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Two-Sided Methods for Solving the Polynomial Equation

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On the base of the well-known methods of Weierstrass-Dochev and Ehrlich we propose two two-sided methods with raised rate of convergence for simultaneous determination of the roots of a polynomial in the case when all the roots are real and simple.

1. Introduction

Let the algebraic equation

$$f(x) \equiv x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0 \tag{1}$$

has only simple roots. In [3] we considered the following generalization of Weierstrass-Dochev's method [1, 5],

$$x_i^{k+1} = x_i^k + \Delta_i^{R+1,k}, \quad \Delta_i^{p,k} = -f(x_i^k) \prod_{\substack{s=1 \\ s \neq i}}^{n} (x_i^k - x_s^k - \Delta_s^{p-1,k})^{-1}, \Delta_i^{0,k} = 0, \tag{2}$$

$$i = 1,2, \dots, n, \quad p = 1,2, \dots, R+1, \quad k = 0,1, \dots$$

for simultaneous determination of the roots of the equation (1). The rate of convergence of (2) is $R+2$, i. e. $|x_i - x_i^k| \leqq cq^{(R+2)^k}$, $0 < q < 1$, C—absolute constant.

2. Two-sided methods

In the case when the equation (1) has only simple and real roots we propose two-sided variant of (2),

$$\bar{x}_i^{k+1} = \bar{x}_i^k + \bar{\Delta}_i^{R+1,k}, \quad \underline{x}_i^{k+1} = \underline{x}_i^k + \underline{\Delta}_i^{R+1,k}, \tag{3}$$

$$\bar{\Delta}_i^{p,k} = -f(\bar{x}_i^k) \Big(\prod_{l=1}^{i-1} (\bar{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{p-1,k})_{-1} \prod_{l=i+1}^{n} (\bar{x}_i^k - \bar{x}_l^k - \bar{\Delta}_l^{p-1,k})\Big)^{-1}, \ \bar{\Delta}_i^{o,k} = 0,$$

$$\underline{\Delta}_i^{p,k} = -f(\underline{x}_i^k)\Big(\prod_{l=1}^{i-1} (\underline{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{p-1,k}) \prod_{l=i+1}^{n} (\underline{x}_i^k - \bar{x}_l^k - \bar{\Delta}_l^{p,1,k})\Big)^{-1}, \quad \underline{\Delta}_i^{0,k}=0,$$

$$i=1,\ 2,\ \dots,\ n,\quad p=1,\ 2,\ \dots,\ R+1,\quad k=0,\ 1,\ \dots$$

Let us note that some other two-sided methods with quadratic and cubic rate of convergence can be found in [4].

Theorem 1. *Let the initial approximations* $\{\bar{x}_i^0\}_{i=1}^n$, $\{\underline{x}_i^0\}_{i=1}^n$ *of the roots* $\{x_i\}_{i=1}^n$ *of the equation* (1) *satisfy the conditions* $\underline{x}_1^0 \leqq x_1 \leqq \bar{x}_1^0 < \underline{x}_2^0 \leqq x_2 \leqq \bar{x}_2^0 < \dots < \underline{x}_n^0 \leqq x_n \leqq \bar{x}_n^0$. *Then the method* (3) *gives* $\underline{x}_1^k \leqq x_1 \leqq \bar{x}_1^k < \underline{x}_2^k \leqq x_2 \leqq \bar{x}_2^k < \dots < \underline{x}_n^k \leqq x_n \leqq \bar{x}_n^k$, $k=0,\ 1,\ 2,\dots$

Proof. We shall prove the theorem by induction. When $k=0$ the assertion holds. Let us assume that the theorem is true for some k and from this assumption we shall get the theorem for $k+1$. First we prove

(4) $$0 \leqq \underline{\Delta}_l^{p,k} \leqq x_l - \underline{x}_l^k,\quad x_l - \bar{x}_l^k \leqq \bar{\Delta}_l^{p,k} \leqq 0,\quad p=0,\ 1,\dots,R+1,\quad l=1,\ 2,\dots,\ n.$$

Indeed $\underline{\Delta}_l^{0,k} = \bar{\Delta}_l^{0,k} = 0$, and if (4) is valid for some p, then

$$\underline{\Delta}_l^{p+1,k} = -(\underline{x}_l^k - x_l) \prod_{s=1}^{l-1} (\underline{x}_l^k - x_s)(\underline{x}_l^k - \underline{x}_s^k - \underline{\Delta}_s^{p,k})^{-1} \prod_{s=l+1}^{n} (\underline{x}_l^k - x_s)(\underline{x}_l^k$$

$$- \bar{x}_s^k - \bar{\Delta}_s^{p,k})^{-1} = (x_l - \underline{x}_l^k) \prod_{s=1}^{l-1} (\underline{x}_l^k - x_s)(\underline{x}_l^k - x_s + x_s$$

$$- \underline{x}_s^k - \underline{\Delta}_s^{p,k})^{-1} \prod_{s=l+1}^{n} (\underline{x}_l^k - x_s)(\underline{x}_l^k - x_s + x_s - \bar{x}_s^k - \bar{\Delta}_s^{p,k})^{-1}.$$

The above equality gives $0 \leqq \underline{\Delta}_l^{p+l,k} \leqq x_l - \underline{x}_l^k$, as

$$0 \leqq \prod_{s=1}^{l-1} (\underline{x}_l^k - x_s)(\underline{x}_l^k - x_s + x_s - \underline{x}_s^k - \underline{\Delta}_s^{p,k})^{-1} \leqq 1,$$

$$0 \leqq \prod_{s=l+1}^{n} (\underline{x}_l^k - x_s)(\underline{x}_l^k - x_s + x_s - \bar{x}_s^k - \bar{\Delta}_s^{p,k})^{-1} \leqq 1.$$

Analogously $x_l - \bar{x}_l^k \leqq \bar{\Delta}_l^{p+1,k} \leqq 0$. So the inequalities (4) are proved.

