input value for the subroutine used by the user of this factorization method.

Remark 3. Assuming that

$$\frac{f(t_i^k)}{\prod\limits_{j \neq i}^{n} (t_i^k - t_j^k - \Delta_j^{R,k})} \cdot \frac{f(t_{i+m}^k)}{\prod\limits_{j \neq i+m}^{n} (t_{i+m}^k - t_j^k - \Delta_j^{R,k})}$$

is small enough (in other words, all starting approximations are taken to be sufficiently close to the zeroes), we have the following:

Algorithm 3 ([8]).

$$\begin{aligned} p_i^{k+1} &= p_i^k + \frac{f(t_i^k)}{\prod\limits_{j \neq i}^{n} (t_i^k - t_j^k - \Delta_j^{R,k})} + \frac{f(t_{i+m}^k)}{\prod\limits_{j \neq i+m}^{n} (t_{i+m}^k - t_j^k - \Delta_j^{R,k})}, \\ q_i^{k+1} &= q_i^k - t_{i+m}^k \frac{f(t_i^k)}{\prod\limits_{j \neq i}^{n} (t_i^k - t_j^k - \Delta_j^{R,k})} - t_i^k \frac{f(t_{i+m}^k)}{\prod\limits_{j \neq i+m}^{n} (t_{i+m}^k - t_j^k - \Delta_j^{R,k})}, \end{aligned} \tag{5}$$

$$i = 1, 2, \ldots, m; \quad k = 0, 1, \ldots.$$

For other results see Zheng [10], Sendov, Andreev and Kyurkchiev [11], Iliev and Kyurkchiev [12], Kyurkchiev ([13, 14]), Kyurkchiev and Mahdi [15], Luk [16], Zheng and Kyurkchiev [17].

Of particular interest is the task for the partial factorization (see e.g. Zheng, Kyurkchiev and Iliev [18]).

In this article, we will construct other methods for polynomial factorization which are based on known methods for simultaneous finding of all simple zeroes – as an example, the Tanabe method [19, 20].

2. Main results. The following method for simultaneous finding of all simple zeroes of $f(x)$ is known as Tanabe method:

$$x_i^{k+1} = x_i^k - \frac{f(x_i^k)}{\prod\limits_{j\neq i}^{n}(x_i^k - x_j^k)}\left(1 - \sum_{j\neq i}^{n}\frac{f(x_j^k)}{(x_i^k - x_j^k)\prod\limits_{s\neq j}^{n}(x_j^k - x_s^k)}\right), \tag{6}$$

$$i = 1, 2, \ldots n; \quad k = 0, 1, 2, \ldots .$$

On the basis of this iteration process, we will formulate the following new algorithm for factorization:

Algorithm A. In the terms of factorization (2) the following parallel algorithm for polynomial decomposition of products of quadratic factors is offered:

$$\begin{aligned}
p_i^{k+1} &= p_i^k + \frac{f(t_i^k)}{\prod\limits_{j\neq i}^{n}(t_i^k - t_j^k)}\left(1 - \sum_{j\neq i}^{n}\frac{f(t_j^k)}{(t_i^k - t_j^k)\prod\limits_{s\neq j}^{n}(t_j^k - t_s^k)}\right) \\
&\quad + \frac{f(t_{i+m}^k)}{\prod\limits_{j\neq i+m}^{n}(t_{i+m}^k - t_j^k)}\left(1 - \sum_{j\neq i+m}^{n}\frac{f(t_j^k)}{(t_{i+m}^k - t_j^k)\prod\limits_{s\neq j}^{n}(t_j^k - t_s^k)}\right), \\
q_i^{k+1} &= q_i^k - t_i^k\frac{f(t_{i+m}^k)}{\prod\limits_{j\neq i+m}^{n}(t_{i+m}^k - t_j^k)}\left(1 - \sum_{j\neq i+m}^{n}\frac{f(t_j^k)}{(t_{i+m}^k - t_j^k)\prod\limits_{s\neq j}^{n}(t_j^k - t_s^k)}\right) \\
&\quad - t_{i+m}^k\frac{f(t_i^k)}{\prod\limits_{j\neq i}^{n}(t_i^k - t_j^k)}\left(1 - \sum_{j\neq i}^{n}\frac{f(t_j^k)}{(t_i^k - t_j^k)\prod\limits_{s\neq j}^{n}(t_j^k - t_s^k)}\right),
\end{aligned} \tag{7}$$

