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Regularity, Nearly Regularity and Extensions of Borel Measures Valued in Partially Ordered Semigroups

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1. Introduction and some general preliminary results

A commutative semigroup G with identity O, is said to be a partially ordered semigroup, if it is endowed with a partial ordering $\leqq$ satisfying the conditions:

(i) If x, y, z are elements of G with $x<y (x\leqq y$ and $x\neq y)$ then $x+z<y+z$;

(ii) $x+\sup E=\sup(x+E)$, whenever there exist sup E (the supremum of E *in* G) *and* $\sup(x+E)$, $E\subseteq G$, $x\in G$.

G is monotone complete if every majorised increasing directed family in G has a supremum in G.

Let G be a partialy ordered semigroup and $\mathcal{R}$ a ring of subsets of X. The function $m:\mathcal{R}\to G$ is said to be a measure on $\mathcal{R}$, if m is positive on $\mathcal{R}$ ($m(A)\geqq 0$, for every A in $\mathcal{R}$) and $m(\cup_{n\in N} A_n)=\sup\{\Sigma_{n=1}^{n} m(A_i)\ n\in N\}$ whenever $(A_n)_{n\in N}$ is a disjoint sequence of elements of $\mathcal{R}$ with $(\cup_{n\in N} A_n)\in\mathcal{R}$. m is order bounded (o-bounded) on $\mathcal{R}$, if there exists an element $g\in G$ such that: $0\leqq m(A)\leqq g$, whenever $A\in\mathcal{R}$.

We shall denote by $Q(\mathcal{R})$ the algebra of all subsets $A\subset X$ which have the property that $(A\cap B)\in\mathcal{R}$ for every $\mathcal{B}\in R$ ([1] p. 6). By $S(\mathcal{R})$ we shall denote the σ-ring generated by $\mathcal{R}$. We also set $\mathcal{R}_\sigma:=\{F\subseteq X$, there exists a sequence $(A_n)_{n\in N}$ in $\mathcal{R}$ with $F=\cup_{n\in N} A_n\}$.

Let $\mathcal{T}$ be another ring of subsets of X. $\mathcal{R}$ is an ideal in $\mathcal{T}$ if $\mathcal{R}\subseteq\mathcal{T}$ and the relations $A\in\mathcal{T}$, $B\in\mathcal{R}$ *imply* $(A\cap B)\in\mathcal{R}$.

Theorem 1.1. *Let $\mathcal{R}$, $\mathcal{T}$ be rings of subsets of X and $m:\mathcal{R}\to G$ an o-bounded measure on $\mathcal{R}$. Suppose that $\mathcal{R}$ is an ideal in $\mathcal{T}$ and $\mathcal{G}$ is a monotone complete partially ordered semigroup. Then there exists a measure $l:\mathcal{T}\to\mathcal{G}$ which extends m on $\mathcal{T}$.*

P r o o f. Let $E\in\mathcal{T}$. Thus the directed (by inclusion) family $\{m(A\cap E): A\in\mathcal{R}\}$ is increasing and o-bounded.

Hence it is defined a set function $l:\mathscr{T}\to G$ by the equality: $l(E):=\sup\{m(A\cap E):A\in\mathscr{R}\}$, whenever $E\in\mathscr{T}$. Lemma 2.2 in [8] guarantees that l is a measure on $\mathscr{T}$.

Corollary 1.2. *Let $m:\mathscr{R}\to G$ be an o-bounded measure on $\mathscr{R}$. If G is monotone complete partially ordered semigroup, then there exists an o-bounded measure $l:Q(\mathscr{R})\to G$ which extends m on the algebra $Q(\mathscr{R})$.*

Theorem 1.3. *Let $m:\mathscr{R}\to G$ be an o-bounded measure on $\mathscr{R}$. If G is monotone complete partially ordered semigroup there exists a measure $l:\mathscr{R}_\sigma\to G$ which extends m on $\mathscr{R}_\sigma$. If $\mathscr{R}$ is a semitribe* ([4] p. 3) *this extension is unique.*

Proof. For every $F\in\mathscr{R}_\sigma$ the directed family:

$\{m(A):A\subseteq F,\ A\in\mathscr{R}\}$ is increasing and o-bounded.