Now using (3) and (4), we obtain

$$\bar{x}_i^{k+1} - x_i = (\bar{x}_i^k - x_i)\Big(1 - \prod_{l=1}^{i-1} (\bar{x}_i^k - x_l)(\bar{x}_i^k - x_l + x_l - \underline{x}_l^k$$

$$-\underline{\Delta}_l^{R,k})^{-1}\prod_{l=i+1}^{n}(\bar{x}_i^k-x_l)(\bar{x}_i^k-x_l+x_l-\bar{x}_l^k-\bar{\Delta}_l^{R,k})^{-1})\geqq 0,$$

$$\underline{x}_i^{k+1}-x_i=(\underline{x}_i^k-x_i)(1-\prod_{l=1}^{i-1}(\underline{x}_i^k-x_l)(\underline{x}_i^k-x_l+x_l-\underline{x}_l^k$$

$$-\underline{\Delta}_l^{R,k})^{-1}\prod_{l=i+1}^{n}(\underline{x}_i^k-x_l)(\underline{x}_i^k-x_l+x_l-\bar{x}_l^k-\bar{\Delta}_l^{R,k})^{-1})\leqq 0$$

$$\bar{x}_{i+1}^{k+1}-\bar{x}_i^{k+1}=\underline{x}_{i+1}^k-\bar{x}_i^k-(\underline{x}_{i+1}^k-x_{i+1})\prod_{l=1}^{i}(\underline{x}_{i+1}^k-x_l)(\underline{x}_{i+1}^k-x_l+x_l$$

$$-\underline{x}_l^k-\underline{\Delta}_l^{R,k})^{-1}\prod_{l=i+2}^{n}(x_{i+1}^k-x_l)(\underline{x}_{i+1}^k-x_l+x_l-\bar{x}_l^k-\bar{\Delta}_l^{R,k})^{-1}$$

$$+(\bar{x}_i^k-x_i)\prod_{l=1}^{i-1}(\bar{x}_i^k-x_l)(\bar{x}_i^k-x_l+x_l-\underline{x}_l^k-\underline{\Delta}_l^{R,k})^{-1}\prod_{l=i+1}^{n}(\bar{x}_i^k-x_l)(\bar{x}_i^k$$

$$-x_l+x_l-\bar{x}_l^k-\bar{\Delta}_l^{R,k})^{-1})\geqq 0,$$

for $i=1, 2, \ldots, n$.

The theorem is proved.

The next theorem gives the order of convergence of the suggested method.

Theorem 2. *Let* $0<q<1$, $d=\min_{i\neq j}|x_i-x_j|$ *and the constant* o *satisfies* $0<c\leqq d/(2n^2+1)$. *If the initial approaximations* $\{\underline{x}_i^0\}_{i=1}^n$, $\{\bar{x}_i^0\}_{i=1}^n$ *of the roots* $\{x_i\}_{i=1}^n$ *of the equation* (1) *satisfy* $0\leqq x_i-\underline{x}_i^0\leqq cq$, $0\leqq\bar{x}_i^0-x_i\leqq cq$, $i=1, 2, \ldots, n$, $0\leqq\underline{x}_{i+1}^0-\bar{x}_i^0$, $i=1, 2, \ldots, n-1$ *then for every* $k=0, 1, 2, \ldots$, *we have*

$$0\leqq\bar{x}_i^k-x_i\leqq cq^{(R+2)^k},\quad 0\leqq x_i-\underline{x}_i^k\leqq cq^{(R+2)^k},\quad i=1, 2, \ldots, n.$$

Proof. Again we shall prove the theorem by induction. Obviously the theorem is true for $k=0$. Let us assume the theorem is true for some k. Then

(5) $$\bar{x}_i^{k+1}-x_i=(\bar{x}_i^k-x_i)(1-\prod_{l-1}^{i-1}(\bar{x}_i^k-x_l)(\bar{x}_i^k-\underline{x}_l^k$$

$$-\underline{\Delta}_l^{R,k})^{-1}\prod_{l=i+1}^{n}(\bar{x}_i^k-x_l)(\bar{x}_i^k-\bar{x}_l^k-\bar{\Delta}_l^{R,k})^{-1})=(\bar{x}_i^k-x_i)(1-\bar{A}_{i,R}^k\,\bar{B}_{i,R}^k),$$

$$x_i - \underline{x}_i^{k+1} = (x_i - \underline{x}_i^k)\,\Big(1 - \prod_{l=1}^{i-1} (\underline{x}_i^k - x_l)\,(\underline{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{R,k})^{-1} \prod_{l=i+1}^{n} (\underline{x}_i^k - x_l)\,(\underline{x}_i^k - \bar{x}_l^k - \bar{\Delta}_l^{R,k})^{-1} = (x_i - \underline{x}_i^k)\,(1 - \underline{A}_{i,R}^k\, \underline{B}_{i,R}^k).$$

