Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica
Balkanica

New Series Vol. 1, 1987, Fasc. 1

Duality Methods for Constrained Best Interpolation

Asen L. Dontchev

Presented by P. Kenderov

We study problems for convex and monotone interpolation with minimal L_p norm of k-th derivative. Our approach is based on Lagrange duality theory from optimal control.

1. Introduction

Given points (t_i, y_i), $i=1, \dots, n$ in R^2, $0 \leqq t_1 \dots t_n \leqq 1$, define the class of functions

$$F=\{ f, f^{(k)} \in L_p(0, 1), f(t_i)=y_i, i=1, \dots, n\}, \tag{1}$$

where $f^{(k)}$ denotes k-th derivative of f, $n>k$. The problem of best interpolation is

$$\text{Minimize } \|f^{(k)}\|_p \qquad \text{subject to } f \in F. \tag{2}$$

J. Favard [3] considered this problem for $p=\infty$. C. De Boor [1] extended Favard's work to $1<p\leqq\infty$. For the case $k=p=2$ U. Hornung added new constraints for the derivatives; in [5] he treated a monotone best interpolation problem in which $f' \geqq 0$ and in [6] he solved a convex best interpolation problem with $f'' \geqq 0$. Hornung's results concerning the convex interpolation were developed further by G. Iliev and W. Pillul [7], C. A. Micchelli et al. [9] and A. Dontchev et al. [2] ·

For $p=k=2$ the best interpolation problem can be rewritten as

$$\text{Minimize } \int u^2 \qquad \text{subject to: } x_1'=x_2, \quad x_2'=u, \quad x_1(t_i)=y_i, \quad i=1, \dots, n.$$

This problem has a well-known interpretation in optimal control: it describes the motion of a car with unit mass equipped with two rocket engines one on each end. The car runs on the level and its position is x_1 and the velocity is x_2. The control u represents the force on the car due to firing either engine. The car should arrive to the station y_i at the moment t_i. The problem is to minimize the energy (fuel consumption). Then "monotone interpolation" $(x_2 \geqq 0)$ means that the car moves in one direction only. In the "convex" case $(u \geqq 0)$ there is one engine only.

If the costs are connected with the changes of the control (the engines regime) then the next derivative is to be taken into account. Then $u'=w$ and the norm $\|w\|_2$ should be minimal. In terms of the problem (2) we have $k=3$ and $p=2$. This particular problem with monotonicity and convexity constraints is studied in Section 3.

The rocket car interpretation suggests that optimal control methods can be used for solving best interpolation problems. We should note that this connection was not fully exploited in the above mentioned papers.

In the present paper we study constrained best interpolation problems applying duality methods from optimal control. In Section 2 we consider problems containing constraints for that derivative only, which norm is minimized.
An extension of one of the characterization theorems by C. A. Micchelli et al. [9] is obtained. We study also the continuity properties of the solution with respect to the data. Section 3 deals with the monotone best interpolation problem for $p=2$. We present a short proof of Hornung's result for $k=2$ and solve the problem for $k=3$.

2. Constrained k-th derivative

It is known that the interpolation conditions in (1) can be equivalently written as affine constraints for the k-th derivative $\int f^{(k)} M_{j,k}=d_j$, $j=1,\dots,N$, where $N=n-k$, $M_{j,k}$ are the appropriately normalized B-splines determined by $t_t, \dots, t_{j+k}$ and d_j are the divided differences of y_j at $t_j, \dots, t_{j+k}$.

Now we introduce the following more general problem. Let T_i, $i=0, 1, 2, 3$ be disjoint measurable sets in [0, 1], $\cup_{i=0}^{3} T_i=[0, 1]$ and let a_i and b_j be L_p functions on T_i and T_j, respectively. Define

$$\bar{a}=\begin{cases} -\infty, & t\in T_0\\ a_1, & t\in T_1\\ -\infty, & t\in T_2\\ a_3, & t\in T_3\end{cases}, \qquad \bar{b}=\begin{cases} +\infty, & t\in T_0\\ +\infty, & t\in T_1\\ b_2, & t\in T_2\\ b_3, & t\in T_3\end{cases}$$

and let $C=\{u\in L_p(0, 1),\ \bar{a}\leqq u\leqq \bar{b}\}$.

Let $1<p<\infty$, $1/p+1/q=1$, $\psi_i\in L_q(0, 1)$, c_i be fixed numbers, $i=1, \dots, m$. Define the set

$$D=\{u\in L_p(0, 1),\ \int \psi_i u=c_i,\quad i=1, \dots, m\}$$

and let $E=C\cap D$.

