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On a Theorem of Ju. Brudnyi

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In 1970 Ju. A. Brudnyi proved the following theorem [1], which is widely used in approximation theory.

Theorem 1 (Ju. Brudnyi). *Let f be continuous in $[0, 1]$ and let n be a prescribed natural number. Then there exists a family of functions $\{f_h: 0<h\leqq 1/n\}$ such that*

$$\|f-f_h\|\leqq A_n\omega_n(f;\ h), \tag{1}$$

$$\|f_h^{(n)}\|\leqq B_n h^{-n}\omega_n(f;\ h), \tag{2}$$

where A_n and B_n depend only on n.

Here ω denotes the modulus of continuity:

$$\omega_n(f;\ \delta):=\sup\{|\Delta_h^n f(t)|: t,\ t+nh\in[0,\ 1];\ \ |h|\leqq\delta\} \tag{3}$$

with Δ_h^n the n-th difference with step size h:

$$\Delta_h^n f(x)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}f(x+ih). \tag{4}$$

The norm in (1) and (2) is the uniform norm. In fact Theorem 1 holds for integral norms [1] as well, but we will confine ourselves to the uniform norm case.

The purpose of the present paper is to find the estimates of the constants A_n and B_n.

1. An Estimate of B_n when $A_n=1$

We will use the modified Steklov's function [2]:

$$S_{n,h}(f;\ x)=h^{-n}\int_0^h\ldots\int_0^h f(x-nhx+t_1\ +\ \ldots\ +\ t_n)\,dt_1,\ \ldots\ ,\ dt_n.$$

It is directly seen, that if f is given on the segment $[0, 1]$, then $S_{n,h}(f)$ is defined on $[0, 1]$, when $h\in(0,\ 1/n]$, as well. It can be easily checked, that following equalities hold

(5) $S_{n,\theta h}(f;\ x)=h^{-n}\int_0^h \dots \int_0^h f(x+\theta(-nhx+t_1+\ \dots\ +t_n))\,dt_1 \dots dt_n$

for $\theta\in[0,\ 1]$, $S_{n,0}(f;\ x)=f(x)$ and

(6) $$S_{n,h}^{(n)}=h^{-n}(1-nh)^n\ \Delta_h^n\ f(x-nhx).$$

Theorem 2. *Let the function f be bounded and integrable on the segment* [0, 1]. *Then for every natural n and for every* $h\in(0,\ 1/n]$, *there exists a function* f_h, *defined on the segment* [0, 1] *with bounded n-th derivative, for which the following estimates hold*

(7) $$\|f-f_h\|\leqq\omega_n(f;\ h),$$

(8) $$\|f_h^{(n)}\|\leqq(n+1)\,n^n\,h^{-n}\,\omega_n(f;\ h).$$

Proof. Denote

(9) $$f_h(x)=\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}S_{n,ih/n}(f;\ x).$$

Using (5) and (9), taking into account, that $S_{n,0}(f;\ x)=f(x)$, we obtain

(10) $$f(x)-f_h(x)=\sum_{i=0}^{n}(-1)^i\binom{n}{i}S_{n,ih/n}(f;\ x)$$

$$=h^{-n}\int_0^h \dots \int_0^h \sum_{i=0}^{n}(-1)^i\binom{n}{i}f(x+\frac{i}{n}(-nhx+t_1+\ \dots\ +t_n))\,dt_1 \dots dt_n$$

$$=h^{-n}\int_0^h \dots \int_0^h (-1)^n\,\Delta_\xi^n f(x)\,dt_1 \dots dt_n,$$

where $\xi=\frac{1}{n}(-nhx+t_1+\ \dots\ +t_n)$; $\quad -h\leqq\xi\leqq h$.

From (10) it follows that

$$|f(x)-f_h(x)|\leqq\sup\ \{|\Delta_\xi^n f(x)|:|\xi|\leqq h\}=\omega_n(f;\ h),$$

thus (7) is proved.

