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Semilinear Elliptic Equation with Asymptotes. Some Multiplicity and Existence Results

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In the present paper the existence of multiple solutions for a weakly nonlinear elliptic boundary value problem is studied. The main result (Theorem 1) states that under certain restrictions for the growth of the nonlinear part and its behaviour at infinity, which are related to the eigenfunctions of the linear part, there are at least four solutions for an unbounded open set of functions in the right-hand side of the equation.

0. Introduction

The purpose of the present paper is to study the existence and the multiplicity of solutions of the semi-linear elliptic equation

$$\Delta u+f(x,u)=y(x) \quad \text{in} \quad \Omega, \qquad u=0 \quad \text{on} \quad \Omega \tag{0.1}$$

under some appropriate restrictions on the functions $y(x)$.

Our basic assumptions on the nonlinear term are that it is "linear" at infinity in the sense that we suppose the existence of the limits $f(x,s)/s$ for $s\to\pm\infty$. We do not suppose them to be equal, however, so that the problem is not asymptotically linear. The results are derived by studying the behaviour of an "asymptotic operator", and the existence, as well as the multiplicity results are then deduced from the properties of this operator only, without making use of critical point theory as usual in similar cases.

Semi-linear problems of this kind have been subject to numerous studies, see for example [1]–[5] and the references they include.

Our multiplicity result (Theorem 1 below) treats the case when the function $f(x,s)$ ultimately increases and is a sharpening of the results of a previous paper [6]. Making additional hypotheses on the limits of $f(x,s)/s$ and their behaviour with respect to the first two eigenvalues of the Laplace operator, we obtain that for $y(x)$ in some open unbounded set the equation (0.1) has at least four solutions. In [5] some hints towards the possibility of obtaining similar result are given, but we believe our method to be different, being closer to the one employed in [7], where a case of a resonance with just one eigenvalue is studied.

As regards the existence result, i. e. the solvability of (0.1) for any function $y(x)$ in the appropriate function space (Theorem 2), it is an easy consequence of the preparatory work done for proving the multiplicity. Under the hypothesis that $f(x, s)$ ultimately decreases, it is related to some particular cases of the results of [1], [2] and other works.

It should be noted, that the results can be easily extended to more general second order elliptic operators with regular coefficients. Also, more precise information about the range of (0.1) can be obtained combining with the results and methods used in some of the above cited papers, those of [5] for example.

1. Preliminaries and statement of the results

Let Ω be a bounder region in R^n with boundary $\partial\Omega$, and let H be the Hilbert space $H_0^1(\Omega)$ with the scalar product

$$\langle u, v\rangle = \int_\Omega \nabla u . \nabla v dx$$

and corresponding norm $\|u\|_1^2 = \langle u, u\rangle$. The norm and the scalar product in $L^2(\Omega)$ will be denoted by $\|u\|_0$ and (u, v), respectively, while $|\cdot|$ will denote various finite dimensional Euclidean norms.

It is well known that under the above assumptions the Laplace operator has infinitely many eigenvalues $0<\lambda_1<\lambda_2\leqq\lambda_3 \ldots$ and a corresponding orthonormal system of eigenfunctions $v_1, v_2, v_3, \ldots$ that are related by the following variational characteristic

$$\lambda_k = \min \{\| u \|_1^2/\| u \|_0^2 : u\in H,\ u\neq 0,\ (u, v_i)=0,\ i=1, \ldots k-1\}.$$

Moreover, it is known that $v_1(x)$ does not change sign in Ω and in the sequel we shall always assume that $v_1(x)>0$ in Ω.

For the function $f(x, s)$ we shall always assume that

(i) $f(x, s)$: $\Omega\times R\to R$ is measurable in x and Lipschitz continuous in s, i.e.

$$(1.1)\qquad |f(x, s)-f(x, t)|\leqq K|s-t|$$

for almost all $x\in\Omega$, and $f(x, 0)\in L^2(\Omega)$.

