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Iterative Procedure for Solving Stochastic Differential Equations

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In this paper we describe a method for an approximation of the strong solution of Ito'stochastic differential equation. We construct a sequence of processes converging almost surely to the solution of the equation. Similar method for deterministic ordinary differential equations is considered in [4].

1. Introduction.

It is well known that some classes of stochastic processes can be obtained as solutions of stochastic differential equations. There is a great number of papers and books in this field (see e. g. [2], [3]). However, from purely theoretical point of view, and much more from the point of view of various applications, it is very important to find suitable method for finding at least of approximate solutions of the stochastic differential equations (SDE).

In this paper we consider a class of SDE involving stochastic integrals in the sense of K. Ito. We describe an interative procedure such that a sequence of stochastic processes will converge with probability 1 to the strong solution of the original SDE. In some sense, the idea of this investigation goes back to the paper of Zuber [4] treating deterministic ordinary differential equations. Some indications without details were announced recently in [5].

Let us introduce some notations. Throughout the paper, (Ω, F, P) is a fixed complete probability space. All random variables and stochastic processes will be defined on this space.

Let $W=(W_t, F_t)$, $t \geqq 0$, be a standard Wiener process, where (F_t) is a filtration satisfying the usual conditions. We have given the real-valued finctions $a(t, x)$, $a_n(t, x)$, $b(t, x)$ and $b_n(t, x)$, $n=1, 2, \dots$, defined on $[0, T] \times R$ and measurable with respect to the product σ-algebra $B_{[0,T]} \times B$, where $T=\text{const}>0$, $R=(-\infty, \infty)$. Following the traditions of the classical theory of SDE (see [1], [3]), we suppose that each of the functions a, a_n, b, b_n satisfies the global Lipschitz's condition in the second argument and when $x \to \infty$ grows not faster than the linear function, i. e. for some positive constant L we have

$$|a(t, x)-a(t, y)| \leqq L|x-y|, \quad a^2(t, x) \leqq L^2(1+x^2) \tag{1}$$

and analogously for the rest functions.

Our main purpose here is to study the SDE

$$X_t = \eta + \int_0^t a(s, X_s)\,ds + \int_0^t b(s, X_s)\,dW_s, \tag{2}$$

where η is a random variable not depending on W with $E\{|\eta|^2\}<\infty$ and $\int_0^t b(\cdot)dW_s$ is the stochastic integral in the sense of Ito (see [2], [3]). Let us note that the conditions on the drift coefficient $a(\cdot)$ and diffusion coefficient $b(\cdot)$ guarantee the existence and uniquencess of the strong solution $X=(X_t, F_t)$, $t\in[0, T]$, of the SDE (2).

Now we shall define the sequence $X^{(n)}=\{X_t^{(n)}, t\in[0,T]\}$, $n=1, 2, \ldots$, of stochastic processes as follows. Using the pair of functions $\{a_n, b_n\}$ and the Wiener process W, we consider a SDE whose solution will be denoted by $X^{(n+1)}$, i. e.

$$X_t^{(n+1)}=\eta+\int_0^t a_n(s, X_s^{(n+1)})\,ds+\int_0^t b_n(s, X_s^{(n+1)})\,dW_s. \tag{3}$$

It is quite natural to expect that if the pair of functions $\{a_n, b_n\}$ is close in some sense to $\{a, b\}$, then the processes $X^{(n)}$ will tend to X as $n\to\infty$. We shall establish results of this type, but first let introduce the following condition

$$\sum_{n=1}^{\infty} \sup_{t,x} \{|a(t,x)-a_n(t,x)|+|b(t,x)-b_n(t,x)|\}<\infty. \tag{4}$$

Obviously, (4) implies that $a_n(t,x)\to a(t,x)$ and $b_n(t,x)\to b(t,x)$ as $n\to\infty$ uniformly in t,x. The condition (4) will be used essentially to prove our main result (see Section 2). But this condition will be modified and weakened in Section 3 in order to cover other interesting cases.