$$i = 1, 2, \ldots, m; \quad k = 0, 1, \ldots .$$

Remark 4. In constructing the iteration for q_i^{k+1} in (7) the following magnitude is ignored:

$$\frac{f(t_{i+m}^k)}{\prod\limits_{j\neq i+m}^{n}(t_{i+m}^k - t_j^k)} \left(1 - \sum_{j\neq i+m}^{n} \frac{f(t_j^k)}{(t_{i+m}^k - t_j^k)\prod\limits_{s\neq j}^{n}(t_j^k - t_s^k)}\right) \frac{f(t_i^k)}{\prod\limits_{j\neq i}^{n}(t_i^k - t_j^k)}$$

$$\times \left(1 - \sum_{j\neq i}^{n} \frac{f(t_j^k)}{(t_i^k - t_j^k)\prod\limits_{s\neq j}^{n}(t_j^k - t_s^k)}\right),$$

which is sufficiently small, without influence to the speed of the proposed Algorithm A.

Remark 5. A more compact record of algorithm (7) is as follows:

$$\begin{aligned} p_i^{k+1} &= p_i^k + W_i^k\left(1 - \sum_{j\neq i}^{n}\frac{W_j^k}{t_i^k - t_j^k}\right) + W_{i+m}^k\left(1 - \sum_{j\neq i+m}^{n}\frac{W_j^k}{t_{i+m}^k - t_j^k}\right), \\ q_i^{k+1} &= q_i^k - t_i^k W_{i+m}^k\left(1 - \sum_{j\neq i+m}^{n}\frac{W_j^k}{t_{i+m}^k - t_j^k}\right) - t_{i+m}^k W_i^k\left(1 - \sum_{j\neq i}^{n}\frac{W_j^k}{t_i^k - t_j^k}\right), \end{aligned} \tag{8}$$

$$i = 1, 2, \dots, m; \quad k = 0, 1, \dots,$$

where we have used the symbol

$$W_i^k = \frac{f(t_i^k)}{\prod\limits_{j\neq i}^{n}(t_i^k - t_j^k)}, \quad i = 1, 2, \dots.$$

In the literature several proofs for the order of convergence of the Tanabe method with some restrictions imposed on the system of initial approximations for the polynomial zeroes are known. For other results related to the study of SOR-Tanabe's method, see [21–24].

One can find a meticulous proof on the important issue of "guaranteed convergence", in the monograph of Petkovic, Herceg and Ilic [25] (see, also [26]).

3. Auxiliary result. For the purposes of this article, namely for evaluation of the error of the new factorization method of type (7), based on Tanabe iteration, a new convergence proof of method (6) is adduced.

For the error of Tanabe method $x_i^{k+1} - x_i$ from (6) we have

$$
\begin{aligned}
x_i^{k+1} - x_i &= x_i^k - x_i \\
&- (x_i^k - x_i)\prod_{j\neq i}^{n}\frac{x_i^k - x_j}{x_i^k - x_j^k}\left(1 - \sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{l\neq s}^{n}\frac{x_s^k - x_l}{x_s^k - x_l^k}\right) \\
&= (x_i^k - x_i)\left(1-\prod_{j\neq i}^{n}\frac{x_i^k - x_j}{x_i^k - x_j^k}\left(1-\sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{l\neq s}^{n}\frac{x_s^k - x_l}{x_s^k - x_l^k}\right)\right).
\end{aligned}
\tag{9}
$$

The following presentation is known [27]:

$$
\prod_{j\neq i}^{n}\frac{x_i^k - x_j}{x_i^k - x_j^k} - 1 = \sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{j\neq i}^{s-1}\frac{x_i^k - x_j}{x_i^k - x_j^k}. \tag{10}
$$