(ii) The limits

$$(1.2)\qquad f_\pm(x) = \lim_{s\to\pm\infty} f(x, s)/s$$

exist uniformly with respect to $x\in\Omega$.

The condition (i) implies that there is a function $k(x)\in L^2(\Omega)$ such that

$$(1.3)\qquad |f(x, s)|\leqq k(x)+K|s|$$

for all $s\in R$, that $f^\pm\in L^\infty(\Omega)$ and that there exists a constant M, such that the function

$$(1.4)\qquad f(x, s)-Ms$$

is nonincreasing. (In fact the statement is true for every $M > K$ but in the sequel we shall make additional assumptions on the constant M.)

It is known that any function that is measurable in x and continuous in s and satisfies growth restrictions similar to (1.3) defines a continuous and bounded operator from $L^2(\Omega)$ into $L^2(\Omega)$, the so-called Niemitzkii operator, corresponding to f (cf. [8]). This enables us to define in the usual manner an operator $F: H \to H$ implicitly by

$$\langle F\,u, v\rangle = \int_\Omega f(x, u) v\,dx \quad \text{for every} \quad v \in H \tag{1.5}$$

the resulting operator being bounded, continuous, even Lipschitz continuous under the assumption (1.1) and compact. Now we can study the abstract equation in H

$$u - F(u) = g \tag{1.6}$$

for $g \in H$ instead of the problem (0.1) for $y \in H^{-1}(\Omega)$.

Let now V_k be the finite dimensional space spanned by the first k eigenfunctions $v_1, \ldots, v_k$, and let Q_k be the $L^2(\Omega)$ orthogonal projection of H on V_k and $P_k = Id_H - Q_k$ be the projection on W_k, the $L^2(\Omega)$ orthogonal complement of V_k in H. In the sequel we shall make frequent use of the following elementary facts about P_k and Q_k.

$$\langle u, v_i\rangle = \lambda_i (u, v_i) \qquad u \in H, \; i = 1, 2, \ldots$$

$$\langle P_k u, w\rangle = \langle u, w\rangle \qquad u \in H, \; w \in W_k,$$

$$Q_k u = \sum_{i=1}^{k} (u, v_i)\, v_i.$$

We are now in a position to state our main results.

Theorem 1. *Let us assume* (i), (ii) *and furthermore let $\lambda_2 < \lambda_3$, let the constant M in* (1.4) *be such that $M < \lambda_3$. $f^\pm(x) = l^\pm = \text{const}$, such that $l^- < \lambda_1 < \lambda_2 < l^+ < \lambda_3$. Then for every $g = \mu v_2 + \tilde{g}$ with $\tilde{g} \in W_2$ there exists a constant $\tau(\mu)$, such that for $t > \tau(\mu)$, the equation*

$$-u + F(u) = tv_1 + g$$

has at least four solutions. Moreover, the set of the functions of the form $tv_1 + g$, for which the above statement holds has nonempty interior.

Theorem 2. *Let us assume* (i), (ii) *and let the functions f^+ and f^-* (1.2) *be such that*

$$\max\{f^+, f^-\} \leq \lambda_1 - \varepsilon. \tag{1.7}$$

Then the equation $-u + F(u) = g$ has a solution for every $g \in H$.

2. Some auxiliary results

In the present section some facts about a special type of operators that arise as asymptotes and the necessary topological results are established.

For $\alpha(x)$, $\beta(x) \in L^\infty(\Omega)$, let $g(x,s)$ be the function

$$g(x,s)=\alpha(x)s+\beta(x)|s| \tag{2.1}$$

and let $G: H\to H$ be the corresponding operator defined by

$$\langle Gu, v\rangle = \int_\Omega g(x,u)v\,dx \qquad v\in H. \tag{2.2}$$

The following lemma treats the behaviour of the operator $u-G(u)$ on the subspaces W_k.