For $\bar{A}_{i,R}^k$, $\bar{B}_{i,R}^k$, $\underline{A}_{i,R}^k$, $\underline{B}_{i,R}^k$ we get

$$(6)\quad \bar{A}_{i,R}^k = (\bar{x}_i^k - x_{i-1})\,(\bar{x}_i^k - \underline{x}_{i-1}^k - \underline{\Delta}_{i-1}^{R,k})^{-1} \prod_{l=1}^{i-1} (\bar{x}_i^k - x_l)\,(\bar{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{R,k})^{-1}$$

$$- \prod_{l=1}^{i-2} (\bar{x}_i^k - x_l)\,(\bar{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{R,k})^{-1} + \prod_{l=1}^{i-2} (\bar{x}_i^k - x_l)\,(\bar{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{R,k})^{-1}$$

$$= \prod_{l=1}^{i-2} (\bar{x}_i^k - x_l)\,(\bar{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{R,k})^{-1} + (\underline{x}_{i-1}^k - x_{i-1} + \underline{\Delta}_{i-1}^{R,k})\,(\bar{x}_i^k - \underline{x}_{i-1}^k$$

$$- \underline{\Delta}_{i-1}^{R,k})^{-1} \prod_{l=1}^{i-2} (\bar{x}_i^k - x_l)\,(\bar{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{R,k})^{-1} = \;.\;.\;.\; = 1 + \sum_{p=1}^{i-1} (\underline{x}_p^k - x_p - \underline{\Delta}_p^{R,k})\,(\bar{x}_i^k - \underline{x}_p^k$$

$$- \underline{\Delta}_p^{R,k})^{-1} \prod_{l=1}^{p-1} (\bar{x}_i^k - x_l)\,(\bar{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{R,k})^{-1} = 1 + \bar{M}_{i,R}^k;$$

$$\bar{B}_{i,R}^k = 1 + \sum_{p=i+1}^{n} (\bar{x}_p^k - x_p - \bar{\Delta}_p^{R,k})\,(\bar{x}_i^k - \bar{x}_p^k - \bar{\Delta}_p^{R,k})^{-1} \prod_{l=p+1}^{n} (\bar{x}_i^k - x_l)\,(\bar{x}_i^k$$

$$- \bar{x}_l^k - \bar{\Delta}_l^{R,k})^{-1} = 1 + \bar{N}_{i,R}^k,$$

$$\underline{A}_{i,R}^k = 1 + \sum_{p=1}^{i-1} (\underline{x}_p^k - x_p - \underline{\Delta}_p^{R,k})\,(\underline{x}_i^k - \underline{x}_p^k - \underline{\Delta}_p^{R,k})^{-1} \prod_{l=1}^{p-1} (\underline{x}_i^k$$

$$- x_l)\,\,(\underline{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{R,k})^{-1} = 1 + \underline{M}_{i,R}^k,$$

$$\underline{B}_{i,R}^k = 1 + \sum_{p=i+1}^{n} (\bar{x}_p^k - x_p - \bar{\Delta}_p^{R,k})\,(\underline{x}_i^k - \bar{x}_p^k$$

$$- \bar{\Delta}_p^{R,k})^{-1} \prod_{l=p+1}^{n} (\underline{x}_i^k - x_l)\,(\underline{x}_i^k - \bar{x}_l^k - \bar{\Delta}_l^{R,k})^{-1} = 1 + \underline{N}_{i,R}^k.$$

In above notations we set $\underline{A}_{1,p}^k = \bar{A}_{1,p}^k = \underline{B}_{n,p}^k = \bar{B}_{n,p}^k = 1$, $\underline{M}_{1,p}^k = \bar{M}_{1,p}^k = \underline{N}_{n,p}^k = \bar{N}_{n,p}^k = 0$, $p = 0, 1, \ldots, R$, $k = 0, 1, 2, \ldots$

It follows from (5) that

$$(7)\quad \begin{aligned} \bar{x}_i^{k+1} - x_i &= (\bar{x}_i^k - x_i)\,(-\bar{M}_{i,R}^k - \bar{N}_{i,R}^k - \bar{M}_{i,R}^k\, \bar{N}_{i,R}^k), \\ x_i - \underline{x}_i^{k+1} &= (x_i - \underline{x}_i^k)\,(-\underline{M}_{i,R}^k - \underline{N}_{i,R}^k - \underline{M}_{i,R}^k\, \underline{N}_{i,R}^k). \end{aligned}$$

From the definition we have

(8) $\quad -1\leqq \underline{M}^k_{i,p}\leqq 0,\ -1\leqq \bar{M}^k_{i,p}\leqq 0,\ -1\leqq \underline{N}^k_{i,p}\leqq 0,\ -1\leqq \bar{N}^k_{i,p}\leqq 0,$