2. Main result and proof.

Now we shall formulate the main result of this paper.

Theorem. *Let the functions a, a_n, b, b_n, $n=1, 2, \ldots$, be defined as above and the condition (4) be fulfilled. Then the sequence of processes $\{X^{(n)}, n=1, 2, \ldots\}$ converges with probability 1 to the process X as $n\to\infty$.*

Proof. Denote

$$\varepsilon_n=E\{\sup_t[|a(t, X_t^{(n)})-a_n(t, X_t^{(n)})|^2+|b(t, X_t^{(n)})-b_n(t, X_t^{(n)})|^2]\}. \tag{5}$$

By the subtraction of the equations (2) and (3) and adding some terms, we have

$$\begin{aligned} X_t-X_t^{(n+1)}&=\int_0^t [a(s,X_s)-a(s,X_s^{(n)})]\,ds+\int_0^t [a(s, X_s^{(n)})-a_n(s, X_s^{(n)})]\,ds \\ &+\int_0^t [a_n(s,X_s^{(n)})-a_n(s,X_s)]\,ds+\int_0^t [a_n(s, X_s)-a_n(s, X_s^{(n+1)})]\,ds \\ &+\int_0^t [b(s, X_s)-b(s, X_s^{(n)})]\,dW_s+\int_0^t [b(s, X_s^{(n)})-b_n(s, X_s^{(n)})]\,dW_s \\ &+\int_0^t [b_n(s, X_s^{(n)})-b_n(s, X_s)]\,dW_s+\int_0^t [b_n(s, X_s^{(n)})-b_n(s, X_s^{(n+1)})]\,dW_s. \end{aligned} \tag{6}$$

Now we have to estimate each of these integrals. First, using the Cauchy-Schwartz inequality and (5), we get

$$E\{\int_0^t [a(s, X_s^{(n)})-a_n(s, X_s^{(n)})]\,ds\}^2 \leqq t\int_0^t E\{|a(s, X_s^{(n)})-a_n(s, X_s^{(n)})|^2\}$$

$$\leqq T\int_0^t E\{\sup_t |a(s, X_s^{(n)})-a_n(s, X_s^{(n)})|^2\}\,ds \leqq T\varepsilon_n t.$$

Applying one of the basic properties of the stochastic integrals (see [1], [2], [3]), we obtain

$$E\{\int_0^t [b(s, X_s^{(n)})-b_n(s, X_s^{(n)})]\,dW_s\}^2 = \int_0^t E\{|b(s, X_s^{(n)})-b_n(s, X_s^{(n)})|^2\}\,ds \leqq \varepsilon_n t.$$

These inequalities and the Lipschitz's conditions for the functions a, a_n, b, b_n, imply that

$$E\{|X_t-X_t^{(n+1)}|^2\} \leqq 2\cdot 8\cdot(T+1)L^2\int_0^t E\{|X_s-X_s^{(n)}|^2\}\,ds$$

$$+8\cdot(T+1)L^2\int_0^t E\{|X_s-X_s^{(n+1)}|^2\}\,ds+8\cdot(T+1)\varepsilon_n t.$$

Denote $8\cdot(T+1)L^2=\alpha$, $8(T+1)=\beta$. If we apply the well-known Gronwall-Bellman's inequality, we come to the estimation

$$E\{|X_t-X_t^{(n+1)}|^2\} \leqq 2\alpha\int_0^t E\{|X_s-X_s^{(n)}|^2\}\,ds+\beta\varepsilon_n t$$

$$+\alpha\int_0^t [2\alpha\int_0^s E\{|X_u-X_u^{(n)}|^2\}\,du+\beta\varepsilon_n s]\,e^{\alpha(t-s)}ds.$$