Lemma 1. *For* $\alpha(x)+|\beta(x)|\leqq\lambda_{k+1}-\varepsilon$, *the operator*

$$A_k(v+w)\equiv w-P_kG(v+w) \tag{2.3}$$

is monotone and coercive for every $v\in V_k$ *fixed. In particular, for every* $v\in V_k$ *there exists an unique solution* $w=w(v)$ of the equation $A_k(v+w)=0$, *which is a Lipschitz function of* v.

Proof. For every w_1, $w_2\in W_k$ we have

$$\langle A_k(v+w_1)-A_k(v+w_2),\ w_1-w_2\rangle = \| w_1-w_2\|_1^2-\int_\Omega \alpha(x)\,(w_1-w_2)^2dx$$

$$-\int_\Omega \beta(x)(|v+w_1|-|v+w_2|)\,(w_1-w_2)\,dx.$$

We estimate the last integral from above to obtain

$$\left|\int_\Omega \beta(x)(|v+w_1|-|v+w_2|)(w_1-w_2)\,dx\right| \leqq \int_\Omega |\beta(x)|(w_1-w_2)^2dx.$$

Now, $\|w\|_1^2\geqq\delta\|w\|_1^2+(1-\delta)\lambda_{k+1}\|w\|_0^2$ for $w\in W_k$ and for δ small enough we obtain

$$\begin{aligned}&\langle A_k(v+w_1)-A_k(v+w_2),\ w_1-w_2\rangle\geqq\delta\|w_1-w_2\|_1^2\\ &+\int_\Omega((1-\delta)\lambda_{k+1}-\alpha(x)-|\beta(x)|)\,(w_1-w_2)^2dx\geqq\delta\|w_1-w_2\|_1^2.\end{aligned} \tag{2.4}$$

For $w_2=0$, (2.4) implies $\langle A_k(v+w)-A_k(v),\ w\rangle\geqq\delta\|w\|_1^2$.

The existence and the uniqueness of the solution are now standard from the theory of monotone operators (cf. [9] or [10]). To prove the Lipschitz continuity, let $w_i=w(v_i)$, $i=1, 2$ be solutions of $A_k(v_i+w)=0$. Elementary computations, similar to the above give

$$0\geqq\delta\|w_1-w_2\|_1^2-\int_\Omega(|\alpha(x)|+|\beta(x)|)\,|v_1-v_2\|w_1-w_2|\,dx,$$

i. e. $\| w_1 - w_2 \|_1 \leqq C\delta^{-1} \| v_1 - v_2 \|_0$ for some constant C.

Next we make some topological considerations.

Let $S^n = \{x \in R^{n+1} : |x| = 1\}$ and $\varphi : S^n \to S^n$ be continuous. If $U \subset S^n$ is an open set, then for every $y \in S^n$ with $y \bar{\in} \varphi(\partial U)$, the topological degree $\deg(\varphi, U, p)$ is defined. In the particular case $U = S^n$, the degree does not depent on p and we write simply $\deg \varphi$. For $U \neq S^n$, the definition of deg is similar to the one usually given for bounded open sets in R^n. (cf. [11] IV. 5 for a more algebraic-topological definition.)

For $0 < \alpha < \beta$, let $U_{\alpha,\beta}$ be the open set in R^{n+1} defined by

$$U_{\alpha,\beta} = \{ y \in R^{n+1} : y = \tau x, \quad x \in U \subset S^n, \quad \alpha < \tau < \beta\} \tag{2.5}$$

(in the sequel we shall make use also of the obvious notation τV for the points of the form $y = \tau x$, $x \in V$.) The function $\Phi(y)$ defined by

$$\Phi(y) = \tau\varphi(x) \quad \text{for} \quad y = \tau x \, (\alpha < \tau < \beta) \tag{2.6}$$

is continuous from $U_{\alpha,\beta}$ in R^{n+1}. Moreover, if $\deg(\varphi, U, p)$ is defined, then $\deg(\Phi, U_{\alpha,\beta}, \tau p)$ is defined for $\alpha < \tau < \beta$ too and the two are equal. To prove this it is enough to observe that $p \bar{\in} \varphi(\partial U)$ implies $\tau p \bar{\in} \Phi(\partial U_{\alpha,\beta})$ for $\alpha < \tau < \beta$ and that in fact $\Phi(y)$ is the Descartes product of the mapping φ with the identity of the interval (α, β) (cf. [9], 1.4.6).