In this section we consider the problem

$$\text{(3)}\qquad \text{Minimize } \|u\|_p \qquad \text{subject to } u\in E.$$

For $k=2$, $T_1=[0, 1]$, $a_1=0$ and $\psi_i=M_{i,2}$ we have the convex best interpolation problem studied by U. Hornung [6]. If $a_1=0$, $b_2=0$ and $T_3=\emptyset$ we come to the problem in [9]. In [2] it is assumed that $T_3=[0, 1]$.

If $\psi_i = M_{i,k}$ then (3) is a best interpolation problem with constrained k-th derivative. Observe that in the monotone case the derivatives in the constrains and in the functional are different.

If $E \neq \emptyset$ then (3) has unique solution. For $T_1=[0, 1]$, $a_1=0$ the question when $E \neq \emptyset$ is the well-known Markov's problem of moments for ψ_i. If $\psi_i = M_{i,2}$, $m=N$ then $E \neq \emptyset$ when the divided differences $d_i > 0$. For $k > 2$ the set of feasible d_i becomes more sophisticated, see K. Ivanov [8].

In the sequel we assume that:

(4) There exist a set $\Delta \subset [0, 1]$ of positive measure and a function $\bar{u} \in E$ so that $\bar{a} < \bar{u} < \bar{b}$ almost everywhere on Δ and

$$|\sum_{i=1}^{m} \alpha_i \psi_i|^q = 0 \qquad \text{implies} \qquad \sum_{i=1}^{m} \alpha_i^2 = 0.$$

For $\psi_i = M_{i,2}$, $T_1 = [0, 1]$ and $a_1 = 0$ the condition (4) will be fulfilled if $d_i > 0$, $i = 1, \ldots, N$.

Introduce Lagrange functional

$$L(u, \beta) = \frac{1}{p}(\| u \|_p)^p - \sum_{i=1}^{m} \beta_i (\int \psi_i u - c_i).$$

Then the dual problem associated with (3) becomes

(5) Maximize $\mathscr{L}(\beta) = \inf(L(u, \beta),\ u \in C)$ subject to $\beta \in R^m$.

Theorem 1. *There exists a solution* $\hat{\beta} = (\hat{\beta}_1, \ldots, \hat{\beta}_m)$ *of the dual problem* (5) *such that the unique solution* $\hat{u}$ *of* (3) *is given by*

$$\hat{u} = \begin{cases} \bar{a} & \text{for } w < \bar{a}, \\ w & \text{for } \bar{b} \geq w \geq \bar{a}, \\ \bar{b} & \text{for } w > \bar{b}, \end{cases}$$

where $w = |\sum_{i=1}^m \hat{\beta}_i \psi_i|^{q-1} \operatorname{sign}(\sum_{i=1}^m \hat{\beta}_i \psi_i)$.

The proof uses standard techniques from convex analysis and optimal control and is presented in the Appendix.

Corollary 1. ([9]. Proposition 2.1). *If* $T_1 = [0, 1]$ *and* $a_1 = 0$ *then* $\hat{u} = (\sum_{i=1}^m \hat{\beta}_i \psi_i)_+^{q-1}$.

In this case β is a solution of the following finite dimensional maximization problem

$$\text{Maximize } (-\frac{1}{q}\int(\sum_{i=1}^{m} \beta_i \psi_i)_+^q + \sum_{i=1}^{m} \beta_i c_i) \quad \text{subject to } \beta = (\beta_1, \ldots, \beta_m) \in R^m.$$

In particular, for the problem of convex interpolation with $k=2$ and $p=2$ we obtain

(6) Maximize $(-0.5\int(\sum_{i=1}^{N} \beta_i M_{i,2})_+^2 + \sum_{i=1}^{N} \beta_i d_i)$ subject to $\beta \in R^N$.

The dual functional is concave, hence β solves (6) iff it is a solution of the following system is nonlinear equations

$$\int M_{i,2}(\beta_{i-1} M_{i-1,2}+\beta_i M_{i,2}+\beta_{i+1} M_{i+1,2})_+=d_i. \tag{7}$$

G. Iliev and W. Pollul [7] proved convergence of the following iterative procedure for solving (7)

$$\int M_{i,2}(\beta_{i-1}^{r-1} M_{i-1,2}+\beta_i^r M_{i,2}+\beta_{i+1}^{r-1} M_{i+1,2})_+=d_i.$$

This scheme is exactly the sequential coordinate descent method applied to the dual problem (6). The duality approach gives a number of possibilities to apply other unconstrained maximization methods. For example the simple gradient method will have the following iteration

$$\beta_i^{r+1}=\beta_i^r-\int M_{i,2}\,(\beta_{i-1}^r M_{i-1,2}+\beta_i^r M_{i,2}+\beta_{i+1}^r M_{i+1,2})_+ +d_i.$$

As in [9] one can relax the condition (4) determining the largest measurable set $\Omega\subset[0,\ 1]$ for which $u\in E$ implies $u=a_1$ on $T_1\cap\Omega$, $u=b_2$ on $T_2\cap\Omega$ and $u=a_3=b_3$ on $T_3\cap\Omega$. Then the above consideration will be confined to the complement of Ω.