Then (9) can be written in the following way:

$$
\begin{aligned}
x_i^{k+1} - x_i = (x_i^k - x_i)&\left(1 - \left(1 + \sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{j\neq i}^{s-1}\frac{x_i^k - x_j}{x_i^k - x_j^k}\right)\right. \\
&\times\left.\left(1 - \sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{l\neq s}^{n}\frac{x_s^k - x_l}{x_s^k - x_l^k}\right)\right) \\
= (x_i^k - x_i)&\left(\sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{l\neq s}^{n}\frac{x_s^k - x_l}{x_s^k - x_l^k} - \sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{j\neq i}^{s-1}\frac{x_i^k - x_j}{x_i^k - x_j^k}\right. \\
&+\left.\sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{j\neq i}^{s-1}\frac{x_i^k - x_j}{x_i^k - x_j^k}\sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{l\neq s}^{n}\frac{x_s^k - x_l}{x_s^k - x_l^k}\right) \\
= (x_i^k - x_i)&\left(A + \sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{j\neq i}^{s-1}\frac{x_i^k - x_j}{x_i^k - x_j^k}\sum_{s\neq i}^{n}\frac{x_s^k - x_s}{x_i^k - x_s^k}\prod_{l\neq s}^{n}\frac{x_s^k - x_l}{x_s^k - x_l^k}\right).
\end{aligned}
\tag{11}
$$

The expression A can be written in the following way:

$$A = \sum_{s\neq i}^{n} \frac{x_s^k - x_s}{x_i^k - x_s^k} \prod_{l\neq s}^{n} \frac{x_s^k - x_l}{x_s^k - x_l^k} - \sum_{s\neq i}^{n} \frac{x_s^k - x_s}{x_i^k - x_s^k} \prod_{j\neq i}^{s-1} \frac{x_i^k - x_j}{x_i^k - x_j^k}$$

$$= \sum_{s\neq i}^{n} \frac{x_s^k - x_s}{x_i^k - x_s^k} \left(\prod_{l\neq s}^{n} \frac{x_s^k - x_l}{x_s^k - x_l^k} - \prod_{j\neq i}^{s-1} \frac{x_i^k - x_j}{x_i^k - x_j^k} + 1 - 1 \right) \tag{12}$$

$$= \sum_{s\neq i}^{n} \frac{x_s^k - x_s}{x_i^k - x_s^k} \left(\prod_{l\neq s}^{n} \frac{x_s^k - x_l}{x_s^k - x_l^k} - 1 \right) - \sum_{s\neq i}^{n} \frac{x_s^k - x_s}{x_i^k - x_s^k} \left(\prod_{j\neq i}^{s-1} \frac{x_i^k - x_j}{x_i^k - x_j^k} - 1 \right).$$

In the second equality of (12) 1 is added and removed in order (10) to be applied again.

Finally, in this way we get A for

$$\begin{aligned} A = {} & \sum_{s\neq i}^{n} \frac{x_s^k - x_s}{x_i^k - x_s^k} \sum_{t\neq s}^{n} \frac{x_t^k - x_t}{x_s^k - x_t^k} \prod_{l\neq s}^{t-1} \frac{x_s^k - x_l}{x_s^k - x_l^k} \\ & - \sum_{s\neq i}^{n} \frac{x_s^k - x_s}{x_i^k - x_s^k} \sum_{t\neq i}^{s-1} \frac{x_t^k - x_t}{x_i^k - x_t^k} \prod_{j\neq i}^{t-1} \frac{x_i^k - x_j}{x_i^k - x_j^k}. \end{aligned} \tag{13}$$

From (11) and (13) it can be easily reasoned that Tanabe iteration has cubic order of convergence.

Now we will adduce a precise analysis of this statement.

4. Convergence theorems. The following statement is valid:

Theorem A. *Let*

$$0 < q < 1, \quad d = \min_{i\neq j} |x_i - x_j|$$

and $c > 0$ be a number such that

$$0 < \frac{cBn}{d - 2c} \leq 1, \tag{14}$$

where B is a root of nonlinear equation

$$B^2 - e^{\frac{1}{B}} - e^{\frac{2}{B}} = 0. \tag{15}$$

If the initial approximations $\{x_i^0\}_{i=1}^n$ of the roots $\{x_i\}_{i=1}^n$ of $f(x)$ satisfy the inequalities

$$|x_i - x_i^0| \leq cq, \quad i = 1, 2, \ldots, n,$$

then the following estimate for the Tanabe method

$$|x_i^k - x_i| \le cq^{3^k}, \tag{16}$$

$$i = 1, 2, \dots, n; \quad k = 0, 1, 2, \dots,$$

holds true.

(A more precise value for B is $B \approx 2.06375$).