Let now $\psi : R^{n+1} \to R^{n+1}$ be homogeneous, i. e. $\psi(\lambda x) = \lambda\psi(x)$ for $\lambda > 0$ and $\psi(x) \neq 0$ for $x \neq 0$. Then we can define a mapping $\varphi : S^n \to S^n$ by $\varphi(x) = \psi(x)/|\psi(x)|$ for $x \neq 0$. Let U and V be open subsets of S^n and let $\deg(\varphi, U, p) = \text{const} \neq 0$ for $p \in V$. Then there exists $t_0 > 0$ such that $\psi(R^{n+1}) \supset tV$ for $t \geqq t_0$.

Indeed, let $m = \min_{\bar{U}} |\psi(x)|$, $M = \max_{\bar{U}} |\psi(x)|$. Then for $\lambda > 0$ we have

$$\min_{\lambda\bar{U}} |\psi(x)| = \lambda m, \quad \max_{\lambda\bar{U}} |\psi(x)| = \lambda M. \tag{2.7}$$

Let now λ_0 be so big that $\lambda_0 m > M$. For every $p \in R^{n+1}$ such that

$$p/|p| \in V, \quad M < |p| < \lambda_0 m \tag{2.8}$$

we have that $\deg(\psi, U_{1,\lambda_0}, p) = \deg(\varphi, U, p/|p|) \neq 0$, i. e. the degree in the left side is defined and equality holds. (U_{1,λ_0} is defined in (2.5).) We first prove that for p satisfying (2.8), $p \bar{\in} \psi(\partial U_{1,\lambda_0})$. Indeed, $p = \psi(\tau x)$ with $1 \leqq \tau \leqq \lambda_0$, $x \in \partial U$ gives

$$p/|p| = \psi(\tau x)/|\psi(\tau x)| = \varphi(x)$$

that contradicts (2.8). The assumption $p = \psi(y)$ with $y \in U$ or $y \in \lambda_0 U$ contradicts the second part of (2.8) because of (2.7) and the choice of λ_0. As we noted above $\deg(\Phi, U_{1,\lambda_0}, p) = \deg(\varphi, U, p/|p|)$ where Φ is defined by (2.6). It is now easy to see, that under the homotopy

$$F(x, t) = t\psi(x) + (1-t)\Phi(x), \quad x \in U_{1,\lambda_0}, \quad t \in [0, 1]$$

we have

$$V_{M,\lambda_0 m} \cap F(\partial U_{1,\lambda_0}, t) = \qquad t \in [0, 1]$$

since $\Phi(x)$ and $\psi(x)$ lie on the same ray through the origin and the set $V_{M,\lambda_0 m}$, i. e. the constant λ_0 in (2.8) was chosen in a manner to guarantee this fact. The rest of the proof follows from the fact that ψ is homogeneous and from $\psi(U_{1,\lambda_0})\supset V_{M,\lambda_0 m}$.

The particular case $U=V=S^n$ is even more easy. It is enough to consider ψ and Φ on big balls.

Now we are ready to treat the case we are really interested in.

Lemma 2. *Let $f: R^{n+1}\to R^{n+1}$ be such that the limits $\psi(x)=\lim_{t\to\infty} f(tx)/t$ exist uniformly for x in compact sets and let $\psi(x)\neq 0$ for $x\neq 0$. Let $U, V\subset S^n$ be such that for $\varphi(x)=\psi(x)/|\psi(x)|$, $p\in V$, $\deg(\varphi, U, p)=\text{const}\neq 0$.*

Then there exists $t_0>0$, such that $f(R^{n+1})\supset \cup_{t>t_0} tV$. If $U=V=S^n$, then f is surjective.