$p=0,\ 1,\ 2,\ \ldots,\ R,\ i=1,\ 2,\ \ldots,\ n.$ For example

$$\bar{M}^k_{i,p}=\sum_{s=1}^{i-1}(\underline{x}^k_s-x_s+\underline{\Delta}^{p,k}_s)(\bar{x}^k_i-\underline{x}^k_s-\underline{\Delta}^{p,k}_s)^{-1}\prod_{l=1}^{s-1}(\bar{x}^k_i-x_l)(\bar{x}^k_i-\underline{x}^k_l-\underline{\Delta}^{p,k}_l)^{-1}$$

and in view of (4) sign $((\underline{x}^k_s-x_s+\underline{\Delta}^{p,k}_s)(\bar{x}^k_i-\underline{x}^k_s-\underline{\Delta}^{p,k}_s)^{-1})=-1,\ 0<((\bar{x}^k_i-x_l)$ $(\bar{x}^k_i-\underline{x}^k_l-\underline{\Delta}^{p,k}_l)^{-1})<1,\ l<s<i.$ On the other side

$$|\bar{M}^k_{i,p}|\leqq\sum_{s=1}^{i-1}|(\underline{x}^k_s-x_s+\underline{\Delta}^{p,k}_s)(\bar{x}^k_i-\underline{x}^k_s-\underline{\Delta}^{p,k}_s)^{-1}|\leqq\sum_{s=1}^{i-1}(x_s-\underline{x}^k_s)(\bar{x}^k_i-\underline{x}^k_s)^{-1}$$

$=\sum_{s=1}^{i-1}(x_s-\underline{x}^k_s)(\bar{x}^k_i-x_i+x_i-x_s+x_s-\underline{x}^k_s)^{-1}\leqq\sum_{s=1}^{i-1}d^{-1}cq^{(R+2)^k}\leqq d^{-1}ncq<1.$ In similar way the other inequalities in (8) can be proved. Now

(9) $$\bar{M}^k_{i,R}=\sum_{s=1}^{i-1}(\bar{x}^k_i-\underline{x}^k_s-\underline{\Delta}^{R,k}_s)^{-1}\Big[\underline{x}^k_s-x_s-(\underline{x}^k_s-x_s)\prod_{t=1}^{s-1}(\underline{x}^k_s$$

$$-x_t)(\underline{x}^k_s-\underline{x}^k_t-\underline{\Delta}^{R-1,k}_t)^{-1}\prod_{t=s+1}^{n}(\underline{x}^k_s-x_t)(\underline{x}^k_s-\bar{x}^k_t-\bar{\Delta}^{R-1,k}_t)^{-1}\Big]\prod_{p=1}^{s-1}(\bar{x}^k_i$$

$$-x_p)(\bar{x}^k_i-\underline{x}^k_p-\underline{\Delta}^{R,k}_p)^{-1}=\sum_{s=1}^{i-1}((\underline{x}^k_s-x_s)(\bar{x}^k_i-\underline{x}^k_s-\underline{\Delta}^{R,k}_s)^{-1})(1$$

$$-\underline{A}^k_{s,R-1}\underline{B}^k_{s,R-1})\prod_{p=1}^{s-1}(\bar{x}^k_i-x_p)(\bar{x}^k_i-\underline{x}^k_p-\underline{\Delta}^{R,k}_p)^{-1},$$

$$\bar{N}^k_{i,R}=\sum_{s=i+1}^{n}((\bar{x}^k_s-x_s)(\bar{x}^k_i-\bar{x}^k_s-\bar{\Delta}^{R,k}_s)^{-1})(1$$

$$-\bar{A}^k_{s,R-1}\bar{B}^k_{s,R-1})\prod_{p=s+1}^{n}(\bar{x}^k_i-x_p)(\bar{x}^k_i-\bar{x}^k_p-\bar{\Delta}^{R,k}_p)^{-1},$$

$$\underline{M}^k_{i,R}=\sum_{s=1}^{i-1}((\underline{x}^k_s-x_s)(\underline{x}^k_i-\underline{x}^k_s-\underline{\Delta}^{R,k}_s)^{-1})(1$$

$$-\underline{A}^k_{s,R-1}\underline{B}^k_{s,R-1})\prod_{p=1}^{s-1}(\underline{x}^k_i-x_p)(\underline{x}^k_i-\underline{x}^k_p-\underline{\Delta}^{R,k}_p)^{-1},$$

$$\underline{N}^k_{i,R}=\sum_{s=i+1}^{n}((\bar{x}^k_s-x_s)(\underline{x}^k_i-\bar{x}^k_s-\bar{\Delta}^{R,k}_s)^{-1})(1-\bar{A}^k_{s,R-1}\bar{B}^k_{s,R-1})$$

$$\prod_{p=s+1}^{n}(\underline{x}^k_i-x_p)(\underline{x}^k_i-\bar{x}^k_p-\bar{\Delta}^{R,k}_p)^{-1},$$

and if we set

(10) $$\bar{S}^k_{i,p}=\bar{A}^k_{i,p}\bar{B}^k_{i,p},\quad \underline{S}^k_{i,p}=\underline{A}^k_{i,p}\underline{B}^k_{i,p},$$

for $i=1, 2, \ldots, n,\ p=0, 1, \ldots, R,\ k=0, 1, 2, \ldots$, then from (5) and (10) it follows

(11) $$\bar{x}^{k+1}_i-x_i=(\bar{x}^k_i-x_i)\,(1-\bar{S}^k_{i,R}),\quad x_i-\underline{x}^{k+1}_i=(x_i-\underline{x}^k_i)\,(1-\underline{S}^k_{i,R}).$$