Proof. We will do the proof of (16) by induction.

Assume that inequalities (16) for some $k = m$ are satisfied.

Considering the conditions and symbols in the theorem, we obtain

$$|x_i^m - x_j^m| \ge |x_i - x_j| - |x_i^m - x_i| - |x_j^m - x_j| > d - 2c,$$

$$\left| \prod_{j \ne i}^{n} \frac{x_i^m - x_j}{x_i^m - x_j^m} \right| \le \left| \prod_{j \ne i}^{n} \left(1 + \frac{x_j^m - x_j}{x_i^m - x_j^m} \right) \right| \le \left(1 + \frac{c}{d - 2c} \right)^n \le \left(1 + \frac{1}{Bn} \right)^n \le e^{\frac{1}{B}},$$

$$\left| \sum_{j \ne i}^{n} \frac{x_j^m - x_j}{x_i^m - x_j^m} \prod_{s \ne j}^{n} \frac{x_j^m - x_s}{x_j^m - x_s^m} \right| \le \frac{ncq^{3m}}{d - 2c} e^{\frac{1}{B}} \le \frac{q^{3m}}{B} e^{\frac{1}{B}}.$$

Then from (11) and (13) we have

$$|x_i^{m+1} - x_i| \le cq^{3m} \left(2n^2 \frac{c^2 q^{2.3^m}}{(d - 2c)^2} e^{\frac{1}{B}} + n^2 \frac{c^2 q^{2.3^m}}{(d - 2c)^2} e^{\frac{2}{B}} \right)$$

$$\le cq^{3m+1} \left(2n^2 \frac{1}{B^2 n^2} e^{\frac{1}{B}} + n^2 \frac{1}{B^2 n^2} e^{\frac{2}{B}} \right) \le cq^{3m+1} \frac{1}{B^2} \left(e^{\frac{1}{B}} + e^{\frac{2}{B}} \right).$$

Having in mind that B is the solution of nonlinear equation (15), finally we get

$$|x_i^{m+1} - x_i| \le cq^{3^{m+1}}.$$

Thus the theorem is proven.

Remark 6. Till now the most accurate result for guaranteed convergence of the Tanabe method is contained in the following theorem:

Theorem (Petkovic, Herceg and Ilic [25])**.** *Tanabe's method* (6) *is convergent under the condition*

$$\omega = \max_{1 \le i \le n} |W_i^0| = \max_{1 \le i \le n} \left| \frac{f(x_i^0)}{\prod_{j \ne i}^{n} (x_i^0 - x_j^0)} \right| < \frac{d}{3n}. \tag{17}$$

In our case we obtain ω from Theorem A

$$\omega^* \le ce^{\frac{1}{B}} \le \frac{d}{2 + 2.06375n} e^{\frac{1}{B}}.$$

Hence we can conclude that $\omega^* > \omega$ for each n, i.e. we have proved the theorem for convergence of Tanabe method in weaker limits on starting vector approximations for the polynomial roots.

This can be seen immediately in the construction of Theorem A and condition (14)

$$|x_i - x_i^0| \le cq \le c \le \frac{d}{2 + 2.06375n}.$$

Remark 7. Probably the theorem for the convergence of the Tanabe method proven in this article may be further refined using a design technique proposed by PROINOV in [28].

Now we can formulate the following theorem for convergence of Algorithm A – (7):

Theorem B. *Let the conditions of Theorem A are satisfied. If the initial approximations* $\{t_i^0\}_{i=1}^n$ *satisfy the inequalities* $|t_i - t_i^0| \le cq, \quad i = 1, 2, \ldots, n$, and

$$L = \max_i |t^{k+1} + t_i|,$$

then

$$\frac{1}{n}\sum_{i=1}^{m} |p_i^{k+1} - p_i| \le cq^{3^k},$$
$$\frac{1}{nL}\sum_{i=1}^{m} |q_i^{k+1} - q_i| \le \frac{1}{2}cq^{3^k}, \tag{18}$$

where

$$c \le \frac{d}{2 + 2.06375n},$$

i.e. the sequences $\{p_i^k\}$ and $\{q_i^k\}$, *generated by Algorithm A –* (7) *are convergent with cubic order of convergence.*