Our next result is concerned with the sensitivity of the solution.

Theorem 2. *Let $p\geqq 2$, the condition* (4) *holds for some $c^0=(c_1^0, c_2^0,\dots, c_m^0)$ and let $\hat{u}^0$ be the solution of* (3) *corresponding to c^0. There exists a neighbourhood $N(c^0)$ of c^0 so that* (4) *holds for every $c\in N(c^0)$ and if $\hat{u}^c$ is the solution of* (3) *corresponding to c then $\|\hat{u}^c-\hat{u}^0\|_\infty=0(|c-c^0|^{1/p})$.*

The proof is given in the Appendix.

Remark. For $k=2$ and $1<p<+\infty$ the superlevel set $\{\beta\in R, \mathcal{L}(\beta)\geqq c\}$ of the dual functional is compact for any $c\in(-\infty,\ +\infty)$. This observation gives us convergence of a large class of nonlinear programming codes. In particular, the convergence of the sequential coordinate descent method (Jacobi iteration) follows from F. Vasil'ev [11], p. 331. The gradient method is convergent as well, moreover, it convergence rate is $0\left(\frac{1}{r}\right)$, see F. Vasil'ev [11], p. 265.

3. Monotone interpolation

In this section we consider the problem of monotone best interpolation

(8) Minimize $\|f^{(k)}\|_2$ subject to $f\in F$ and $f'\geqq 0$,

assuming that the data are strictly monotone, $y_i>y_{i+1}$. Denoting $f^{(k)}=u$, $f^{(j)}=x_j$, $j=1,\ \dots,\ k-1$, then (8) can be rewritten as an optimal control problem

(9) Minimize $\|u\|_2$ subject to $x_1'=x_2'\ ,\ \dots\ ,\ x_{k-1}'=u$.

(10) $\int M_{j,k}u=d_j$, $j=1\ ,\ \dots\ ,\ N$, $x_1\geqq 0$ for all $t\in[0,\ 1]$.

It is known, see S. Young [10], that if the data are strictly monotone then there exists an algebraic polynomial that interpolates (t_i, y_i) and is monotone on $[0, 1]$. Take $\delta>0$ so small that $2^\delta-1<\min_i(y_{i+1}-y_i)$ and let P interpolate $(t_i, y_i-(t_i+1)^\delta+1)$ witch $P'\geqq 0$. Then $f(t)=P(t)+(t+1)^\delta-1$ interpolates (t_i, y_i) with $f'(t)\geqq\delta$ for all $t\in[0, 1]$. In other words, there exist $\delta>0$ and $\bar{u}=f^{(k)}$ that satisfies (10) and is a continuous function so that if $\bar{x}=(\bar{x}_1, \dots, \bar{x}_{k-1})$ corresponds to $\bar{u}$ then $\bar{x}_1\geqq\delta$ for all $t\in[0,1]$. Thus, we can apply the duality theory developed by W. Hager and S. Mitter in [4].

Introduce Lagrange functional

$$L(x, u;\; p, v, \beta)=0.5\,\|u\|_2^2+\int\Big(\sum_{i=1}^{k-2}(x_i'-x_{i+1})p_i+p_{k-1}(x_{k-1}'-u)\Big) \tag{11}$$
$$+\int x_1\,dv+\sum_{j=1}^{N}\beta_j\Big(\int M_{j,k}u-d_j\Big),$$

where $p=(p_1, \dots, p_{k-1})$ and v are functions of bounded variations that are left continuous on $[0, 1]$, v is nonincreasing $v(1)=0$, $\beta=(\beta_1, \dots, \beta_N)\in R^N$. Denoting $\mathscr{L}(p, v, \beta)=\inf(L(x, u; p, v, \beta)$, $x-$ absolutely continuous, $u\in L_2(0, 1))$, then the problem dual to (9) is

$$\text{Maximize } \mathscr{L}(p, v, \beta). \tag{12}$$

The following theorem can be extracted from [4]:

Theorem 3. *There exists a solution $(\hat{p}, \hat{v}, \hat{\beta})$ of (12) so that if $(\hat{x}, \hat{u})$ solves (9), then*

$$0.5\,\|\hat{u}\|_2^2=\mathscr{L}(\hat{p}, \hat{v}, \hat{\beta})=L(\hat{x}, \hat{u};\; \hat{p}, \hat{v}, \hat{\beta}) \tag{13}$$

and

$$\int\hat{x}_1\,d\hat{v}=0. \tag{14}$$

Moreover, $\hat{p}(0)=\hat{p}(1)=0$, $\hat{p}_1=\hat{v}$, $\hat{p}_i$, $i=2, \dots, k-1$ *are absolutely continuous on* $(0, 1]$ *and* $\hat{p}_2'=-\hat{v}$, $\hat{p}_3'=-\hat{p}_2, \dots, \hat{p}_{k-1}'=-\hat{p}_{k-2}$. *The unique solution of* (9) *is given by* $\hat{u}=\hat{p}_{k-1}-\sum_{j=1}^N\hat{\beta}_jM_{j,k}$.