Therefore a set function $l:\mathscr{R}_\sigma\to G$ is defined by the formula: $l(F):=\sup\{m(A):A\subseteq F,\ A\in\mathscr{R}\}$, whenever $F\in\mathscr{R}_\sigma$. Clearly $l(F)\geqq 0$ for every $F\in\mathscr{R}_\sigma$. Now let an increasing sequence $(F)_{n\in N}$ in $\mathscr{B}_\sigma$ with $F_n\uparrow F$ be given. Evidently $F\in\mathscr{R}_\sigma$.

For every $n\in N$ choose a sequence $(A_{m,n})_{m\in N}$ in $\mathscr{R}$ with $A_{m,n}\subseteq A_{m+1,n}$ and $\cup_{m\in N}A_{m,n}=F_n$, $m\in N$. Put $D_m:=\cup_{n\leq m}A_{m,n}$, $m\in N$. Obviously, $D_m\in\mathscr{R}$, $A\cap D_m\uparrow A$ and $D_m\in F$, for every $A\in\mathscr{R}$ with $A\subseteq F$. So $m(A)=\sup\{m(A\cap D_n):n\in N\}$ for every $A\in\mathscr{R}$, $A\subseteq F$ ([8] Proposition 1.2.).

Consequently, $m(A\cap D_n)\leqq m(D_n)\leqq l(F)$, for every $A\in\mathscr{R}$, $A\subseteq F$ and $n\in N$. Thus $m(A)\leqq\sup\{m(D_n):n\in N\}\leqq l(F)$ whenever $A\in\mathscr{R}$, $A\subseteq F$. Hence $l(F)=\sup\{m(D_n):n\in N\}$. On the other hand, by $D_n\subseteq F_n$, it follows that, $m(D_n)\leqq\sup\{l(F_n):n\in N\}\leqq l(F)$, for every $n\in N$. Therefore

$$l(F)=\sup\{m(D_n):n\in N\}=\sup\{l(F_n):n\in N\}. \tag{1}$$

Furthermore, let $E,F\in\mathscr{R}_\sigma$. Pick increasing sequences $(A_n)_{n\in N}$, $(B_n)_{n\in N}$ in $\mathscr{R}_\sigma$ with $A_n\uparrow E$, $B_n\uparrow F$. Then $m(A_n\cup B_n)+m(A_n\cap B_n)=m(A_n)+m(B_n)$, whenever $n\in N$. Hence $l(E\cup F)+l(E\cap F)=l(E)+l(F)$.

In particular, if $E\cap F=\emptyset$ implies $l(E\cap F)=0$. Therefore the last equality gives

$$l(E\cup F)=l(E)+l(F). \tag{2}$$

Now it follows by (1), (2) and Proposition 1.2 in [8] that l is a measure on $\mathscr{R}_\sigma$ (which evidently extends m).

If $\mathscr{R}$ is a semi-tribe, then $\mathscr{R}_\sigma=S(\mathscr{R})$ ([4] p.6 Corollary 1). Now the uniqueness follows by an easy application of the "Lemma of monotone classes" ([1] p.7).

In the sequel of this paper X will denote a locally compact (Hausdorff) space and G a monotone complete partially ordered semigroup. A subset of X is called nearly closed (n-closed) if it contains the closure (in X) of each of its countable subsets. A subset A of X is said to be bounded if there exists a compact subset C of X such that: $A\subseteq C$ (cf. [1] p. 181).

Following S. K. Berberian [1], let $\mathscr{B}$ (resp. $\mathscr{B}_w$) be the σ-ring of Borel (resp. the σ-algebra of weakly Borel) subsets of X. A (G-valued) Borel (resp. weakly Borel) measure is defined to be a measure m on $\mathscr{B}$ (resp. $\mathscr{B}_w$). Now a Borel (resp. weakly Borel) measure m will be called regular if $m(A)=\{\sup m(C):C$ is a compact subset of $A\}$ for every Borel (resp. weakly Borel) subset A of X.

Also m is said to be nearly regular (n-regular) if for every Borel (resp. weakly Borel) subset A of X, $m(A)=\sup\{m(C):C\subseteq A$ and C is a bounded n-closed Borel (resp. weakly Borel) subset of $X\}$. In keeping with [6] p. 98 every regular real valued Borel measure is n-regular but the converse can be failed.

2. Borel and weakly Borel measures

The following results extend previous ones of S. K. Berberian [1-3], R. A. Johnson [6], Jea-Young Lee [7] and can be applied directly to C^*-algebra-valued positive measures.