Let us determine the upper bound for $E\{|X_t-X_t^{(n+2)}|^2\}$, $n=1, 2, \dots$, by induction. For the first approximation we take any random process $X^{(1)}$ adapted to (F_t) and such that $E\{\sup_t |X_t^{(1)}|^2\}<\infty$. It is easy to prove that the sequence $E\{\sup_t |X_t^{(n)}|^2\}$, $n=1, 2, \dots$, and $E\{\sup_t |X_t|^2\}$ are uniformly bounded. Then $\sup_t E\{|X_t^{(1)}|^2\} \leqq c<\infty$, where the constant c depends on $E\{|\eta|^2\}$ and $E\{\sup_t |X_t^{(1)}|^2\}$. Further,

$$E\{|X_t-X_t^{(2)}|^2\} \leqq (2\alpha c+\beta\varepsilon_1)\frac{e^{\alpha t}-1}{\alpha},$$

$$E\{|X_t-X_t^{(3)}|^2\} \leqq 2(2\alpha c+\beta\varepsilon_1)[te^{\alpha t}-\frac{e^{\alpha t}-1}{\alpha}]+\beta\varepsilon_2\frac{e^{\alpha t}-1}{\alpha}.$$

Since $e^{\alpha t}-1>\alpha t$ for each t and $\alpha>0$, then

$$E\{|X_t-X_t^{(3)}|^2\} < [(2\alpha c+\beta\varepsilon_1)2\alpha t+\beta\varepsilon_2]\frac{e^{\alpha t}-1}{\alpha}.$$

Suppose that $E\{|X_t-X_t^{(n+1)}|^2\}$ has an upper bound of the type

(7) $$E\{|X_t-X_t^{(n+1)}|^2\}<[2\alpha c\frac{(2\alpha t)^{n-1}}{(n-1)!}+\beta\sum_{k=1}^{n}\varepsilon_k\frac{(2\alpha t)^{n-k}}{(n-k)!}]\frac{e^{\alpha t}-1}{\alpha}$$

$$=P_{n-1}(2\alpha t)\frac{e^{\alpha t}-1}{\alpha},\quad n=1,\ 2,\ \ldots,$$

where P_{n-1} is a polynomial of degree $n-1$ and we want to determine an upper bound for $E\{|X_t-X_t^{(n+2)}|^2\}$. We have

(8) $$E\{|X_t-X_t^{(n+2)}|^2\}<2\alpha\int_0^t P_{n-1}(2\alpha s)\frac{e^{\alpha s}-1}{\alpha}ds+\beta\varepsilon_{n+1}t$$

$$+\alpha\int_0^t[2\alpha\int_0^s P_{n-1}(2\alpha u)\frac{e^{\alpha u}-1}{\alpha}du+\alpha\varepsilon_{n+1}s]\,e^{\alpha(t-s)}ds$$

$$=2e^{\alpha t}\int_0^t P_{n-1}(2\alpha s)(1-e^{-\alpha s})\,ds+\beta\varepsilon_{n+1}\frac{e^{\alpha t}-1}{\alpha}.$$

Since $\frac{1}{k!}\int_0^t x^k e^{-x}dx=1-e^{-t}[\frac{t^k}{k!}+\frac{t^{k-1}}{(k-1)!}+\ldots+t+1]$ and $\frac{t^k}{k!}+\frac{t^{k-1}}{(k-1)!}+\ldots+t+1-e^t<-\frac{t^{k+1}}{(k+1)!}$ for each $k=0,\ 1,\ \ldots,$ then for the coefficients B_{n-k} of the polynomial $P_{n-1}(2\alpha t)$ we obtain the following bound

$$B_{n-k}=2e^{\alpha t}\int_0^t\frac{(2\alpha s)^{n-k}}{(n-k)!}(1-e^{-\alpha s})\,ds$$

$$=\frac{2^{n-k+1}}{\alpha}[e^{\alpha t}\frac{(\alpha t)^{n-k+1}}{(n-k+1)!}+\frac{(\alpha t)^{n-k}}{(n-k)!}+\qquad+\frac{\alpha t}{1!}+1-e^{\alpha t}]$$

$$<\frac{(2\alpha t)^{n-k+1}}{(n-k+1)}\cdot\frac{e^{\alpha t}-1}{\alpha},\quad k=1,\ 2,\ \ldots,\ n.$$