Proof. It is obvious, that $\psi(x)$ is homogeneous, so we can use the argument above. In fact we have that under appropriate choice of the constants m, M, λ_0, $\psi(U_{1,\lambda_0})\supset V_{M,\lambda_0 m}$ and there exists $\varepsilon>0$, such that dist $(\psi(\partial U_{1,\lambda_0}), V_{M,\lambda_0 m})\geqq\varepsilon$. Let τ be such that for $t>\tau$, $|t^{-1}f(tx)-\psi(x)|<\varepsilon/2$ for $x\in\partial U_{1,\lambda_0}$. Then for $p\in V_{M,\lambda_0 m}$, $t>\tau$ we have $|t^{-1}f(tx)-p|>\varepsilon/2$ for $x\in U_{1,\lambda_0}$ and we can use the usual homotopy to connect the mappings $t^{-1}f(tx)$ and $\psi(x)$. This implies that $f(tU_{1,\lambda_0})\supset tV_{M,\lambda_0 m}$ and the proof is completed with $t_0=\tau M$.

3. Proof of Theorems 1 and 2

We proceed with the proof using global Lyapunov—Schmidt method. We project the equation (1.6) by means of P_k and Q_k, where k is an integer such that $\lambda_{k+1}>M$, to obtain the equivalent system

$$-w+P_k F(v+w)=P_k g\equiv y \tag{3.1}$$

$$-v+Q_k F(v+w)=Q_k g. \tag{3.2}$$

The first equation is invertible for every $v\in V_k$ and $y\in W_k$. In the followig lemma we sum up the properties of the solutions.

Lemma 3. *For every fixed $y\in W_k$ and $v\in V_k$, the equation* (3.1) *has a unique solution $w=w(v,y)$. The mapping $w(v,y)$ is Lipschitz from H in W_k. Moreover, the inequality*

$$\| w(v,y)\|_1\leqq c(1+\|y\|_1+|v|) \tag{3.3}$$

holds.

Proof. [6], lemma 1. The only thing not proved is the Lipschitz continuity, but it is immediate if we suppose the function $f(x,s)$ to be Lipschitz. The proof is also similar to the one in [3], lemma 7.1, being only slightly more technical, because we have to use the condition (1.4) only.

Our next step is to study the asymptotic behaviour of equation (3.2) for $w=w(v,y)$. To this end we prove the following lemma.

Lemma 4. *For every v fixed, the limit $w(v)=\lim_{t\to\infty} t^{-1}w(tv,y)$ exists, is independent of y and satisfies an equation of the type* (2.3)

(3.4) $$w - P_k G(v+w) = 0,$$

where G is defined by (2.1), (2.2) *with*

$$\alpha(x) = \frac{1}{2}(f^+(x) + f^-(x)), \qquad \beta(x) = \frac{1}{2}(f^+(x) - f^-(x)).$$

P r o o f. Let us first note that (1.4) and $M < \lambda_{k+1}$ imply $f^+(x) \leqq M, f^-(x) \leqq M$. Then $\alpha(x) + |\beta(x)| = \max\{f^+(x), f^-(x)\} \leqq M$ and the hypotheses of lemma 1 are verified on the subspace W_k.

The definition of the functions $w(tv, y)$ and (3.3) imply that the family of functions $t^{-1}w(tv, y)$ is bounded in $H = H_0^1(\Omega)$ for $t \to \infty$. By the Sobolev imbedding theorem we can choose a sequence $w_n(x) = t_n^{-1} w(t_n v, y)$, such that

$$w_n \to w_0 \text{ weakly in } H, \quad w_n \to w_0 \text{ strongly in } L^2(\Omega), \ w_n(x) \to w_0(x) \text{ a. e. in } \Omega.$$

We have too $\| w_n \|_1 \leqq C$ for some constant. From (1.3) it follows now that the functions