From (9) and (6) we compute

$$f_h^{(n)}(x)=\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}\frac{n^n}{i^n h^n}(1-ih)^n\,\Delta_{ih/n}^n f(x-ihx)$$

and therefore

$$\|f_h^{(n)}\|\leqq h^{-n}\,\omega_n(f;\ h)\,n^n\sum_{i=1}^{n}\binom{n}{i}i^{-n}\leqq(n+1)\,n^n\,h^{-n}\,\omega_n(f;\ h),$$

since $\Sigma_{i=1}^{n}\binom{n}{i} i^{-n} \leqq n+1$. □

Corollary 1. *In Theorem* 1 *one can take* $A_n=1$, $B_n=(n+1)n^n$.

2. Estimates of A_n when $B_n=1$

Let us show first, that B_n in Theorem 1 can not be less than 1.

Suppose the converse, that for every continuous function f and for every natural number n, for every choice of $h\in(0, 1/n]$ there exists a function f_h, for which

$$\|f-f_h\| \leqq A_n\omega_n(f;\ h) \text{ and } \|f_h^{(n)}\| \leqq \theta h^{-1}\omega_n(f;\ h),$$

where $0\leqq\theta<1$.

Consider $f(x)=x^n/n!$. Then for every $h\in(0, 1/n]$ a function f_h exists, for which

$$\left|\frac{x^n}{n!}-f_h(x)\right| \leqq A_n h^n \tag{11}$$

and

$$|f_h^{(n)}(x)| \leqq \theta<1. \tag{12}$$

From (11) we obtain

$$\left|\frac{\Delta_\tau^n}{\tau^n}\left(\frac{x^n}{n!}-f_h(x)\right)\right| = |1-f_h^{(n)}(\xi_\tau)| \leqq 2^n A_n\left(\frac{h}{\tau}\right)^n. \tag{13}$$

We fix τ and choose h so small that

$$2^n A_n\left(\frac{h}{\tau}\right)^n < 1-\theta. \tag{14}$$

Then, from (13) and (14), it follows that

$$1-f_h^{(n)}(\xi_\tau)<1-\theta$$

and $f_h^{(n)}(\xi_\tau)>\theta$, that contradicts (12). □

Denote A_n^* the smallest number, for which Theorem 1 holds for $B_n=1$ and $A_n=A_n^*$.

Till now the question for the existence of such A_n^* for every natural n is open.

We will show, that A_n^* exists for $n=1$ and $n=2$.

Theorem 3. $A_1^*\leqq 1$, $A_2^*\leqq 9/8$.

Proof. To prove, that $A_1^*\leqq 1$, it is sufficient to take for f_h the linear interpolation spline $S_1(f)$ on equidistant knots with a stepsize h: $S_1(f;\ ih)$

$=f(ih)$; $i=0, 1, 2, \ldots, m=[1/h]$; $S_1(f;\ 1)=f(1)$. $S_1(f)$ is linear on every interval $[ih, (i+1)h]$; $i=0, 1, 2, \ldots, m-1$ and on the interval $[mh, 1]$.

It can be checked directly, that

$$\|f-S_1(f)\| \leqq \omega_1(f;\ h), \quad \|S_1'(f)\| \leqq h^{-1}\omega_1(f;\ h).$$

To prove the existence of A_2^*, one cannot use the quadratic interpolation spline. For this purpose we construct the quadratic spline $S_2(f;\ x)$, satisfying the following conditions:

$$S_2(f;\ ih+\frac{h}{2})=\frac{1}{2}(f(ih)+f(ih+h))$$

and

$$S_2(f;\ x)=S_1(f;\ x) \text{ for } x\in[0, \frac{h}{2}]\cup[mh+\frac{h}{2}, 1].$$

$$S_2(f;\ x)=\frac{(x-ih)^2}{2h^2}\Delta_h^2\ f(ih-h)+\frac{x-ih}{2h}(f(ih+h)-f(ih-h))+f(ih)+\frac{1}{8}\Delta_h^2 f(ih-h);$$

$$\text{for } x\in[ih-\frac{h}{2}, ih+\frac{h}{2}].$$

One can check immediately that $\|S_2''(f)\| \leqq h^{-2}\omega_2(f;\ h)$ and

$$|f(x)-S_2(f;\ x)| \leqq |f(x)-S_1(f;\ x)|+|S_1(f;\ x)-S_2(f;\ x)| \leqq \omega_2(f;\ h)$$

$$+\frac{1}{8}\omega_2(f;\ h)=\frac{9}{8}\omega_2(f;h).$$

Thus it is proved, that A_2^* exists and does not exceed 9/8.

