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Berge-equilibrium in stochastic differential games

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Two-person nonzero-sum games where the dynamics is described by Ito stochastic differential equations are considered. Berge-equilibrium strategies are introduced and sufficient conditions for them are established. Linear-quadratic games with and without a state and control dependent noise are studied.

1. Introduction

In this paper two-person nonzero-sum stochastic differential games are considered. We follow the approach of W. Fleming and R. Rishel [4] to the optimal control of stochastic dynamic systems, but applied to situations of conflicts. As solutions of the games Berge-equilibrium is introduced. This notion is based on the ideas of the classical book [1]. Sufficient conditions in the form of dynamic programming equations are found. The approach chosen here makes possible further application of the approximation and numerical procedures developed by F. Chernousko and V. Kolmanovskii [2]. Let us mention that martingale methods can also be applied to problems similar to those treated below and we intend to use them as well. Two types of linear-quadratic games are considered as examples. Here the strategies guaranteeing the Berge-equilibrium are given in an explicit form. The results presented in this paper are further development of some ideas and results announced earlier in [7].

2. Formalization of the game

Let us consider the game $\Gamma=\langle\{1, 2\}, \Sigma, \{\mathcal{U}_1, \mathcal{U}_2\}, \{\mathcal{J}_1, \mathcal{J}_2\}\rangle$. Here $\{1, 2\}$ is the set of players, the evolution of the dynamic system Σ is described by a stochastic differential equation of the type

$$(2.1)\quad d\xi(t)=f(t, \xi(t), u_1, u_2)\,dt+\sigma(t, \xi(t), u_1, u_2)\,dw(t),\ t\in[t_0, T]$$

with initial condition $\xi(t_0)=\xi_0$, $0\leqq t_0<T$. The process $w=\{w(t), t\in[t_0, T]\}$ is a standard m-dimensional Wiener process defined on some complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and adapted to a given family $\mathsf{F}=\{\mathcal{F}_t, t\in[t_0, T]\}$ of

nondecreasing sub-σ-algebras of R. $\xi \in \mathrm{R}^n$ is the state vector process and $u_i \in U_i \subset \mathrm{R}^{\nu_i}$ is the control of the i-th player, $i=1, 2$. Now let us make the following assumptions about the functions $f(t, x, u_1, u_2)$ and $\sigma(t, x, u_1, u_2)$. Suppose

$$f\colon [t_0, T] \times \mathrm{R}^n \times U_1 \times U_2 \to \mathrm{R}^n$$

and

$$\sigma\colon [t_0, T] \times \mathrm{R}^n \times U_1 \times U_2 \to \mathrm{R}^n \times \mathrm{R}^m$$

have continuous partial derivatives and let $C>0$ be a constant such that

$$|f(t, 0, 0, 0)| + |\sigma(t, 0, 0, 0)| \leqq C,$$

$$|f_x| + |\sigma_x| + |f_{u_1}| + |\sigma_{u_1}| + |f_{u_2}| + |\sigma_{u_2}| \leqq C$$

where $|\cdot|$ is a general symbol for the norm in the corresponding space.

Each player has perfect observations of the state vector $\xi(t)$ at every moment $t \in [t_0, T]$ and constructs his strategy in the game Γ as an admissible feedback control of the following type $u_i(t) = u_i(t, \xi(t))$ where $u_i(\cdot\,,\cdot)\colon [t_0, T] \times \mathrm{R}^n \to U_i$ is a Borel function satisfying the conditions:

(i) There exists a constant $M_i > 0$ such that $|u_i(t, x)| \leqq M_i(1+|x|)$ for all $t \in [t_0, T]$, $x \in \mathrm{R}^n$;

(ii) For each bounded set $B \subset \mathrm{R}^n$ and $T^* \in (t_0, T)$ there exists a constant $K_i > 0$ such that for arbitrary $x, y \in B$ and $t \in [t_0, T^*]$, $|u_i(t, x) - u_i(t, y)| \leqq K_i |x - y|$. Denote by $\mathcal{U}_i$ the set of strategies of the i-th player, $i=1, 2$.