It is easy to find from (9) that

$$\max_{1\leq i\leq n}|\bar{M}^k_{i,t}|\leqq(d-c)^{-1}ncq^{(R+2)^k}\max_{1\leq p\leq n}|1-\underline{S}^k_{p,t-1}|,$$

$$\max_{1\leq i\leq n}|\bar{N}^k_{i,t}|\leqq(d-c)^{-1}ncq^{(R+2)^k}\max_{1\leq p\leq n}|1-\bar{S}^k_{p,t-1}|,$$

$$\max_{1\leq i\leq n}|\underline{M}^k_{i,t}|\leqq(d-c)^{-1}ncq^{(R+2)^k}\max_{1\leq p\leq n_k}|1-\underline{S}^k_{p,t-1}|,$$

$$\max_{1\leq i\leq n}|\underline{N}^k_{i,t}|\leqq(d-c)^{-1}ncq^{(R+2)^k}\max_{1\leq p\leq n}|1-\bar{S}^k_{p,t-1}|$$

and from the last inequalities, (6) and (8) we obtain

(12) $$\begin{aligned}|1-\bar{S}^k_{i,p}|&=|1-\bar{A}^k_{i,p}\,\bar{B}^k_{i,p}|\leqq|\bar{M}^k_{i,p}|+|\bar{N}^k_{i,p}|\\ &\leqq(d-c)^{-1}\,ncq^{(R+2)^k}\Big(\sum_{s=1}^{i-1}|1-\underline{S}^k_{s,p-1}|+\sum_{s=i+1}^{n}|1-\bar{S}^k_{s,p-1}|\Big),\end{aligned}$$

$$|1-\underline{S}^k_{i,p}|=|1-\underline{A}^k_{i,p}\underline{B}^k_{i,p}|\leqq(d-c)^{-1}\,ncq^{(R+2)^k}\Big(\sum_{s=1}^{i-1}|1-\underline{S}^k_{s,p-1}|+\sum_{s=i+1}^{n}|1-\bar{S}^k_{s,p-1}|\Big)$$

Setting $\|1-\bar{S}^k_p\|=\max_{1\leq i\leq n}|1-\bar{S}^k_{i,p}|$, $\|1-\underline{S}^k_p\|=\max_{1\leq i\leq n}|1-\underline{S}^k_{i,p}|$ for $p=0, 1, \ldots, R,\ k=0, 1, \ldots$, it follows from (12)

$$\|1-\bar{S}^k_p\|\leqq(d-c)^{-1}\,n^2cq^{(R+2)^k}\,(\|1-\underline{S}^k_{p-1}\|+\|1-\bar{S}^k_{p-1}\|),$$

$$\|1-\underline{S}^k_p\|\leqq(d-c)^{-1}\,n^2cq^{(R+2)^k}\,(\|1-\underline{S}^k_{p-1}\|+\|1-\bar{S}^p_{p-1}\|)$$

and under the assumption that $n^2c/(d-c)\leqq1/2$ the above inequalities give

(13) $$\|1-\bar{S}^k_p\|\leqq\frac{1}{2}q^{(R+2)^k}\,(\|1-\underline{S}^k_{p-1}\|+\|1-\bar{S}^k_{p-1}\|),$$

$$\|1-\underline{S}^k_p\|\leqq\frac{1}{2}q^{(R+2)^k}\,(\|1-\underline{S}^k_{p-1}\|+\|1-\bar{S}^k_{p-1}\|).$$

As $\|1-\bar{S}^k_0\|\leqq q^{(R+2)^k}$, $\|1-\underline{S}^k_0\|\leqq q^{(R+2)^k}$, applying recursively (13), we have

$$\|1-\bar{S}^k_R\|\leqq2^{-1}\,q^{2(R+2)^k}\,(\|1-\underline{S}^k_{R-2}\|+\|1-\bar{S}^k_{R-2}\|)$$

$$\leqq \ldots \leqq 2^{-1} q^{R(R+2)^k} (\| 1-\bar{S}_0^k \| + \| 1-\underline{S}_0^k \|) \leqq q^{(R+1)(R+2)^k}$$

and analogously $\| 1-\underline{S}_R^k \| \leqq q^{(R+1)(R+2)^k}$.

The last two inequalities and (11) give

$$0 \leqq \bar{x}_i^{k+1} - x_i \leqq c q^{(R+2)^k} q^{(R+1)(R+2)^k} = c q^{(R+2)^{k+1}},$$

$$0 \leqq x_i - \underline{x}_i^{k+1} \leqq c q^{(R+2)^k} . q^{(R+1)(R+2)^k} = c q^{(R+2)^{k+1}}.$$

The theorem is proved.