Integrating by parts in (11) and using (13) and (14), we obtain the following explicit form of the dual problem

$$\text{Maximize } \Big(-0.5\,\|r-\sum_{j=1}^{N}\beta_jM_{j,k}\|_2^2-\sum_{j=1}^{N}\beta_jd_j\Big) \tag{15}$$

$$\text{subject to: } \beta\in R^N,\; r^{(k-2)}=(-1)^kv,\; v-\text{nonincreasing,}$$

$$\text{continuous from the left, } v(1)=0.$$

3.1. Case $k=2$. We have

$$\hat{u}=\hat{v}-\sum_{j=1}^N\hat{\beta}_jM_{j,2}. \tag{16}$$

Lemma 1. *The function $\hat{u}$ is continuous on* $[0, 1)$.

For a proof, see Appendix.

Now we are prepared to present a new, shorter proof of Hornung's characterization theorem in [5].

Let $\hat{x}_1>0$ on some interval $\Delta\subset[0, 1]$. Then from (14) $\hat{v}$ is constant on Δ, hence, by (16) $\hat{u}$ is piecewise linear and $\hat{x}_1$ is a quadratic spline on Δ. If $\hat{x}_1(t)=0$ at some isolated point t, then since $\hat{v}$ is continuous at t it will be constant around t and $\hat{x}_1$ will be the same quadratic spline around t. If $\hat{x}_1$ is zero on the left or on the right of σ only then σ is considered as a new knot. If σ_j, σ_{j+1} are two neighbouring new knots then not more than one old knot t_i may be situated between σ_j and σ_{j+1}, otherwise the strict monotonicity of y_i will be violated. If $\hat{x}_1>0$ on (σ_j, σ_{j+1}) then by the continuity $\hat{u}(\sigma_j)=\hat{u}(\xi)=\hat{u}(\sigma_{j+1})=0$ for some $\xi\in(\sigma_j, \sigma_{j+1})$ ($\hat{x}_1$ has maximum at ξ). Since $\hat{u}$ is precewise linear on (σ_j, σ_{j+1}) at least two fixed knots t_i, t_{i+1} should be between σ_j and σ_{j+1}. This yields that the number of the intervals $[\sigma_j, \sigma_{j+1}]$ on which $\hat{x}_1=0$ does not exceed $[n/2]+1$. If $\hat{x}_1=0$ on $[\sigma_j, \sigma_{j+1}]$ then $\hat{v}$ is piecewise linear but not constant on $[\sigma_j, \sigma_{j+1}]$. Otherwise since $\hat{v}$ is continuous it will have the same value on (t_i, σ_{j+1}), where t_i is the neighbouring from the left and different knot to σ_j, hence $\hat{u}$ will be zero on $[t_i, \sigma_{j+1}]$ which contradicts the definition of σ_j. Denote $\varphi=-(\Sigma_{j=1}^N \hat{\beta}_j M_{j,2})$. Then φ is a step function across t_i and if $\hat{x}_1$ is not zero on (σ_j, σ_{j+1}) then $\hat{x}_1''=\hat{u}'=\varphi$. If $\hat{x}_1=0$ on (σ_j, σ_{j+1}) then $\varphi=-\hat{v}'>0$.

Thus we come to Hornung's characterization theorem:

Theorem 4 ([5]). *The solution of the problem* (8) *for* $k=2$ *is a cubic spline with defect* 1 *and additional* m *(free) knots* σ_j, *where* $m\leqq 2\ [n/2]+2$, *so that:*

(i) *On each interval* $[t_i, t_{i+1}]$ *there are at most two new knots* $\sigma_j<\sigma_{j+1}$ *and the solution* f *is constant on* $[\sigma_j, \sigma_{j+1}]$;

(ii) *Given* $\sigma_j<\sigma_{j+1}$ *either* $\hat{f}'=0$ *on* $[\sigma_j, \sigma_{j+1}]$ *and then there is at most one* $t_i\in(\sigma_j, \sigma_{j+1})$ or $\hat{f}'>0$ *on* (σ_j, σ_{j+1}) *and there are at least two knots* $t_i, t_{i+1}\in(\sigma_j, \sigma_{j+1})$;