The assumptions mentioned above imply the existence and sample path uniqueness of the solution $\xi = \{\xi(t), t \in [t_0, T]\}$ of (2.1) corresponding to the pair of controls $(u_1, u_2) \in \mathcal{U}_1 \times \mathcal{U}_2$. The infinitesimal operator $\mathcal{A}(u_1, u_2)$ of the Markov process ξ under the pair of strategies (u_1, u_2) has the form, see [3]:

$$\mathcal{A}(u_1, u_2)\, V(t, x) = f'(t, x, u_1, u_2)\, V_x(t, x) + \frac{1}{2} \operatorname{tr}[a(t, x; u_1, u_2)\, V_{xx}(t, x)],$$

where $a = \sigma\sigma'$ and prime denotes vector or matrix transposition. Here $V(t, x)$ is a real-valued function with continuous partial derivatives up to second order for all $t \in [t_0, T]$, $x \in \mathrm{R}^n$.

Let L_i, ψ_i be continuous functions satisfying the polynomial growth conditions:

$$|L_i(t, x, u_1, u_2)| \leqq C_i(1 + |x| + |u_1| + |u_2|)^k,$$

$$|\psi_i(t, x)| \leqq C_i(1+|x|)^k,$$

where C_i, k are positive constants. Finally we introduce the cost-function $\mathcal{J}_i(u_1, u_2)$ of the i-th player

$$\mathcal{J}_i(u_1, u_2) = \mathrm{E}_{t_0, \xi_0} \{\psi_i(T, \xi(T)) + \int_{t_0}^{T} L_i(t, \xi(t), u_1, u_2)\, dt\}, \quad i=1, 2.$$

The object of each player in the game Γ is to minimize his own cost-function.

3. Main result

Definition. *Let the pair of strategies* (u_1^b, u_2^b) *be called a pair of Berge-equilibrium strategies for the game* Γ *if*

$$\mathcal{J}_1(u_1^b, u_2) \geqq \mathcal{J}_1(u_1^b, u_2^b) \text{ for each } u_2 \in \mathcal{U}_2$$

and

$$\mathcal{J}_2(u_1, u_2^b) \geqq \mathcal{J}_2(u_1^b, u_2^b) \text{ for each } u_1 \in \mathcal{U}_1.$$

Denote $G_i(t, x, u_1, u_2) = W_t^{(i)}(t, x) + \mathcal{A}(u_1, u_2) W^{(i)}(t, x) + L_i(t, x, u_1, u_2)$ *for all* $t M [t_0, T]$, $x \in \mathrm{R}^n$, $u_1 \in U_1$, $u_2 \in U_2$, $i = 1, 2$.

Theorem. *The pair of strategies* (u_1^b, u_2^b) *is a pair of Berge-equilibrium strategies for the game* Γ *if there exist real-valued functions* $W^{(i)}(t, x)$ *such that for all* $t \in [t_0, T]$, $x \in \mathrm{R}^n$ *and* $i = 1, 2$, *the following conditions jointly hold:*

(a) $W^{(i)}, W_J^{(i)}, W_x^{(i)}, W_{xx}^{(i)}$ *are continuous;*

(b) $G_i(t, x, u_1^b, u_2^b) = 0;$

(c) $G_1(t, x, u_1^b, u_2) \geqq 0$ *for each strategy* $u_2 \in \mathcal{U}_2$ *and*

$G_2(t, x, u_1, u_2^b) \geqq 0$ *for each strategy* $u_1 \in \mathcal{U}_1;$

(d) $W^{(i)}(T, x) = \psi_i(T, x).$

Proof. Let $\xi^b(t), t \in [t_0, T]$ be the sample path of the solution of (2.1) corresponding to the pair of strategies (u_1^b, u_2^b). Ito-Dynkin's formula implies

$$W^{(i)}(t, x) = \mathrm{E}_{t,x}\{W^{(i)}(T, \xi^b(T)) - \int_t^T [W_\tau^{(i)}(\tau, \xi^b(\tau)) + (u_1^b, u_2^b) W^{(i)}(\tau, \xi^b(\tau))]\, dt\},\ i = 1, 2.$$