(3.5) $$g_n(x) = t_n^{-1} f(x, t_n v(x) + t_n w_n(x))$$

have bounded norms in $L^2(\Omega)$, i. e. $\| g_n \|_0 \leqq c'$. It is not difficult to see, as in [6] for instance, that for almost all $x \in \Omega$,

$$g_n(x) \to g(x) \equiv f^+(x) \max\{v + w_0, 0\} + f^-(x) \min\{v + w_0, 0\}$$

or otherwise written

(3.6) $$g(x) = \alpha(x)(v + w_0) + \beta(x)|v + w_0|$$

with $\alpha(x)$ and $\beta(x)$ as in the statement of the lemma. Obviously $g \in L^2(\Omega)$ and we can conclude that $g_n \to g$ weakly in $L^2(\Omega)$ (cf. [10], lemma 1.3). Multiplying the equation which defines w_n, i. e. $-w_n + t_n^{-1} P_k F(t_n v + t_n w_n) = t_n^{-1} y$ with arbitrary $u \in H$, we get

(3.7) $$-\langle w_n, u\rangle + \langle t_n^{-1} F(t_n v + t_n w_n), P_k u\rangle = t_n^{-1}\langle y, u\rangle.$$

The definition (1.5) of F and (3.5) imply

$$\langle t_n^{-1} F(t_n v + t_n w_n), P_k u\rangle = \langle g_n, P_k u\rangle \to \langle g, P_k u\rangle$$

and since $\langle g, P_k u\rangle = \langle G(v + w_0), P_k u\rangle$, (cf. (3.6), (2.2))

$$-\langle w_0, u\rangle + \langle P_k G(v + w_0), u\rangle = 0$$

for every $u \in H$, i. e. $w_0 = w(v)$ is according to Lemma 1 the unique solution of the equation

$$-w + P_k G(v + w) = 0$$

Since every accumulation point of the family $t^{-1} w(tv, y)$ has to satisfy the same equation, the existence of the limit is thus proved. The Lipschitz continuity follows also from Lemma 1.

Proof of Theorem 1. Under the hypotheses of Theorem 1 we have $M<\lambda_3$ and $l^-<l^+\leqq M<\lambda_3$, so we can limit ourselves to the spaces V_2, spanned by v_1 and v_2 its orthogonal complement W_2. Now we shall study the asymptotic behaviour of the operator in (3.2). It follows that $\alpha=(l^++l^-)/2$, $\beta=(l^+-l^-)>0$ and we shall denote again by $g(u)$ and $G(u)$ the corresponding function and operator. Let

$$B(v+w)=-v+QF(v+w)\quad B(v)=B(v+w(v,\,y)). \tag{3.8}$$

We are interested in the limits of $t^{-1}B(tv)$. As previously we see that

$$t^{-1}f(x,tv+tw(v))\to g(v+w(v))$$

weakly in $L^2(\Omega)$ for $t\to\infty$. Since $t^{-1}w(tv,y)\to w(v)$ strongly $L^2(\Omega)$, the Lipschitz continuity of the function $f(x,s)$ implies that

$$t^{-1}f(x,tv+t(\frac{w(tv,\,y)}{t}))\to g(v+w(v))$$

weakly. Since we consider $B(v)$ on a finite-dimensional space, this weak convergence allows to conclude that the limits exist, i. e.

$$B_\infty(v)\equiv\lim_{t\to\infty}t^{-1}B(tv)=-v+Q_2G(v+w(v)).$$

Moreover, the Lipschitz continuity of all the functions $t^{-1}B(tv)$ with respect to v implies that this convergence is uniform on compact sets. Writing in more detail we have that

$$B_\infty(v)=-v+\sum_{i=1}^{2}\frac{1}{\lambda_i}\int_\Omega(\alpha(v+w(v))+\beta\,|\,v+w(v)\,|\,v_i(x)\,dx\,v_i.$$

Scalar multiplication with v_1 gives

$$\langle B_\infty(v),v_1\rangle=\frac{1}{\lambda_1}\int_\Omega\{(-\lambda_1+\alpha)\,(v+w(v))+\beta\,|\,v+w(v)\,|\,\}\,v_1\,dx>0$$

since $\beta>|\,\alpha-\lambda_1\,|$ and hence it is easily checked that the expression in the integral is always positive for $v\neq0$. The last inequality shows that $B_\infty(v)\neq0$ for $v\neq0$, but it shows also that $\deg\,(B_\infty,\,S^1)=0$.