(iii) *There exists a step function* φ, $\varphi>0$ *on* (σ_j, σ_{j+1}) *where* $\hat{f}'=0$ *and* $\varphi=\hat{f}^{(3)}$ *on* (σ_j, σ_{j+1}) *where* $\hat{f}$ *is not constant.*

3.2. Case $k=3$. Consider (9) with $k=3$. Then from Theorem 3

$$x_1'=x_2,\quad x_2'=u,\quad \hat{u}=\hat{p}-\sum_{j=1}^N \beta_j M_{j,3},\quad \hat{p}'=-\hat{v},\quad \hat{p}(1)=0$$

and the dual problem is

$$\text{(17)}\qquad \text{Maximize } \left(-0.5\,\|p-\sum_{j=1}^N \beta_j, M_{j,3}\|_2^2-\sum_{j=1}^N \beta_j, d_j\right).$$

Lemma 2. *Let* $\hat{x}_1(\sigma)=0$ *for some* $\sigma\in(0, 1)$. *Then either* $\hat{u}(\sigma)=0$ *or there exists* $\delta>0$ *so that* $\hat{v}$ *is constant on* $(\sigma-\delta, \sigma+\delta)$.

The proof is presented the Appendix.

Theorem 5. *The solution* $\hat{f}$ *of the problem* (8) *with* $k=3$ *is a 5-th order spline with defect* 2 *and additional* m *(free) knots* σ_j $m\leqq 2\ [n/2]+2$ *so that:*

(i) *On each interval* $[t_i, t_{i+1}]$ *there are at most two new knots* $\sigma_j\leqq\sigma_{j+1}$ *and* $f'=\hat{f}''=\hat{f}^{(3)}=0$ *on* $[\sigma_j, \sigma_{j+1}]$;

(ii) *Given* $\sigma_j<\sigma_{i+1}$ *either* $\hat{f}'=0$ on $[\sigma_j, \sigma_{j+1}]$ *and then there is at most one* $t_i\in[\sigma_j, \sigma_{j+1})$ *or there are at least two* t_i, t_{i+1} *in* (σ_j, σ_{j+1});

(iii) $f^{(4)}$ *may have jumps at new knots only.*

Proof. If $\hat{x}_1>0$ on some $\Delta\subset[0, 1]$ then by (14) $\hat{v}$ is constant and hence $\hat{u}$ is a quadratic spline and $\hat{x}_1$ is a spline of order 4. If $\hat{x}_1=0$ on Δ then $\hat{u}=0$ on Δ hence $\hat{p}$ is a quadratic spline and $\hat{v}$ is piecewise linear on Δ. As before, let $\hat{x}_1>0$ on (p, τ), $\hat{x}_1(\sigma)=\hat{x}_1(\tau)=0$. If $\hat{u}(\sigma)>0$ then from Lemma 2 $\hat{v}$ is constant around σ and $\hat{x}_1$ will be the same polynomial around σ, thus σ is not a new knot. The same conslusion can be obtained for τ. Then $\hat{u}(\sigma)=\hat{u}(\tau)=0$. There exist at least two additional points ζ, η in (σ, τ) so that $\hat{u}(\zeta)=\hat{u}(\eta)=0$. This implies that there are at least two old knots t_i, t_{i+1} between σ and τ. Hence, between every two new knots σ_j, $\sigma_{j+1}\hat{x}_1$ is either identically zero or a spline of 4-th order with at least two knots t_i. We do not know whether $\hat{v}$ is continuous at the new knots, hence $\hat{x}_1$ is not more than C^2 function.

If we impose additional conditions for f, f' and f'' at $t=0$ and $t=1$ the above characterization will be complete, i. e. the number of conditions will be equal to the number of the parameters.

3.3 Convex interpolation. Replacing $f'\geqq 0$ by $f''\geqq 0$ the problem (8) becomes a best convex interpolation problem with minimal L_2 norm of the k-th derivative. For $k=2$ this problem was solved by U. Hornung [6], see the previous section. Now we can characterize the solution for $k=3$ and $k=4$. This follows from the observation that the convex best interpolation problem can be written as (9) with $M_{j,k}$ replaced by $M_{j,k+1}$.

Consider the case $k=3$ assuming that the second divided differences are strictly positive. By repeating the proof of Theorem 4 taking into account that $\hat{u}$ is a quadratic spline we get the following result.