Taking into account conditions (b) and (d), we get

$$W^{(i)}(t, x) = \mathrm{E}_{t,x}\{\psi_i(T, \xi^b(T)) + \int_t^T L_i(\tau, \xi^b(\tau), u_1^b, u_2^b)\, d\tau\},\ i = 1, 2,$$

and hence

$$W^{(i)}(t_0, \xi_0) = \mathrm{E}_{t_0,\xi_0}\{\psi_i(T, \xi^b(T)) + \int_t^T L_i(t, \xi^b(t), u_1^b, u_2^b)\, dt\},\ i = 1, 2.$$

Now let $\xi^{1b}(t)$ and $\xi^{2b}(t)$, $t \in [t_0, T]$ be the sample paths of the solutions of (2.1) coresponding to the pairs of strategies (u_1^b, u_2), and (u_1, u_2^b), respectively. Ito-Dynkin's formula in conjuction this time with conditions (c) and (d) implies

$$W^{(1)}(t_0, \xi_0) \leqq \mathrm{E}_{t_0,\xi_0}\{\psi_1(T, \xi^{1b}(T)) + \int_{t_0}^T L_1(t, \xi^{1b}(t), u_1^b, u_2)\, dt\}$$

and

$$W^{(2)}(t_0, \xi_0) \leqq \mathsf{E}_{t_0,\xi_0}\{\psi_2(T, \xi^{2b}(T)) + \int_{t_0}^{T} L_2(t, \xi^{2b}(t), u_1, u_2^b)\,dt\}.$$

Thus we get

$$\mathscr{J}_1(u_1^b, u_2) \geqq W^{(1)}(t_0, \xi_0) = \mathscr{J}_1(u_1^b, u_2^b) \text{ for each } u_2 \in \mathscr{U}_2$$

and

$$\mathscr{J}_2(u_1, u_2^b) \geqq W^{(2)}(t_0, \xi_0) = \mathscr{J}_2(u_1^b, u_2^b) \text{ for each } u_1 \in \mathscr{U}_1.$$

This means that (u_1^b, u_2^b) is a pair of Berge-equilibrium strategies for the game Γ. □

Remark. Note that $\{W^{(i)}(t, x), i=1, 2\}$ is a solution of a system of dynamic programming equations of the type

$$\min_{u_2} G_1(t, x, u_1^b, u_2)=0, \quad \min_{u_1} G_2(t, x, u_1, u_2^b)=0$$

with boundary conditions $W^{(i)}(T, x)=\psi_1(T, x)$, $i=1,2$, for all $t \in [t_0, T]$, $x \in \mathsf{R}^n$.

4. Linear-quadratic game with state and control independent noise

Let us consider game Γ where the evolution of the dynamic system Σ is described by the linear stochastic differential equation $d\xi(t)=[A(t)\xi(t)+B_1(t)u_2 + B_2(t)]\; dt+\sigma(t)\; dw(t),\; t \in T]$ with initial condition $\xi(t_0)=\xi_0$. Here ξ_0, w, u_1, u_2 are the same as in Section 2. $A(t)$ is an $n \times n$-matrix, $\sigma(t)$ is an $n \times m$-matrix and $B_1(t)$ and $B_2(t)$ are $n \times v_1$ — and $n \times v_2$-matrices, respectively. The cost-function $\mathscr{J}_i(u_1, u_2)$ of the i-th player is given by the following quadratic functional:

$$_i(u_1, u_2)=\mathsf{E}_{t_0,\xi_0}\{\xi'(T)D_i\xi(T)+\int_{t_0}^{T}[\xi'(t)\xi(t)\;\; M_i(t)+u_1'(t)N_1^{(i)}(t)u_1(t)$$
$$+u_2'(t)N_2^{(i)}(t)u_2(t)]\,dt\}, \quad i=1,2.$$

Here D_i. $M_i(t)$, $N_1^{(i)}(t)$, $N_2^{(i)}(t)$ are symmetric matrices with dimensions $n \times n, n \times n$, $v_i \times v_i . v_i \times v_i$, respectively, $i=1, 2$. Let D_i be consta nt and $A(t), B_i(t), \sigma(t), M_i(t)$, $N_1^{(i)}(t), N_2^{(i)}(t)$ be continuous, $i=1, 2$.