To continue the study of $B_\infty(v)$ we need know a little more about the regularity of the functions $w\,(v)$. We arrive at the higher regularity of these functions by standard bootstrap argument. Indeed $w=w(v)$ is a variational solution in $H_0^1\,(\Omega)$ of the equation

$$\Delta w=-P_2(\alpha(v+w)+\beta\,|\,v+w\,|),$$

where the right-hand side belongs to $L^{2^+}(\Omega)$ $(1/2^*=1/2-1/n)$ by the Sobolev imbedding theorem. Now $w\in H^{2,\,2^*}(\Omega)$ and the right-hand side is again more regular. In a finite number of steps we arrive at $w \in L^\infty(\Omega)$ and that is all we need.

Next we note that $w(v_1)=w(-v_1)=0$. This follows immediately from (3.4) substituting 0 for w, and v_1 and $-v_1$ for v, using the fact that now $\alpha(x)$ and $\beta(x)$ are constants, that $|v_1|=|-v_1|=v_1$ and the uniqueness assertion of lemma 1.

Now we use a consequence of the maximum principle ([5], lemma 1), that in the present situation states that there exists $\varepsilon>0$, such that if $w \in L^\infty(\Omega)$, $\Delta w \in L^\infty(\Omega)$ and $\|w\|_\infty<\varepsilon$, $\|\Delta w\|_\infty<\varepsilon$, then $v_1(x)+w(x)>0$ in Ω. Of course, $-v_1(x)+w(x)<0$ in Ω for such functions w. But the regularity considerations above, together with the continuity of $w(v)$ with respect to v, and $w(v_1)=w(-v_1)=0$ imply that we are in this situation, i.e. if we write

$$v(\varphi)=v_1 \cos \varphi+v_2 \sin \varphi,$$

there are intervals $(-\varepsilon, \varepsilon)$ and $(\pi-\varepsilon, \pi+\varepsilon)$, such that for φ in these intervals we have $v(\varphi)+w(v)>0$ and $v(\varphi)+w(v)<0$ in Ω, respectively.

The constant sign of the above functions, together with $\alpha,\beta=\text{const}$, $w(v) \perp v$ and (3.1) give

$$B_\infty(v)=(\frac{l^+}{\lambda_1}-1)\,v_1 \cos \varphi+(\frac{l^-}{\lambda_2}-1)\,v_2 \sin \varphi$$

and

$$B_\infty(v)=-(1-\frac{l^+}{\lambda_1})\,v_1 \cos \varphi-(1-\frac{l^-}{\lambda_2})\,v_2 \sin \varphi$$

for φ in $(-\varepsilon, \varepsilon)$ and $(\pi-\varepsilon, \pi+\varepsilon)$, respectively. If we put $U^+=(-\varepsilon, \varepsilon)$, $U^-=(\pi-\varepsilon, \pi+\varepsilon)$, since $\lambda_2<l^+$ and $l^-<\lambda_1$ it is easy to calculate

$$\deg(B_\infty/|B_\infty|,\ U^+,\ v_1)=\deg(B_\infty/|B_\infty|,\ U^-,\ v_1)=1 \tag{3.10}$$

and the same is true for v in some neighbourhood V of v_1, such that the four endpoints of the above segments of S^1 do not belong to V. Then from simple geometrical considerations on S^1, taking into account $\langle B_\infty(v), v_1\rangle>0$ for every $v\neq 0$ and the fact that the segments $U'=(\varepsilon, \pi-\varepsilon)$, $U''=(\pi+\varepsilon, -\varepsilon)$ have the same endpoints as U^+, U^- we find that

$$\deg(B_\infty/|B_\infty|,\ U',\ v)=\deg(B_\infty/|B_\infty|,\ U'',\ v)=-1 \tag{3.11}$$