Theorem 6. *The solution $\hat{f}$ of the problems for convex interpolation with minimal L_2 norm of the 3-rd derivative is a 4-th order spline with defect 1 with fixed knots t_i and additional m free knots σ_j, $m\leqq 2n+2$ so that:*

(1) *In each interval $[t_i, t_{i+1}]$ there are at most two new knots $\sigma_j<\sigma_{j+1}$ and $\hat{f}''=0$ on $[\sigma_j, \sigma_{j+1}]$;*

(ii) *Given $\sigma_j\lesssim\sigma_{j+1}$ either $\hat{f}''=0$ on $[\sigma_j, \sigma_{j+1}]$ and then there is at most one $t_i\in(\sigma_j, \sigma_{j+1})$ or $\hat{f}''>0$ on (σ_j, σ_{j+1}) and there is at least one $t_i\in(\sigma_j, \sigma_{j+1})$.*

The case $k=4$ can be solved similarly, on the basis of Theorem 5.

Appendix

Proof of Theorem 1. Denote

$$L_1(u, \lambda)=\frac{\lambda_0}{p}(\| u\|_p)^p-\sum_{j=1}^{m}\lambda_j\ (\int\psi_j u-c_j),$$

where $\lambda_0\geqq 0$ and let $\mathscr{L}_1(\lambda)=\inf(L_1(u, \lambda), u\in C)$. One can easily see that if $u\in E$ then

$$\text{(A1)}\qquad \frac{\lambda_0}{p}(\| u\|_p)^p\geqq\mathscr{L}_1(\lambda)$$

for every $\lambda=(\lambda_0, \ldots, \lambda_m)$ with $\lambda_0\geqq 0$. Using the theorem for separation of convex sets, one can show that there exists $\bar{\lambda}$ $\Sigma_{j=0}^m\bar{\lambda}_j^2>0$ such that if $\bar{u}$ solves (3) then

$$\text{(A2)}\qquad \frac{\bar{\lambda}_0}{p}(\| u\|_p)^p\leqq L_1(u,\bar{\lambda})\quad\text{for all } u\in C.$$

Thus, by (A1) and (A2) we conclude that $\bar{\lambda}$ solves the problem

$$\text{Maximize } \mathscr{L}_1(\lambda) \quad \text{subject to } \lambda \in R^{m+1}.$$

Then $\hat{u}$ is the unique solution of the problem

$$\text{(A3)} \qquad \text{Minimize } \int H(u, t) \quad \text{subject to } \bar{a} \leqq u \leqq \bar{b},$$

where

$$\text{(A4)} \qquad H(u, t) = (\bar{\lambda}_0/p)\,|u|^p - \Sigma_{j=1}^m \bar{\lambda}_j(\psi_j(t)u - c_j).$$

We will show that

$$\text{(A5)} \qquad H(\hat{u}(t), t) = \min(H(u, t), \bar{a}(t) \leqq u \leqq \bar{b}(t))$$

for a. e. $t \in [0, 1]$. Let M be the intersection of the sets of Lebesgue points of $H(\hat{u}(\cdot) \ldots)$ and ψ_j. Suppose that $H(z, s) < H(\hat{u}(s), s)$ for some $s \in M$ and $z \in [\bar{a}(s). \bar{b}(s)]$. Then $\int_{s-\delta}^{s+\delta} H(\hat{u}(t), t)\,dt = H(\hat{u}(s), s)\delta + o(\delta)$ and $\int_{s-\delta}^{s+\delta} H(z, t)\,dt = H(z, t)\delta + o(\delta)$.
Thus, for small $\delta > 0$

$$\int_{s-\delta}^{s+\delta} H(z, t) < \int_{s-\delta}^{s+\delta} H(\hat{u}(t), t).$$

The function

$$u^*(t) = \begin{cases} z & \text{for } t \in [s-\delta, s+\delta], \\ \hat{u}(t) & \text{otherwise,} \end{cases}$$

applied to (A3) gives smaller value of the functional than $\hat{u}$, which is a contradiction. Then (A5) holds.
Now we show $\bar{\lambda}_0 > 0$ using the regularity condition (4). Let $\bar{\lambda}_0 = 0$. For $\varepsilon > 0$ denote

$$\text{(A6)} \qquad u_\varepsilon = \begin{cases} \bar{u} + \varepsilon \left| \sum_{j=1}^m \bar{\lambda}_j \psi_j \right|^{q/p} \operatorname{sign} \left(\sum_{j=1}^m \bar{\lambda}_j \psi_j \right), & t \in \Delta, \\ \bar{u} & \text{otherwise.} \end{cases}$$

For small $\varepsilon > 0$ one has $u_\varepsilon \in C$. By (A2) $0 \leqq -\Sigma_{j=1}^m \bar{\lambda}_j(\int \psi_j u_\varepsilon - c_j)$, which yields

$$0 \leqq -\varepsilon \int_\Delta \left| \sum_{j=1}^m \bar{\lambda}_j \psi_j \right|^q.$$