Let us assume that the strategies of the i-th player are functions of the form $u_i(t, x)=F_i(t)x$ where $F_i(t)$ are continuous $v_i \times n$-matrices, $i=1, 2$.

Now consider the functions

$$G_i(t, x, u_1, u_2)=W_t^{(i)}(t, x)+\mathscr{A}.(u_1, u_2)W^{(i)}(t, x)+L_i(t, x, u_1, u_2)$$

$$=W_t^{(i)}(t, x)+[A(t)x+B_1(t)u_1+B_2(t)u_2]'\,W_x^{(i)}(t, x)+\frac{1}{2}\operatorname{tr}[a(t)\,W_{xx}^{(i)}(t, x)]$$

$$+x'M_i(t)x+u_1'\,N_1^{(i)}(t)u_1+u_2'\,N_2^{(i)}(t)u_2$$

for all $t \in [t_0, T]$, $x \in \mathsf{R}^n$, $u_1 \in U_1$, $u_2 \in U_2$, $i=1, 2$. Taking into consideration the remark (Section 3) we have

$$\left.\frac{\partial G_1}{\partial \mathcal{U}_2}\right|_{u_2=u_2^b}=B_2'(t)\,W_x^{(1)}(t,\ x)+2\,N_2^{(1)}(t)\,u_2^b=0$$

and

$$\left.\frac{\partial G_2}{\partial \mathcal{U}_1}\right|_{u_1=u_1^b}=B_1'(t)\,W_x^{(2)}(t,\ x)+2\,N_1^{(2)}(t)\,u_1^b=0,$$

i. e.

$$u_2^b=-\frac{1}{2}[N_2^{(1)}(t)]^{-1}B_2'(t)\,W_x^{(1)}(t,\ x)\ \text{ and }\ u_1^b=-\frac{1}{2}[N_1^{(2)}(t)]^{-1}\ B_1'(t)\ W_x^{(2)}(t,\ x).$$

Further put u_i^b, $i=1, 2$ in $G_i(t, x, u_1^b, u_2^b)=0$, $i=1, 2$, and search $W^{(i)}(t, x)$ in the following special form $W^{(i)}(t, x)=x'\theta_i(t)x+r_i(t)$ where $\theta_1(t)$ is a symmetric $n\times n$-matrix with $\theta_i(T)=D_i$ and $r_i(t)$—a scalar function, $i=1,\ 2.$ Then

$$\text{(4.1)}\quad u_1^b=-[N_1^{(2)}(t)]^{-1}B_1'(t)\theta_2(t)x,\ \ u_2^b=-[N_2^{(1)}(t)]^{-1}B_2'(t)\theta_1(t)x$$

and the system of equations for $W^{(i)}(t, x)$, $i=1,\ 2$ transforms in

$$\begin{aligned}x'\,\{\theta_i(t)&+A'(t)\,\theta_1(t)+\theta_i(t)\,A(t)+M_i(t)-\theta_2(t)\,B_1(t)\,[N_2^{(1)}(t)]^{-1}B_1'(t)\,\theta_1(t)\\&-\theta_i(t)\,B_1(t)[N_1^{(2)}(t)]^{-1}B_1'(t)\theta_2(t)-\theta_1(t)\,B_2(t)\,[N_2^{(1)}(t)]^{-1}B_2'(t)\theta_i(t)\\&-\theta_i(t)\,B_2(t)\,[N_2^{(1)}(t)]^{-1}B_2'(t)\theta_1(t)+\theta_2(t)\,B_1(t)[N_1^{(2)}(t)]^{-1}N_1^{(i)}(t)[N_1^{(2)}(t)]^{-1}B_1'(t)\theta_2(t)\\&+\theta_1(t)\,B_2(t)[N_2^{(1)}(t)]^{-1}N_2^{(i)}(t)[N_2^{(1)}(t)]^{-1}B_2'(t)\theta_1(t)\}x+\dot r_i(t)\\&+\mathrm{tr}[a(t)\theta_1(t)]=0,\quad i=1,\ 2.\end{aligned}$$