for every $v \in V$, the same set as before. Now we apply lemma 2 with respect to each of the sets U^+, U^-, U', U'' and all the mappings (3.8). It is to be noted that the sets we obtained above do not depend on $y \in W_2$, so we can affirm that there exists $\tau>0$, such that

$$B(\bigcup_{t>0} tu) \supset \bigcup_{t>\tau} tV$$

where u is any of the four sets above. The last is equivalent to the statement that for any p with $p/|p| \in V$, $|p|>\tau$, the equation

$$B(v+w(v, y))=p \tag{3.12}$$

has at least four solutions, each being in one of the "cones" generated by the sets U above. The conclusions of the theorem are now just a restatement of this fact, taking into account that finding solutions of (3.1), (3.2) is equivalent to finding solutions of (3.12) only, the last part of the theorem being obvious from the geometry of the set $\cup_{t<\tau} tV$.

Remark. It is not difficult to see that one can treat likewise the case when $\lambda_k < l^+ \leqq M < \lambda_{k+1}$ for some k, without further restrictions on the multiplicity of λ_i for $i<k$. In fact, the only point for which the two-dimensionality is essential is the existence of two sets U', U'', such that (3.11) holds. The existence of sets U^+, U^-, open neigbourhoods of v_1 and $-v_1$ on S^{k-1}, such that (3.10) holds and $\deg(B_\infty/|B_\infty|, S^{k-1})=0$ follow repeating verbatim the same argument, but therefrom we can deduce only for $U=S^{k-1}\setminus(U^+\cup U^-)$

$$\deg(B_\infty/|B_\infty|, U, v_1)=-2$$

which ensures the existence of one more solution for $p\in V$(some neighbourhood of v_1), being intuitively clear, that there should be two solutions in U for "most" $p\in V$.

Proof of Theorem 2. For $\alpha(x)=\frac{1}{2}(f^+(x)+f^-(x))$ $\beta(x)=\frac{1}{2}(f^+(x)-f^-(x))$, (1.7) implies that $\alpha(x)+|\beta(x)|\leqq\lambda_1-\varepsilon$. From Lemma 1 it follows that the operators

$$A_k(v+w)=w-P_k G(v+w)$$

are monotone and coercive for any k, and in particular for $P_0\equiv Id_H$. Let now k be such that $\lambda_{k+1}>M$, the existence of M such that (1.4) holds being implied by (1.3). We use again the splitting (3.1), (3.2) and now Lemmas 3 and 4 imply that

$$w_k=w_k(v)=\lim_{t\to\infty} t^{-1} w(tv, y)$$

for $v\in V_k$, $y\in W_k$ satisfies the equation $A_k(v+w_k)=0$. Now

$$\delta\|v\|_1^2\leqq\delta(\|v\|_1^2+\|w\|_1^2)\leqq\langle A_0(v+w), v+w\rangle$$

for every $v\in V_k$ and $w\in W_k$, whence

$$\delta\|v\|_1^2\leqq\langle A_0(v+w_k(v)), v+w_k(v)\rangle=\langle A_0(v+w_k(v), w_k(v)\rangle+\langle A_0(v+w_k(v), v\rangle$$

$$=\langle P_k A_0(v+w_k(v)), w_k(v)\rangle+\langle Q_k A_0(v+w_k(v)), v\rangle=\langle B_\infty(v), v\rangle$$

since $P_k A_0=A_k$. The last inequality shows that $B_\infty(v)\neq 0$ for $v\neq 0$ and moreover $\deg(B_\infty/|B_\infty|, S^k)=1$. According to the last part of lemma 2 this is enough to guarantee the surjectivity of the mapping $B(v)$.

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7	1 from bottom	$V_{M,\lambda_0 m} \cap F(\partial U_{1,\lambda_0}, t) =$	$V_{M,\lambda_0 m} \cap F(\partial U_{1,\lambda_0}, t) = \emptyset$