From (8) and the last relation it follows that

$$E\{|X_t-X_t^{(n+2)}|^2\}<[(2\alpha c+\beta\varepsilon_1)\frac{(2\alpha t)^n}{n!}+\beta\varepsilon_2\frac{(2\alpha t)^{n-1}}{(n-1)!}$$

$$+\ \ldots\ +\beta\varepsilon_n\frac{2\alpha t}{1!}+\beta\varepsilon_{n+1}]\frac{e^{\alpha t}-1}{\alpha}=P_n(2\alpha t)\frac{e^{\alpha t}-1}{\alpha}.$$

This means that (7) is an upper bound of $E\{|X_t-X_t^{(n+1)}|^2\}$ for each $n=1,2,\ldots$

We need the following inequality for stochastic integrals:

(9) $$E\{\sup_{t:t\in[0,T]}\int_0^t X(s,\ \omega)\,dW_s\}^2\leq 4\int_0^T E\{|X(t,\ \omega)|^2\}\,dt,$$

if the last integral is finite (see [1], [2], [3]).

From (6), (7) and (9) we find

$$E\{\sup_t |X_t - X_t^{(n+1)}|^2\} \leqq 8(T+4)L^2 [2 \int_0^T E\{|X_s - X_s^{(n)}|^2\} ds$$

$$+ \int_0^T E\{|X_s - X_s^{(n+1)}|^2\} ds + 8T(T+4)\varepsilon_n$$

$$< c_1 P_{n-2}(2\alpha T) + c_2 P_{n-1}(2\alpha T) + c_3 \varepsilon_n, \quad n = 1, 2, \ldots,$$

where c_1, c_2 and c_3 are constants. Now we are interested if the series $\Sigma_{n=1}^{\infty} P\{\sup_t |X_t - X_t^{(n+1)}| \geqq \varepsilon\}$ is convergent. According to the Chebyshev's inequality, we have

$$(10) \quad \sum_{n=1}^{\infty} P\{\sup_t |X_t - X_t^{(n+1)}| \geqq \varepsilon\} \leqq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} E\{\sup_t |X_t - X_t^{(n+1)}|^2\}.$$

The condition (4) implies that $\Sigma_{n=1}^{\infty} \varepsilon_n < \infty$. Since

$$\sum_{n=1}^{\infty} P_{n-1}(2\alpha t) = \sum_{n=1}^{\infty} [2\alpha c \frac{(2\alpha T)^{n-1}}{(n-1)!} + \beta \sum_{k=1}^{\infty} \varepsilon_k \frac{(2\alpha T)^{n-k}}{(n-k)!}]$$

$$< 2\alpha c \sum_{n=1}^{\infty} \frac{(2\alpha T)^{n-1}}{(n-1)!} + \beta \sum_{n=1}^{\infty} \varepsilon_n \cdot \sum_{n=1}^{\infty} \frac{(2\alpha T)^{n-1}}{(n-1)!} < \infty,$$

it is clear that the series in the right side, and hence in the left side of (10), are convergent. By the Borel-Cantelli's lemma and the Weierstrass' uniform convergence theorem, it follows that the sequence $\{X_t^{(n)}, n = 1, 2, \ldots\}$ almost surely converges uniformly in t, $t \in [0, T]$, to the random proces X. Thus the Theorem is proved.