Thus we come to the following

Proposition. *Let the matrices D_i, $M_i(t)$ be nonnegative definite and $N_j^{(i)}(t)$ be positive definite $(i\neq j)$, $i=1, 2$. Then (u_1^b, u_2^b) given by (4.1) is a pair of Berge-equilibrium strategies if $\{\theta_2(t), i=1, 2\}$ is the solution of the system of matrix differential equations of Riccati's type*

$$\begin{aligned}&\dot\theta_1(t)+A'(t)\theta_1(t)+\theta_1(t)\,A(t)-\theta_1(t)\,B_2(t)\,[N_2^{(1)}(t)]^{-1}B_2'(t)\theta_1(t)\\&-\theta_1(t)\,B_1(t)\,[N_1^{(2)}(t)]^{-1}B_1'(t)\theta_2(t)-\theta_2(t)\,B_1(t)\,[N_1^{(2)}(t)]^{-1}B_1'(t)\theta_1(t)\\&+M_1(t)+\theta_2(t)\,B_1(t)\,[N_1^{(2)}(t)]^{-1}N_1^{(1)}(t)\,[N_1^{(2)}(t)]^{-1}B_1'(t)\theta_2(t)=0,\\&\dot\theta_2(t)+A'(t)\,\theta_2(t)+\theta_2(t)\,A(t)-\theta_2(t)\,B_1(t)\,[N_1^{(2)}(t)]^{-1}B_1'(t)\theta_2(t)\\&-\theta_2(t)\,B_2(t)\,[N_2^{(1)}(t)]^{-1}B_2'(t)\theta_1(t)-\theta_1(t)\,B_2(t)\,[N_2^{(1)}(t)]^{-1}B_2'(t)\theta_2(t)\\&+M_2(t)+\theta_1(t)\,B_2(t)\,[N_2^{(1)}(t)]^{-1}N_2^{(2)}(t)\,[N_2^{(1)}(t)]^{-1}B_2'(t)\theta_1(t)=0\end{aligned}$$

with boundary conditions $\theta_i(T)=D_i$, $i=1,\ 2$ and

$$r_i(t)=\int_t^T\mathrm{tr}\,[a(\tau)\,\theta_i(\tau)]\,d\tau,\quad i=1,\ 2.$$

5. Linear-quadratic game with state and control dependent noise

Now let the evolution of the dynamic system Σ of the game Γ be described by the linear stochastic differential equation

$$d\xi(t)=[A(t)\xi(t)+B_1(t)u_1+B_2(t)u_2]dt+\sigma(t,\ \xi(t),\ u_1,\ u_2)dw(t),\quad t\in[t_0,T]$$

with initial condition $\xi(t_0)=\xi_0$. Here $\xi\in\mathsf{R}$ is the state process, $w=\{w(t),\ t\in[t_0,T]\}$ is a 4-dimensional standard Wiener process and $u_i\in U_i\subset\mathsf{R}$ is the control of the i-th player, $i=1,\ 2$. $\sigma(t,\ \xi,\ u_1,\ u_2)$ is an 1×4-matrix of the form

$$\sigma=(\sigma_0(t)\xi\quad \sigma_1(t)u_1\quad \sigma_2(t)u_2\quad \sigma_3(t)).$$