Proposition 2.1. *Let m be a regular Borel (resp. weakly Borel) measure. Then m is n-Borel (resp. m-weakly Borel) measure.*

P r o o f. Let $A\in\mathscr{B}$ (resp. $A\in\mathscr{B}_w$). Since the family of bounded n-closed Borel (resp. weakly Borel) subsets F of A is increasing directed and $m(F)\leqq m(A)$, for all F, there exists $\sup\{m(F): F\subseteq A$, F is bounded n-closed Borel (resp. weakly Borel)$\}$. Hence $m(A)=\sup\{m(C),\ C\subseteq A,\ C \text{ compact}\}\leqq\sup\{m(F): F\subseteq A,\ F$ bounded n-closed Borel (resp. weakly Borel$\}\leqq m(A)$.

Since the class of Borel sets coincides with the class of all σ-bounded weakly Borel sets, we easily verify (using similar arguments) the following estimate.

Proposition 2.2. *Let m be an n-regular weakly Borel measure. Then* $m(A)=\sup\{m(D): D\subseteq A,\ D\in\mathscr{B}\}$ *for every* $A\in\mathscr{B}_w$. Now we state the following.

Theorem 2.3. *Let m be an o-bounded regular Borel measure. Then there exists a unique o-bounded regular weakly Borel measure* m_w *which extends m.*

P r o o f. Consider the increasing majorised directed family of measures: $m_M: Q(\mathscr{B})\to X$, $M\in\mathscr{B}$ with $m_M(A):=m(A\cap M)$, for every $A\in Q(\mathscr{B})$. Since $B_w\subseteq Q(\mathscr{B})$ we define the set function $m_w: \mathscr{B}_w\to G$ by the formula:

$$m_w(A):=\sup\{m(A\cap M): \mathrm{M}\in\mathscr{B}\},\quad A\in\mathscr{B}_w.$$

By Corollary 1.2 it follows easily that m_w is a weakly Borel measure. Next we prove the regularity of m_w. Let $A\in\mathscr{B}_w$. Then

$$\begin{aligned} m_w(A)&=\sup\{m(A\cap M): M\in\mathscr{B}\}=\sup\{m(M): \subseteq \mathrm{A},\ \mathrm{M}\in\mathscr{B}\}\\ &=\sup\{\sup\{m(C): C\subseteq M,\ C \text{ compact}\}: M\subseteq A,\ M\in\mathscr{B}\}\\ &=\sup\{m(C): C\subseteq A,\ C \text{ compact}\}. \end{aligned}$$

([9] p. 12 Theorem I. 6. 1). The uniqueness follows easily by regularity condition. Similarly we can prove the following

Theorem 2.4. *Let m be an o-bounded n-regular Borel measure. Then there exists an unique o-bounded n-regular weakly Borel measure* m_w *which extends m.*

Analogous results may be developed for Baire and weakly Baire measures if we define regularity (resp. nearly regularity) in the obvious way. In particular we state the corresponding result of 2.3.

Theorem 2.5. *Let m be an o-bounded regular Baire measure. Then there exists an unique o-bounded regular weakly Baire measure* m_w *which extends m.*

References

1. S. K. Berberian. Measure and integration. New York, 1965.
2. S. K. Berberian. On the extension of Borel measures. *Proc. Amer. Math. Soc.*, **16**, 1965, 415-418.
3. S. K. Berberian. Notes on spectral theory. Amsterdam, 1966.
4. N. Dinculeanu. Vector measures. (Internat. Ser. of Monographs in Pure and Appl. Math. 95). New York, 1967.
5. N. Dunford, J. T. Schwartz. Linear operators I. New York, 1967.
6. R. A. Johnson. Nearly Borel sets an product measures. *Pacific J. Math.*, **87**, 1980, 97-109.
7. Jea-Young Lee. Nearly and anti-nearly regular weakly Borel measures. (Thesis). Young nam University.
8. P. K. Pavlakos. The Lebesgue decomposition theorem for partially ordered semigroup-valued measures. *Proc. Amer. Math. Soc.*, **71**,1978, 207-211.
9. B. Z. Vulikh. Introduction to the theory of partially ordered spaces. Moscow, 1961. (English transl. Groningen, 1967).
10. J. D. M. Wright. Measures with values in partially ordered spaces: regularity and σ-additivity. – In: Measure Theory. (*Lecture notes in Mathematics*, **54**). Berlin, 1976, 267-276.

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