The Ehrlich's method, [2], and its generalisation can be used for obtaining another two-sided method for simultaneous determination of the roots of the equation (1):

$$\bar{x}_i^{k+1} = \bar{x}_i^k + \bar{\Delta}_i^{R+1,k}, \quad \underline{x}_i^{k+1} = \underline{x}_i^k + \underline{\Delta}_i^{R+1,k},$$

$$(14) \qquad \bar{\Delta}_i^{p,k} = -f(\bar{x}_i^k)/(f'(\bar{x}_i^k) - f(\bar{x}_i^k) \sum_{\substack{l=1 \\ l \neq i}}^{n} (\bar{x}_i^k - \underline{x}_l^k - \underline{\Delta}_l^{p-1,k})^{-1}),$$

$$\underline{\Delta}_i^{p,k} = -f(\underline{x}_i^k)/(f'(\underline{x}_i^k) - f(\underline{x}_i^k) \sum_{\substack{l=1 \\ l \neq i}}^{n} (\underline{x}_i^k - \bar{x}_l^k - \bar{\Delta}_l^{p-1,k})^{-1}),$$

$\bar{\Delta}_i^{0,k} = \underline{\Delta}_i^{0,k} = 0, \quad i=1, 2, \ldots, n, \quad p=1, \ldots, R+1, \quad k=0, 1, 2, \ldots$

Theorem 3. *Under the conditions in Theorem* 1 *the method* (14) *gives*

$$\underline{x}_1^k \leqq x_1 \leqq \bar{x}_1^k < \underline{x}_2^k \leqq x_2 \leqq \bar{x}_2^k < \ldots < \underline{x}_n^k \leqq x_n \leqq \bar{x}_n^k, \quad k=0, 1, 2, \ldots$$

P r o o f. When $k=0$ the assertion holds true. Let it be true for some k. We shall prove it is true for $k+1$. It follows from (14)

$$(15) \qquad \begin{aligned} \bar{x}_i^{k+1} - x_i &= (\bar{x}_i^k - x_i)(1 - \underline{A}_i^{R,k}), \\ \underline{x}_i^{k+1} - x_i &= (\underline{x}_i^k - x_i)(1 - \bar{A}_i^{R,k}), \quad i=1, 2, \ldots, n, \end{aligned}$$

and

$$(16) \qquad \bar{\Delta}_l^{p+1,k} = (x_l - \bar{x}_l^k)\underline{A}_l^{p,k}, \quad \underline{\Delta}_l^{p+1,k} = (x_l - \underline{x}_l^k)\bar{A}_l^{p,k}, \quad p=0, 1, \ldots, R,$$

where

$$\underline{A}_i^{p,k} = 1/(1 + (\bar{x}_i^k - x_i) \sum_{\substack{s=1 \\ s \neq i}}^{n} (x_s - \underline{x}_s^k - \underline{\Delta}_s^{p,k})/(\bar{x}_i^k - x_s)(\bar{x}_i^k - \underline{x}_s^k - \underline{\Delta}_s^{p,k}),$$

$$(17) \qquad \bar{A}_i^{p,k} = 1/(1 + (\underline{x}_i^k - x_i) \sum_{\substack{s=1 \\ s \neq i}}^{n} (x_s - \bar{x}_s^k - \bar{\Delta}_s^{p,k})/(\underline{x}_i^k - x_s)(\underline{x}_i^k - \bar{x}_s^k - \bar{\Delta}_s^{p,k})),$$

$$p=1, \ldots, R.$$

Since $\bar{\Delta}_i^{0,k}=\underline{\Delta}_i^{0,k}=0$, $i=1, 2, \ldots, n$, (16) and (17) give

$$0 \leqq \underline{A}_i^{p,k} \leqq 1, \quad 0 \leqq \bar{A}_i^{p,k} \leqq 1, \tag{18}$$

$$0 \geqq \bar{\Delta}_i^{p,k} \geqq x_i - \bar{x}_i^k, \quad 0 \leqq \underline{\Delta}_i^{p,k} \leqq x_i - \underline{x}_i^k, \tag{19}$$

for $i=1, 2, \ldots, n$, $p=0, 1, \ldots, R$.

For $k+1$ the assertion of the theorem follows from (15) and (18).

Theorem 4. *Let* $0<q<1$, $d=\min\limits_{i\neq j} |x_i-x_j|$ *and the constant c is chosen so that*

$$0<c<d/(3+\sqrt{n}). \tag{20}$$

If the initial approximations $\{\underline{x}_i^0\}_{i=1}^n$, $\{\bar{x}_i^0\}_{i=1}^n$ *of the roots* $\{x_i\}_{i=1}^n$ *of the equation* (1) *satisfy* $0 \leqq x_i-\underline{x}_i^0 \leqq cq$, $0 \leqq \bar{x}_i^0 - x_i \leqq cq$, $i=1, \ldots, n$, $\bar{x}_i^0 \leqq \underline{x}_{i+1}^0$, $i=1, 2, \ldots, n-1$, *and we apply the method* (14), *then for every* $k=0, 1, 2, \ldots$, *the following inequalities*

$$0 \leqq \bar{x}_i^k - x_i \leqq cq^{(2R+3)^k}, \quad 0 \leqq x_i - \underline{x}_i^k \leqq cq^{(2R+3)^k}, \quad i=1, 2, \ldots, n \tag{21}$$

hold true.

