Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica
Balkanica
New Series Vol. 1, 1987, Fasc. 1

A Characterization of Smooth Spaces

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In this paper we correlate the notion of the semi-inner-product with that of the generalized-inner-product on a real linear normed space to characterize the smooth spaces.

The notion of the semi-inner product is due to G. Lumer [3]. Purpose of this notion is the transfer of type arguments over Hilbert space to that of Banach spaces. He defined a semi-inner-product (s.i.p.) on a complex (real) vector space X as a complex (real) function $[x, y]$ on $X\times X$ with the following properties:

(i) $[x+y, z]=[x, z]+[y, z]$ and
$[\lambda x, y]=\lambda[x, y]$ for all $x, y, z\in X$ and every complex (real) λ,
(ii) $[x, x]>0$ for every $x\in X\setminus\{0\}$ and
(iii) $|[x, y]|^2\leqq[x, x]\cdot[y, y]$ for all $x, y\in X$.

A vector space with a s.i.p. is called a semi-inner product space (s.i.p.s.). Of course a s.i.p.s. is a normed vector space with norm $\|x\|=[x, x]^{1/2}$ ([3]).

On the other side, let X be a normed linear space and X' be the dual of X. Now for each $x\in X$, by the Hahn-Banach theorem, there exists at least one (and we choose exactly one) element $W_x\in X'$ with the properties $W_x(x)=\|x\|^2$ and $\|W_x\|=\|x\|$.

Given any such mapping W from X into X' (and there exist in general infinitely many such mappings for a given X), it is verified that $[x, y]:=W_y(x)$ defines a s.i.p. on X.

Moreover, in [2] it is proved that every normed vector space can be represented as a s. i. p. space with the homogeneity property, namely:

(iv) $[x, \lambda y]=\bar{\lambda}[x, y]$ for each complex (real) λ, where $\bar{\lambda}$ denotes the conjugate of λ.

For our aim the following Lemma ([4], [5]) is useful.

Lemma: *If X is any s. i. p. s. and $x, y\in X$, then it is valid:*

$$(1)\qquad \|y\|\frac{d^-}{dt}\bigg|_{t=0^-}\{\|y+tx\|\}\leqq \operatorname{Re}[x, y]\leqq\|y\|\frac{d^+}{dt}\bigg|_{t=0^+}\{\|y+tx\|\},$$

where d^-/dt, d^+/dt denote left-hand and right-hand derivatives with respect to the real variable t. If on the others the norm is differentiable, then it is valid:

$$(2) \qquad [x,\ y] = \| y \| \frac{d}{dt}\bigg|_{t=0} \{\| y+tx \| + i \| y-itx \|\}.$$

On the other hand, one has that every real linear normed space is a generalized-inner-product space (g. i. p. s.) in the following sense due to R. A. Tapia [6]:

Let X be a real linear normed space, $f\colon X \to \mathrm{R}$ with $f(x) = \frac{1}{2} \| x \|^2$ and $\langle \ldots \rangle$: $X \times X \to \mathrm{R}$ with $\langle x,\ y \rangle = f'_+(x)\ (y) = \lim_{t\to 0^+} ((f(x+ty)-f(x))/t)$ (the first right-hand Gâteaux derivative of f at x in the direction y). Then one obtains that:

(a) $\langle x,\ y \rangle$ is well defined,
(b) $\| x \| = \langle x,\ x \rangle^{1/2}$,
(c) $| \langle x,\ y \rangle | \leqq \| x \| \cdot \| y \|$ and
(d) if X is an inner product space with inner product (x, y), then $\langle x, y \rangle = (x, y)$.
From the above one has that:

$$(3) \qquad \langle y,\ x \rangle = \| y \| \lim_{t\to 0+} \frac{\| y+tx \| - \| y \|}{t}, \text{ where } x,\ y \in X.$$

Now if X is a real linear normed space and $[x, y]$ is a s. i. p. and $\langle x, y \rangle$ is the g. i. p. on it, then one obtains from (1) and (3) that: $[x,\ y] \leqq \langle y,\ x \rangle$ for all $x,\ y \in X$.