3. Remark

The assertion of the Theorem can be proved under other conditions which in some sense are weaker than the condition (4). For example, we can suppose that all the functions a, a_n, b, b_n, are random functions and to require that conditions (1) and (4) are fulfilled almost surely. Moreover, condition (4) can be replaced by

$$(11) \qquad \sum_{n=1}^{\infty} E\{\sup_t [|a(t, X_t^{(n)}) - a_n(t, X_t^{(n)})|^2 + |b(t, X_t^{(n)}) - b_n(t, X_t^{(n)})|^2]\} < \infty,$$

where the process $X^{(n)}$ is produced by the pair of functions $\{a_{n-1}, b_{n-1}\}$ and the Wiener process.

The Theorem gives us an idea how to construct a sequence of succesive approximations which almost surely converges to the solution of the SDE (2). Notice especially that each approximation is an element of the space $c[0, T]$ of the continuous functions on $[0, T]$. So we can find an ε-approximation of the solution X ofthe original equation (2), i. e. the random process $X^{(n)}$, where $n = n(\varepsilon)$ such that

$$P\{\sup_t |X_t - X_t^{(n)}| < \varepsilon\} = 1.$$

To find an ε-approximation of the solution X, we shall construct a sequence of random processes $X^{(1)}, X^{(2)}, \ldots, X^{(n)}$ in the following way. Let $X^{(1)}$ be an arbitrary random process with $X_0^{(1)} = \eta$ and $E\{\sup_t |X_t^{(1)}|^2\} < \infty$. Next, we choose the functions $a_1(t, x)$ and $b_1(t, x)$ such that condition (1) is satisfied and such that $\sup_{t,x}\{|a(t, x) - a_1(t, x)| + |b(t, x) - b_1(t, x)|\} \leqq c_1 < \infty$. Inductively, if we have $X^{(n-1)}$, we choose the functions $a_{n-1}(t, x)$ and $b_{n-1}(t, x)$ such that the condition (1) is satisfied, and such that $\sup_{t,x}\{|a(t, x) - a_{n-1}(t, x)| + |b(t, x) - b_{n-1}(t,x)|\} \leqq c_{n-1} < \infty$, where c_{n-1} is $(n-1)$-th term of any convergent series $\Sigma_{n=1}^{\infty} c_n$. Now the process $X^{(n)}$ is defined as a solution of the equation

$$dX_t^{(n)} = a_{n-1}(t, X_t^{(n)})\,dt + b_{n-1}(t, X_t^{(n)})\,dW_t, \quad X_t^{(n)} = \eta.$$

Analogously to the consideration in [4], it is natural to use the notion Z-algorithm for this iterative procedure. The sequence of functions $\{a_n(t, x), b_n(t, x)\}$, $n = 1, 2, \ldots$, will be called a determined sequence. Obviously, Z-algorithm can be effectuvely used only if our choice of determined sequence is good enough, i. e. if SDE (3) can be solved. Also, it is clear that the rate of convergence in the Z-algorithm depends on the choice of the starting process and determined sequence. Recall that we can solve explicitly any linear stochastic differential equation (see [1], [2]). This fact leads to the idea to make a linearization of the functions $a(t, x)$ and $b(t, x)$, such that the equations (3) should have the form

$$\text{u(y(}dX_t^{(n+1)} = [A_n(t) + B_n(t)X_t^{(n+1]}]\,dt + [c_n(t) + D_n(t)X_t^{(n+1)}]\,dW_t, \quad X_0^{(n+1)} = \eta.$$

Let us describe a simple form of such linearization. Suppose the functions $a(t, x)$ and $b(t, x)$ satisfy the condition (1). If $\{\alpha_n(t)\}$, $n = 1, 2, \ldots$, and $\{\beta_n(t)\}$, $n = 1, 2, \ldots$, are sequences of uniformly bounded continuous functions on $[0, T]$, then the sequence of pairs of functions

$$\{a_n(t, x), b_n(t, x)\} = \{\alpha_n(t)(x - X_t^{(n)}) + a(t, X_t^{(n)}), \beta_n(t)(x - X_t^{(n)}) + b(t, X_t^{(n)})\}, \; n = 1, 2, \ldots,$$