P r o o f. Again we shall prove the theorem by induction. Under the assumptions the theorem is valid for $k=0$. Let (21) be true for some . Then

$$\bar{x}_i^{k+1}-x_i=(\bar{x}_i^k-x_i)(1-\underline{A}_i^{R,k})=(\bar{x}_i^k-x_i)^2 \underline{A}_i^{R,k} \sum_{\substack{l_0=1 \\ l_0\neq i}}^{n} (x_{l_0}-\underline{x}_{l_0}^k-\underline{\Delta}_{l_0}^{R,k})/(\bar{x}_i^k - x_{l_0})(\bar{x}_i^k-\bar{x}_{l_0}^k-\underline{\Delta}_{l_0}^{R,k})$$

From the last equality and (16) one obtains

$$\bar{x}_i^{k+1}-x_i=(\bar{x}_i^k-x_i)^2 \underline{A}_i^{R,k} \sum_{\substack{l_0=1 \\ l_0\neq i}}^{n} (x_{l_0}-\bar{x}_{l_0}^k)(1-\bar{A}_{l_0}^{R-1,k}{}_{l_0}^{-1,k})/(\bar{x}_i^k-x_{l_0})(\bar{x}_i^k-\underline{x}_{l_0}^k-\underline{\Delta}_{l_0}^{R,k})$$

$$=(\bar{x}_i^k-x_i)^2 \underline{A}_i^{R,k} \sum_{\substack{l_0=1 \\ l_0\neq i}}^{n} (x_{l_0}-\bar{x}_{l_0}^k)^2 \bar{A}_{l_0}^{R-1,k})/(x_{l_0}-\bar{x}_i^k)(\bar{x}_i^k-\underline{x}_{l_0}^k-\underline{\Delta}_{l_0}^{R,k})$$

$$\sum_{\substack{l_1=1 \\ l_0\neq i}}^{n} (x_{l_1}-\bar{x}_{l_1}^k-\bar{\Delta}_{l_1}^{R-1,k})/(\underline{x}_{l_0}^k-x_{l_1})(\underline{x}_{l_0}^k-\bar{x}_{l_1}^k-\bar{\Delta}_{l_1}^{R-1,k}).$$

We can continue to lower the index R in the above expression and shall obtain: if R is even—

$$\bar{x}_i^{k+1}-x_i=(\bar{x}_i^k-x_i)^2 \Delta_i^{R,k} \sum_{\substack{l_0=1\\ l_0\neq i}}^{n} (x_{l_0}-\bar{x}_{l_0}^k)^2 \bar{A}_{l_0}^{R-1,k}/(x_{l_0}-\bar{x}_i^k)(\bar{x}_i^k-\underline{x}_{l_0}^k-\Delta_{l_0}^{R,k}) \ldots$$

$$\ldots \sum_{\substack{l_{R-1}=1\\ l_{R-1}\neq l_{R-2}}}^{n} (x_{l_{R-1}}-\bar{x}_{l_{R-1}}^k)^2 \Delta_{l_{R-1}}^{0,k}/(x_{l_{R-1}}-\underline{x}_{l_{R-2}}^k)(\underline{x}_{l_{R-2}}^k\bar{x}_{l_{R-1}}^k-\bar{\Delta}_{l_{R-1}}^{1,k})$$

$$\sum_{\substack{l_R=1\\ l_R\neq l_{R-1}}}^{n} (\bar{x}_{l_R}-\underline{x}_{l_R}^k)/(\bar{x}_{l_{R-1}}^k-x_{l_R})(\bar{x}_{l_{R-1}}^k-\underline{x}_{l_R}^k);$$

if R is odd –

$$\bar{x}_i^{k+1}-x_i=(\bar{x}_i^k-x_i)^2 \Delta_i^{R,k} \sum_{\substack{l_0=1\\ l_0\neq i}}^{n} (x_{l_0}-\bar{x}_{l_0}^k)^2 \bar{A}_{l_0}^{R-1,k})/(x_{l_0}-\bar{x}_i^k)(\bar{x}_i^k-\underline{x}_{l_0}^k-\underline{\Delta}^{R,k}) \ldots$$

$$\ldots \sum_{\substack{l_{R-1}=1\\ l_{R-1}\neq l_R}}^{n} (x_{l_{R-1}}-\underline{x}_{l_{R-1}}^k)^2 \bar{A}_{l_{R-1}}^{0,k}/(x_{l_{R-1}}-\bar{x}_{l_{R-2}}^k)(\bar{x}_{l_{R-2}}^k-\underline{x}_{l_{R-1}}^k$$

$$-\Delta_{l_{R-1}}^{1,k}) \sum_{\substack{l_R=1\\ l_R\neq l_{R-1}}}^{n} (x_{l_R}-\bar{x}_{l_R}^k)/(\underline{x}_{l_{R-1}}^k-x_{l_R})(\underline{x}_{l_{R-1}}^k-\bar{x}_{l_R}^k).$$

We can get similar expressions for $x_i-\underline{x}_i^{k+1}$ and from (18), (19), (20), above inequalities, min $\{|x_\mu-\bar{x}_\nu^k|,\ |x_\mu-\underline{x}_\nu^k|,\ |\bar{x}_\nu^k-\underline{x}_\mu^k-\underline{\Delta}_\mu^{p,k}|,\ |\underline{x}_\nu^k-\bar{x}_\mu^k-\bar{\Delta}_\mu^{p,k}|\}>d-3c$and the inductive assumption it follows $\bar{x}_i^{k+1}-x_i\leqq(cq^{2R+1})^{2+2R+1}\ ((d-3c)^{-2}\ \ n)^{R+1}\leq cq^{(2R+3)^{k+1}}$, and analogously $x_i-\underline{x}_i^{k+1}\leqq cq^{(2R+3)^{k+1}}$. The theorem is proved.

3. Computer experiments.

It is well known that computer realization of usual Weierstrass–Dochev's method has a perfect numerical properties. The numerical experiments show two-sided method (3) preserves these nice properties.