Henceforth $A(t)$, $B_i(t)$, $i=1,\ 2$, $\sigma_j(t)$, $j=1,\ 2,\ 3$ are functions taking values in R. The cost-functon $\mathscr{I}_i(u_1,\ u_2)$ of the i-th player is the functional

$$\mathscr{I}_i(u_1,\ u_2)=\mathsf{E}_{t_0,\ \xi_0}\{D_i\xi^2(T)+\int_{t_0}^{T}[M_i(t)\xi^2(t)+N_1^{(i)}(t)u_1^2(t)+N_2^{(i)}(t)u_2^2(t)]dt\},\quad i=1,\ 2.$$

Here D_i are constants and $M_i(t)$, $N_j^{(i)}(t)$, $j=1,\ 2$ are real-valued continuous functions, $i=1,\ 2$.

Suppose the strategies of the i-th player are functions of the form $u_i(t,\ x)=F_i(t)x$ where $F_i(t)$ are real-valued continuous functions, $i=1,\ 2$.

In this case $G_i(t,\ x,\ u_1,\ u_2)$ are functicns as follows

$$G_i(t,\ x,\ u_1,\ u_2)=W_t^{(i)}(t,\ x)+\mathscr{A}(u_1,\ u_2)W^{(i)}(t,\ x)+L_i(t,\ x,\ u_1,\ u_2)$$

$$=W_t^{(i)}(t,\ x)+[A(t)x+B_1(t)u_1+B_2(t)u_2]\,W_x^{(i)}(t,x)+\frac{1}{2}[\sigma_0^2(t)x^2+\sigma_1^2(t)u_1^2$$

$$+\sigma_2^2(t)u_2^2+\sigma_3^2(t)]\,W_{xx}^{(i)}(t,\ x)+M_i(t)x^2+N_1^{(i)}(t)u_1^2+N_2^{(i)}(t)u_2^2$$

for all $t\in[t_0,T]$, $x\in\mathsf{R}$, $u_1\in U_1$, $u_2\in U_2$, $i=1,\ 2$. The remark (Section 3) leads now to

$$\left.\frac{\partial G_1}{\partial u_2}\right|_{u_2=u_2^b}=B_2(t)\,W_x^{(1)}(t,\ x)+\sigma_2^2(t)u_2^b\,W_{xx}^{(1)}(t,\ x)+2\,N_2^{(1)}(t)u_2^b=0$$

and

$$\left.\frac{\partial G_2}{\partial u_2}\right|_{u_1=u_1^b}=B_1(t)\,W_x^{(2)}(t,\ x)+\sigma_1^2(t)u_1^b\,W_{xx}^{(2)}(t,\ x)+2\,N_1^{(2)}(t)u_1^b=0,$$

i. e.

$$u_2^b=-[\sigma_2^2(t)\,W_{xx}^{(1)}(t,\ x)+2\,N_2^{(1)}(t)]^{-1}\,B_2(t)\,W_x^{(1)}(t,\ x)$$

and

$$u_i^b=-[\sigma_1^2(t)\,W_{xx}^{(2)}(t,\ x)+2\,N_1^{(2)}(t)]^{-1}\,B_1(t)\,W_x^{(2)}(t,\ x).$$

Further, put u_i^b, $i=1,\ 2$, in the system

$$G_i(t,\ x,\ u_1^b,\ u_2^b)=0,\quad i=1,\ 2, \tag{5.1}$$

and search $W^{(i)}(t,\ x\ )$ in the following special form

$$W^{(i)}(t,\ x)=x^2\theta_i(t)+r_i(t) \tag{5.2}$$

where $\theta_i(t)$ and $r_i(t)$—real-valued functions, $i=1,\ 2$. Then

$$\begin{aligned} u_1^b &= -[\sigma_1^2(t)\theta_2(t)+N_1^{(2)}(t)]^{-1}B_1(t)\theta_2(t)x,\\ u_2^b &= -[\sigma_2^2(t)\theta_1(t)+N_2^{(1)}(t)]^{-1}B_2(t)\theta_1(t)x, \end{aligned} \tag{5.3}$$