Then all $\bar{\lambda}_j$ are zero which is a contradiction. Thus, $\bar{\lambda}_0 > 0$. Take $\beta_j = \bar{\lambda}_j/\bar{\lambda}_0$. Then $\hat{u}(t)$ minimizes

$$\frac{1}{p}|u|^p - \sum_{j=1}^m \beta_j(\psi_j(t)u - c_j)$$

over $[\bar{a}(t), \bar{b}(t)]$. The proof is completed. □

P r o o f o f T h e o r e m 2. The operator $\psi = (\psi_1, \ldots, \psi_m): L_p(\Delta) \to R^m$ is surjective, hence (4) holds in $N(c^0)$. Let $c^k \to c^0$ and $u^k. \bar{\lambda}^k$ correspond to c^k, $k = 0, 1, \ldots$ (we use the notation from the proof of Theorem 1). For small $\varepsilon > 0$ the control u_ε^k defined as in (A6) satisfies $u_\varepsilon^k \in C$. Then, by (A2) for $t \in \Delta$

$$\text{(A7)} \qquad \frac{\bar{\lambda}_0^k}{p}|\hat{u}^k(t)|^p \leqq \frac{\bar{\lambda}_0^k}{p}|u_\varepsilon^k(t)|^p - \varepsilon \left| \sum_{j=1}^m \bar{\lambda}_j^k \psi_j(t) \right|^q.$$

Without loss of generality, let $\bar{\lambda}_j^k \to \bar{\lambda}_j^0$ and $\Sigma_{j=0}^m \bar{\lambda}_j^k = 1$. Clearly, $\hat{u}^k$ is a bounded sequence in $L_p(0, 1)$. Integrating both sides of (A7) in Δ and assuming that $\bar{\lambda}_0^k \to 0$, we come to a contradiction. Thus, liminf $\bar{\lambda}_0^k > 0$. Then we can take $\beta_j^k = \bar{\lambda}_j^k/\bar{\lambda}_0^k$ and obtain limsup $\Sigma_{j=1}^m |\beta_j^k| < +\infty$.
We have

(A8) $$|\hat{u}^k(t)|^p \leqq |\hat{u}^0(t)|^p + p \sum_{j=1}^{m} \beta_j^k (c_j^0 - c_j^k).$$

Using the uniform convexity of $|u|^p$, we get

(A9) $$|\hat{u}^k(t)|^p = |\hat{u}^k(t)|^p + p \sum_{j=1}^{m} \beta_j^0 (\psi_j(t)\hat{u}^k(t) - c_j^k) \geqq |\hat{u}^0(t)|^p + p \sum_{j=1}^{m} \beta_j^0 (c_j^0 - c_j^k)$$
$$+ p(|\hat{u}^0(t)|^{p-2}\hat{u}^0(t) - \sum_{j=1}^{m} \beta_j^0 \psi_j(t))\,(\hat{u}^k(t) - \hat{u}^0(t)) + |\hat{u}^k(t) - \hat{u}^0(t)|\,p/2^{p-2}.$$

Combining (A8) and (A9) and using the well-known minimum condition for convex functions

$$(|\hat{u}^0(t)|^{p-2}\hat{u}^0(t) - \sum_{j=1}^{m} \beta_j^0 \psi_j(t))\,(\hat{u}^k(t) - \hat{u}^0(t)) \geqq 0,$$

we get

$$|\hat{u}^k(t) - \hat{u}^0(t)|^p \leqq 2^{p-2} \sum_{j=1}^{m} (\beta_j^k - \beta_j^0)\,(c_j^0 - c_j^k).$$

Using the boundedness of β^k in this estimate, we complete the proof. □

P r o o f o f L e m m a 1. We know that $\hat{u}$ has bounded variation, hence it has right limit at each $t \in [0, 1]$. If $\hat{x}_1(t) > 0$ then $\hat{u}$ is continuous at t since by (14) $\hat{v}$ is constant around t. Let $\hat{x}_1(t) = 0$. We will show that $\hat{u}(t^+) \geqq 0$. On the contrary, suppose that there exists $\delta > 0$ so that $\hat{u} < 0$ on $(t, t+\delta)$. Then, since $\hat{x}_1(t+\delta) \geqq 0$ we have

$$0 = \hat{x}_1(t) = \hat{x}_1(t+\delta) - \int_t^{t+\sigma} \hat{u} > 0.$$

Thus, $\hat{u}(t^+) \geqq 0$. Similarly, if $\hat{x}_1(t) = 0$ then $\hat{u}(t^-) = \hat{u}(t) \leqq 0$. This means that $\hat{u}(t^+) \geqq \hat{u}(t)$. On the other hand, $\hat{u}(t^+) - \hat{u}(t) = \hat{v}(t^+) - \hat{v}(t) \leqq 0$. Then $\hat{u}(t^+) = \hat{u}(t)$, i. e. $\hat{u}$ is continuous on [0, 1). □