Proof. It is clear that

$$\sum_{i=1}^{m} |p_i^{k+1} - p_i| = \sum_{i=1}^{n} |t_i - t_i^{k+1}|,$$
$$\sum_{i=1}^{m} |q_i^{k+1} - q_i| = \frac{1}{2}\sum_{i=1}^{m} \left(|t_i^{k+1} - t_i||t_{i+m}^{k+1} + t_{i+m}| + |t_{i+m}^{k+1} - t_{i+m}||t_i^{k+1} + t_i|\right). \tag{19}$$

Using Theorem A we have

$$|t_i^{k+1} - t_i| \le cq^{3^k}.$$

From (19) we obtain

$$\sum_{i=1}^{m}|p_i^{k+1}-p_i|\leq cq^{3^k}n$$

and

$$\sum_{i=1}^{m}|q_i^{k+1}-q_i|\leq \frac{1}{2}cq^{3^k}nL.$$

The theorem is proved.

5. Numerical example. Let

$$f(t)=t^4-1=\prod_{j=1}^{2}(t^2+p_jt+q_j).$$

For numerical determination of p_i^{k+1}, q_i^{k+1}, we apply the algorithm A using initial approximations

$$p_1^0=-0.05;\ p_2^0=-0.05I$$

$$q_1^0=-0.765;\ q_2^0=0.765$$

We will make comparison with the results obtained by the method (4) – Algorithm 2 when $R=1$.

I. Algorithm 2 (see, [8])

Table 1

k	p_1^{k+1} / p_2^{k+1}	q_1^{k+1} / q_2^{k+1}
0	$0.23883\times10^{-1}-0.286415\times10^{-1}I$	$-1.11781+0.537260\times10^{-2}I$
	$-0.286415\times10^{-1}+0.238883\times10^{-1}I$	$1.11781+0.537260\times10^{-2}I$
1	$0.471474\times10^{-2}+0.474005\times10^{-2}I$	$-1.01428+0.944577\times10^{-3}I$
	$-0.474005\times10^{-2}+0.471474\times10^{-2}I$	$1.01428+0.944577\times10^{-3}I$
2	$0.130205\times10^{-3}-0.130200\times10^{-3}I$	$-1.00025+0.382490\times10^{-4}I$
	$-0.130200\times10^{-3}+0.130205\times10^{-3}I$	$1.00025+0.382490\times10^{-4}I$
3	$0.648036\times10^{-7}-0.648034\times10^{-7}I$	$-1.00000+0.301639\times10^{-7}I$
	$-0.648034\times10^{-7}+0.648036\times10^{-7}I$	$1.00000+0.301639\times10^{-7}I$
4	$0.101806\times10^{-13}-0.101834\times10^{-13}I$	$-1.00000+0.748425\times10^{-14}I$
	$-0.101834\times10^{-13}+0.101814\times10^{-13}I$	$1.00000+0.748435\times10^{-14}I$

II. On Table 2 we give results of the numerical experiments obtained in Algorithm A.

T a b l e 2

k	p_1^{k+1} p_2^{k+1}	q_1^{k+1} q_2^{k+1}
0	$-0.004610 + 0.004610I$ $0.004610 - 0.004610I$	$-0.965166 - 0.0007482I$ $0.965166 - 0.0007482I$
1	$-0.0000093368230801 + 0.0000093368230801I$ $0.0000093368230801 - 0.0000093368230801I$	$-0.9996514935152646 - 0.0000042638044115I$ $0.9996514935152646 - 0.0000042638044115I$
2	$-1.7028309 \times 10^{-12} + 1.7028952 \times 10^{-12}I$ $1.7028953 \times 10^{-12} - 1.7028309 \times 10^{-12}I$	$-0.9999999695979953 - 6.9994223 \times 10^{-10}I$ $0.9999999695979953 - 6.9994223 \times 10^{-10}I$
3	$-3.082414 \times 10^{-17} - 2.2 \times 10^{-24}I$ $-5. \times 10^{-25} - 3.082414 \times 10^{-17}I$	$-0.9999999999999999 - 1.06398242 \times 10^{-17}I$ $0.9999999999999999 - 1.06398225 \times 10^{-17}I$

Remark 8. From the experiment presented here it can be seen that the results using Algorithm A on the third iteration step are better than those obtained with Algorithm 2 (on the fourth iteration step).

We note that both mentioned algorithms have the same theoretical speed of convergence.

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