The relation between semi-inner-product and generalized-inner-product on a real linear normed space X is now

$$(4) \qquad [x,\ y] \leqq \langle y,\ x \rangle \text{ for all } x,\ y \in X$$

and for all semi-inner-products $[.\ ,\ .]$ on X.

It is easy to show that a normed linear space can be made into a s. i. p. space in a unique way if and only if its unit sphere is smooth. Namely if at each point of the unit surface there is only one supporting hyperplane of the unit ball or, equivalenty: the norm functional of the space has a Gâteaux differential at each point of the unit surface ([1, p. 144]).

Theorem 1. *The real linear normed space X is smooth if and only if for a certain semi-inner-product $[x,\ y]$ on it one obtains $[x,\ y] = \langle y,\ x \rangle$ for all $x, y \in X$.*

Proof. Let the space X be smooth. Then there is exactly one s. i. p., say $[x,\ y]$, on it and one has from (2) and (3) that

$$[x,\ y] = \| y \| \lim_{t\to 0} \frac{\| y+tx \| - \| y \|}{t} = \| y \| \lim_{t\to 0+} \frac{\| y+tx \| - \| y \|}{t} = \langle y,x \rangle.$$

Conversely. We have that $[x, y] = \langle y, x \rangle$ for a certan s.i.p. $[.\ ,.]$ on X and for the g.i.p. $\langle .\ ,. \rangle$ on X. But the s. i. p. $[.\ ,.]$ is linear in the first variable x and this has as consequence the linearity in x of the g. i. p. $\langle y,\ x \rangle$. But the latter is equivalent with the smoothness of the space X ([6]).

Corollary 2: *Let X be a real linear normed space, which is not smooth. Then we have $[.\ ,\ .] \neq \langle .\ ,\ . \rangle$ for every semi-inner-product $[.\ ,\ .]$ on X, where $\langle .\ ,\ . \rangle$ is the generalized-inner-product on X.*

P r o o f. The relation (4), which connects s.i. products with g.i. product, and Theorem 1 gives the assertion.

Theorem 3: *Let X be a real linear normed space, which is smooth. Then the following are equivalent:*
(α): *The space X is an inner product space.*
(β): *The generalized-inner-product $\langle\cdot,\cdot\rangle$ on X is additive on the first variable.*

P r o o f. Let (α) be true, i.e. X is an inner product space with inner product (x, y). Then one obtains $\langle x, y\rangle=(x, y)$, i.e. the g.i.p. is linear in x; namely the proposition (β) is true.

If (β) is valid then, since the generalized-inner-product $\langle.,.\rangle$ on X is additive on the first variable and the space X is smooth, we have that $[x, y]=\langle y, x\rangle$, and with (iv) above one obtains at once the assertion (α).

It is not difficult to verify that a real linear normed space X is smooth if and only if the following situation is true:

$$-\langle x, -y\rangle=\langle x, y\rangle \quad \text{for all} \quad x, y\in X. \tag{5}$$

Theorem 4: *The real linear normed space X is smooth if and only if* $\lim_{t\to 0^-}\langle x+ty, y\rangle=\langle x, y\rangle$ *for all* $x\in X\setminus\{0\}$, $y\in X$.

P r o o f. Let the space X be smooth. Then there is exactly one s.i. p., say $[x, y]$, on it and from Theorem 1 one has that: $[y, x]=\langle x, y\rangle$ for all $x, y\in X$. But $\lim_{t\to 0^-}[y, x+ty]=[y, x]$ ([2], Th. 3) i.e. $\lim_{t\to 0^-}[y, x+ty]=\lim_{t\to 0^-}\langle x+ty, y\rangle=\langle x, y\rangle$.

C o n v e r s e l y : If $\lim_{t\to 0^-}\langle x+ty, y\rangle=\langle x, y\rangle$, then one has that $-\langle x, -y\rangle=\langle x, y\rangle$ and from the relation (5) the space X is smooth.

An immediate consequence of the above theorem is the following corollary.

Corollary 5: *The norm of the space X is uniformly Fréchet differentiable if and only if* $\lim_{t\to 0^-}\langle x+ty, y\rangle=\langle x, y\rangle$, *where the limit is approached uniformly for all x, y from the unit surface of the space X.*

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Received 30.12.1986