P r o o f o f L e m m a 2. Clearly, if $\hat{x}_1(\sigma) = 0$ then $\hat{u}(\sigma) \geqq 0$. Suppose that $\hat{u}(\sigma) > 0$. Then since the minimum of $\hat{x}_1$ at σ is unique in a neighbourhood of σ, σ is isolated. If $\hat{v}$ is continuous at σ, then by (14) $\hat{v}$ is constant on $(\sigma - \delta, \sigma + \delta)$ and the proof is completed. Suppose that $\hat{v}(\sigma) > \hat{v}(\sigma^+)$. We know that $\hat{v}$ is constant on $(\sigma - \delta, \sigma)$ and $(\sigma^+, \sigma + \delta)$. Take an integer $l > 0$ and $\varepsilon > 0$ and let

$$\hat{v} = \begin{cases} -(\hat{v}(\sigma) - \hat{v}(\sigma^+))\,(t - \sigma + \delta)^l/2\delta^\varepsilon + \hat{v}(\sigma), & t \in (\sigma - \delta, \sigma], \\ (\hat{v}(\sigma) - \hat{v}(\sigma^+))\,(\sigma + \delta - t)^l/2\delta^\varepsilon + \hat{v}(\sigma^+), & t \in (\sigma, \sigma + \delta], \\ \hat{v} & \text{otherwise.} \end{cases}$$

The function $\hat{v}$ is decreasing and continuous on $(\sigma - \delta, \sigma + \delta)$. For every $\varepsilon > 0$ one can find l so large that

$$\varepsilon = \int_{\sigma-\varepsilon}^{\sigma} (\hat{v} - \tilde{v}) = \int_{\sigma}^{\sigma+\varepsilon} (\tilde{v} - \hat{v}).$$

Moreover, $\tilde{p}(t) = \hat{p}(\sigma - \delta) + \int_{\sigma-\delta}^{t} \tilde{v}$ satisfies $\tilde{p}(\sigma + \delta) = \hat{p}(\sigma + \delta)$ and $\tilde{p} < \hat{p}$ on $(\sigma - \delta, \sigma + \delta)$. Then, for δ and ε sufficiently small since $\hat{u}(\sigma) > 0$ we have

$$0 \leqq \hat{u} = \tilde{p} - \sum_{j=1}^{m} \hat{\beta}_j M_{j,3} < \hat{u} \qquad \text{on } (\sigma - \delta, \sigma + \delta).$$

We obtain that $\hat{v}$ and $\tilde{p}$ together with $\hat{\beta}$ give greater value of the dual functional in (17). The obtained contradiction means that $\hat{v}$ is constant on $(\sigma - \delta, \sigma + \delta)$. □

A c k n o w l e d g m e n t s. The author wishes to thank Prof. B. Bojanov and Dr. G. Iliev for their helpful comments.

References

1. C. de Boor. On "best" interpolation. *J. Appr. Theory.*, **16**, 1976, 28-42.
2. A. L. Dontchev, G. Iliev, M. M. Konstantinov. Constrained interpolation via optimal control. – In: Mathematics and Education in Mathematics '85. Sofia, 1985, 385-392.
3. J. Favard. Sur l'interpolation. *J. Math. Pures Appl.*, **19**, 1940, 281-306.
4. W. W. Hager, S. K. Mitter. Lagrange duality theory for convex control problems. *SIAM J. Control Opt.*, **14**, 1976, 883-856.
5. U. Hornung. Monotone Spline-Interpolation. – In: Numerische Methoden der Approximationstheorie. Basel, 1978, 172-191.
6. U. Hornung. Interpolation by smooth functions under restriction on the derivatives. *J. Appr. Theory*, **28**, 1980, 227-237.
7. G. Iliev, W. Pollul. Convex interpolation by functions with minimal L_p norm $(1<p<\infty)$ of the k-th derivative. – In: Mathematics and Education in Mathematics , **84**, Sofia, 1984, 31-42.
8. K. Ivanov. Shape preserving interpolation and approximation. *Serdica*, **11**, 1985, 398-410.
9. C. A. Micchelli et al. Constrained L_p approximation. *Constructive Approximation*, **1**, 1985, 93-102.
10. S. Young. Piecewise monotone polynomial approximation. *Bull. Amer. Math. Soc.*, **73**, 1976, 642-643.
11. Ф. П. Васильев. Численные методы решения экстремальных задач. Наука, М. 1980.

Institute of Mathematics
Bulgarian Academy of Sciences
1090 Sofia, P. O. Box 373
BULGARIA

Received 14. 01. 1987