For the algebraic eguation $x^4-26x^3+131x^2-226x+120=0$ with roots $x_1=1$, $x_2=2$, $x_3=3$, $x_4=20$ we take the following initial approximations from below $\underline{x}_1^0=0.85$, $\underline{x}_2^0=1.95$, $\underline{x}_3^0=2.75$,$\underline{x}_4^0=19.05$ and from above $\bar{x}_1^0=1.25$, $\bar{x}_2^0=2.35$, $\bar{x}_3^0=3.15$, $\bar{x}_4^0=20.55$. Using different R, we apply the method (3) and obtain the following results (all evaluations are made with double precision accuracy on a computer EC 1040).

1.$R=0$

$\underline{x}_1^1=$ .954498730964467 $\quad$ $\bar{x}_1^1=$ 1.097476417433123

$\underline{x}_2^1=$ 1.986666819403708	$\bar{x}_2^1=$ 2.101796875000002
$\underline{x}_3^1=$ 2.959201517223533	$\bar{x}_3^1=$ 3.019872485632187
$\underline{x}_4^1=$ 19.975007233308580	$\bar{x}_4^1=$ 20.013300348490290
$\underline{x}_1^2=$ .995535173061370	$\bar{x}_1^2=$ 1.010846577210582
$\underline{x}_2^2=$ 1.999157864998096	$\bar{x}_2^2=$ 2.006224320591363
$\underline{x}_3^2=$ 2.998496675695452	$\bar{x}_3^2=$ 3.000703614301403
$\underline{x}_4^2=$ 19.999861994103510	$\bar{x}_4^2=$ 20.000073286359330
$\underline{x}_1^3=$ .999970929234721	$\bar{x}_1^3=$ 1.000071679038885
$\underline{x}_2^3=$ 1.999995661182392	$\bar{x}_2^3=$ 2.000031906198809
$\underline{x}_3^3=$ 2.999995378444715	$\bar{x}_3^3=$ 3.000002160039772
$\underline{x}_4^3=$ 19.999999948922760	$\bar{x}_4^3=$ 20.000000027123570
$\underline{x}_1^4=$ .999999999041082	$\bar{x}_1^4=$ 1.000000002364611
$\underline{x}_2^4=$ 1.999999999864488	$\bar{x}_2^4=$ 2.000000000996431
$\underline{x}_3^4=$ 2.999999999912770	$\bar{x}_3^4=$ 3.000000000040781
$\underline{x}_4^4=$ 19.999999999999880	$\bar{x}_4^4=$ 20.000000000000000
$\underline{x}_1^5=$ .999999999999999	$\bar{x}_1^5=$ 1.000000000000000
$\underline{x}_2^5=$ 1.999999999999988	$\bar{x}_2^5=$ 2.000000000000000
$\underline{x}_3^5=$ 2.999999999999992	$\bar{x}_3^5=$ 3.000000000000003
$\underline{x}_4^5=$ 19.999999999999930	$\bar{x}_4^5=$ 20.000000000000000

2. $R=2$

$\underline{x}_1^1=$ .996863689949065	$\bar{x}_1^1=$ 1.007748553858683
$\underline{x}_2^1=$ 1.999059323257024	$\bar{x}_2^1=$ 2.006132889928764
$\underline{x}_3^1=$ 2.996980358857402	$\bar{x}_3^1=$ 3.001380135752327
$\underline{x}_4^1=$ 19.998471841811670	$\bar{x}_4^1=$ 20.000813263325410
$\underline{x}_1^2=$ .9999999999492591	$\bar{x}_1^2=$ 1.000000001266907
$\underline{x}_2^2=$ 1.999999999893418	$\bar{x}_2^2=$ 2.000000000691742
$\underline{x}_3^2=$ 2.999999999788932	$\bar{x}_3^2=$ 3.000000000096181
$\underline{x}_4^2=$ 19.999999999986450	$\bar{x}_4^2=$ 20.000000000007150
$\underline{x}_1^3=$ .999999999999999	$\bar{x}_1^3=$ .999999999999999
$\underline{x}_2^3=$ 1.999999999999995	$\bar{x}_2^3=$ 1.999999999999988
$\underline{x}_3^3=$ 2.999999999999998	$\bar{x}_3^3=$ 3.000000000000003
$\underline{x}_4^3=$ 19.999999999999930	$\bar{x}_4^3=$ 20.000000000000000

3. $R=4$

$\underline{x}_1^1=$ .999769968713488	$\bar{x}_1^1=$ 1.000573714993205
$\underline{x}_2^1=$ 1.999935655151968	$\bar{x}_2^1=$ 2.000423999985037
$\underline{x}_3^1=$ 2.999791433154968	$\bar{x}_3^1=$ 3.000094748332660
$\underline{x}_4^1=$ 19.999894150372310	$\bar{x}_4^1=$ 20.000056317671060
$\underline{x}_1^2=$ .999999999999999	$\bar{x}_1^2=$ 1.000000000000000
$\underline{x}_2^2=$ 1.999999999999993	$\bar{x}_2^2=$ 1.999999999999988
$\underline{x}_3^2=$ 3.000000000000003	$\bar{x}_3^2=$ 3.000000000000007
$\underline{x}_4^2=$ 19.999999999999930	$\bar{x}_4^2=$ 20.000000000000000

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