and for equations (5.1) we have

$$\begin{aligned} &x^2\{\dot\theta_i(t)+2A(t)\theta_i(t)-2[\sigma_1^2(t)\theta_2(t)+N_1^{(2)}(t)]^{-1}B_1^2(t)\theta_2(t)\theta_i(t)\\ &-2[\sigma_2^2(t)\theta_1(t)+N_2^{(1)}(t)]^{-1}B_2^2(t)\theta_1(t)\theta_i(t)+M_i(t)+\sigma_0^2(t)\theta_i(t)\\ &+\sigma_1^2(t)\ [\sigma_1^2(t)\theta_2(t)+N_1^{(2)}(t)]^{-2}B_1^2(t)\theta_2^2(t)\theta_i(t)+\sigma_2^2(t)\ [\sigma_2^2(t)\theta_1(t)\\ &+N_2^{(1)}(t)]^{-2}B_2^2(t)\theta_1^2(t)\theta_i(t)+N_1^{(i)}(t)\ [\sigma_1^2(t)\theta_2(t)+N_1^{(2)}(t)]^{-2}B_1^2(t)\theta_2^2(t)\\ &+N_2^{(i)}(t)\ [\sigma_2^2(t)\theta_1(t)+N_2^{(1)}(t)]^{-2}B_2^2(t)\theta_1^2(t)\}+\dot r_i(t)+\sigma_3^2(t)\theta_i(t)=0,\quad i=1,\ 2. \end{aligned}$$

Thus we come to the following

Proposition. *Let D_i be nonnegative constants, $M_i(t)$ be nonnegative functions, $i=1,\ 2$ and $N_2^{(1)}(t)$, $N_1^{(2)}(t)$ be positive functions for each $t\in[t_0,\ T]$. Then the pair of functions $\{W^{(i)}(t,\ x),\ i=1,2\}$, given by (5.2) is the solution of the system (5·1) and u_i^b, $i=1,2$, defined by (5.3) are Berge-equilibrium strategies if $\{\theta_i(t),\ i=1,2\}$ is the solution of the system*

$$\begin{aligned} &\dot\theta_1(t)+2A(t)\theta_1(t)+\sigma_0^2(t)\theta_1(t)-[\sigma_2^2(t)\theta_1(t)+N_2^{(1)}(t)]^{-1}B_2^2(t)\theta_1^2(t)\\ &-2[\sigma_1^2(t)\theta_2(t)+N_1^{(2)}(t)]^{-1}B_1^2(t)\theta_1(t)\theta_2(t)+M_1(t)+[\sigma_1^2(t)\theta_1(t)\\ &+N_1^{(1)}(t)]\ [\sigma_1^2(t)\theta_2(t)+N_1^{(2)}(t)]^{-2}B_1^2(t)\theta_2^2(t)=0,\\ &\dot\theta_2(t)+2A(t)\theta_2(t)+\sigma_0^2(t)\theta_2(t)-[\sigma_1^2(t)\theta_2(t)+N_1^{(2)}(t)]^{-1}B_1^2(t)\theta_2^2(t)\\ &-2\,[\sigma_2^2(t)\theta_1(t)+N_2^{(1)}(t)]^{-1}B_2^2(t)\theta_2(t)\theta_1(t)+M_2(t)+[\sigma_2^2(t)\theta_2(t)\\ &+N_2^{(2)}(t)]\ [\sigma_2^2(t)\theta_1(t)+N_2^{(1)}(t)]^{-2}B_2^2(t)\theta_1^2(t)=0 \end{aligned}$$

with boundary conditions $\theta_i(T)=D_i$, $i=1,\ 2$, and

$$r_i(t)=\int_t^T\sigma_3^2(\tau)\,\theta_i(\tau)dt,\quad i=1,\ 2.$$

6. Conclusions

In this paper we consider Berge-equilibrium in stochastic differential games. In comparison with Nach-equilibrium [5], [6] there are differences in the game and in the analytical aspects of the problem. Berge-equilibrium is a firm background for considering other kinds of solutions of games, including many-player games.

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