is a determined sequence for the Z-algorithm. Indeed,

$$|a_n(t, x) - a_n(t, y)| \leqq \sup_t |\alpha_n(t) \cdot |x - y|, \quad n = 1, 2 \ldots,$$

and analogously for $b_n(t, x)$. Since

$$a(t, X_t^{(n)}) - a_n(t, X_t^{(n)}) = b(t, X_t^{(n)}) - b_n(t, X_t^{(n)}) \equiv 0, \quad n = 1, 2, \ldots,$$

the condition (11) holds, but the condition $a_n^2(t, x) \leqq L^2(1 + x^2)$ does not hold. Because of this fact, we suggest the following procedure. Choose a positive number M. For the (F_t)-adapted random process $X^{(1)}$, which is our first approximation, let

$$\tau_1^M = \begin{cases} \inf\{s : |X_s^{(1)}| > M\} \\ T \text{ if } |X_s^{(1)}| \leqq M \text{ for all } s \in [0, T]. \end{cases}$$

Then τ_1^M is a stopping time with respect to (F_t). Denote $X_{t,M}^{(1)}=X_t^{(1)}$ if $|X_t^{(1)}|<M$ and $X_{t,M}^{(1)}=M$ sgn $X_t^{(1)}$ if $|X_t^{(1)}|\geqq M$;

$$a_1^M(t, x)=\alpha_1(t)\ (x-X_{t,M}^{(1)})+a(t, X_{t,M}^{(1)}),$$

and

$$b_1^M(t, x)=\beta_1(t)\ (x-X_{t,M}^{(1)})+b(t, X_{t,M}^{(1)}).$$

Thus we come to the conclusion that the SDE

$$X_t^{(2)}=\eta+\int_0^t a_1^M(s, X_s^{(2)})\,ds+\int_0^t b_1^M(s, X_s^{(2)})\,dW_s$$

has a unique solution on the interval $[0, \tau_1^M]$.
Let now

$$\tau_2^M=\begin{cases} \inf\{s:|X_s^{(2)}|>M\} \\ T \text{ if } |X_s^{(2)}|\leqq M \text{ for all } s\in[0, \tau_1^M], \end{cases}$$

and define the process $X_{t,M}^{(2)}$ in the same way as $X_{t,M}^{(1)}$. We can continue this procedure and obtain a sequence of processes $X_M^{(n)}$. For a fixed t in $[0, T]$ we can find a sufficiently large M, such that $\tau_n^M>t$, $n=1, 2, \ldots$, almost surely. Since there exists a stopping time $\tau^M=\inf_n \tau_n^M$ (see [3]), then on the interval $[0, \tau^M]$ all conditions of our Theorem are satisfied and therefore the sequence $\{X_M^{(n)}, n=1, 2, \ldots\}$ almost surely converges to the process X as $n\to\infty$. Also, on this interval the relations

$$a_n^M(t, X_{t,M}^{(n)})=a_n(t, X_t^{(n)}),\quad b_n^M(t, X_{t,M}^{(n)})=b_n(t, X_t^{(n)}),\quad n=1, 2, \ldots,$$

are valid. Since $\tau^M\to T$ almost surely as $M\to\infty$ (see [2]), we come to the conclusion that the sequence of random processes $\{X^{(n)}, n=1, 2, \ldots\}$ almost surely converges to the solution X of the SDE (2) for each $t\in[0, T]$.

In particular, if $\alpha_n(t)=\beta_n(t)\equiv 0$ for each $n=1, 2, \ldots$, then the Z-algorithm is reduced to the usual Picard-Lindelöf method of succesive approximations.

Finally, let us note that the classical Chaplygin's method in the theory of the deterministic differential equations (see [6]) could be extended to stochastic differential equations using the results of the present paper. However it will be a subject of a forthcoming paper. Also, we could extend the results of this paper for SDE involving stochastic integrals with respect to any continuous martingale and martingale measure.

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