Up to now we could not prove the existence of A_n^* for $n \geqq 3$.

3. Close value estimates of A_n and B_n

To find other estimates of A_n and B_n we will use the interpolation splines of n-th order, introduced by S. B. Stechkin and Ju. N. Subbotin [3].

Let a uniform net $x_i=ih$; $i=0, 1, 2, \ldots, m$; $h=1/m$ is defined on the segment $[0, 1]$. Suppose $m>n$.

For every function f, defined on the segment $[0, 1]$, we define two polynomials:

$$Q_n(f;\ x)=f(0)+\sum_{s=1}^{n}\frac{\Delta_h^s f(0)}{h^s s!}x(x-h)\ \ldots\ (x-sh+h),$$

$$R_n(f;\ x)=f(1-nh)+\sum_{s=1}^{n}\frac{\Delta_h^s f(1-nh)}{h^s s!}(x-1-nh+h)\ \ldots\ (x-1-nh+sh).$$

Using these polynomials, we define the function

$$F(x)=\begin{cases} Q_n(f;\ x); & x\leqq 0, \\ f(x)\ ; & 0\leqq x\leqq 1, \\ R_n(f;\ x); & x\geqq 1. \end{cases} \tag{15}$$

In [3] it is proved, that there exists a unique interpolation spline of n-th order $S_n(F;\ x)$, which has a bounded n-th derivative, $S_n(F;\ ih)=F(ih)$ $i=0,\ \pm 1,\ \pm 2, \dots$ For the n-th derivative of $S_n(F)$, the following estimate is proved [3]:

$$|S_n^{(n)}(F)\leqq |u_n|^{-1}h^{-n}\omega_n(f;\ h), \tag{16}$$

where

$$u_{2s}=\frac{1}{(2s)!}\sum_{k=0}^{2s}\binom{2s+1}{k}\sum_{p=0}^{2s-k}(-1)^p(p+\tfrac{1}{2})^{2s}, \tag{17}$$

$$u_{2s+1}=\frac{1}{(2s+1)!}\sum_{k=0}^{2s}\binom{2s+1}{k}\sum_{p=0}^{2s-k}(-1)^p(p+1)^{2s+1}, \tag{18}$$

for $s=1,\ 2,\ 3, \dots$

We need the following more precised theorem of Whitney, proved in [4]:

Theorem 4. *If $P_{n-1}(x)$ is an interpolation polynomial of f at the points $(i+1)h$, $(i+2)h, \dots, (i+n)h$, then*

$$|f(x)-P_{n-1}(x)|\leqq 6\omega_n(f;\ h) \text{ for } x\in[ih,\ (i+n+1)h]. \tag{19}$$

From the definition of the spline $S_n(F)$ for the function f and the obtained in [3] representation, it is obvious, that

$$|S_n(F:x)-P_{n-1}(x)\leqq |u_n|^{-1}\omega_n(f;\ h)$$

for $x\in[ih,\ (i+n+1)h]$, where P_{n-1} is the interpolation polynomial in Theorem 4. Hence

$$\|f-S_n(F)\|\leqq(6+|u_n|^{-1})\cdot\omega_n(f;\ h). \tag{20}$$

The inequalities (16) and (20) show, that the following assertion holds:

Corollary 2. *In Theorem 1 for values of the constants A_n and B_n one can take*

$$A_n^{**}=|u_n|^{-1}+6,\quad B_n^{**}=|u_n|^{-1}.$$

We will give several consecutive values of $|u_n|^{-1}$ to compare B_n^{**} with $B_n=(n+1)n^n$ from 1.

n	3	4	5	6	7	8
$\lvert u_n\rvert^{-1}$	3	4.8	7.5	$\frac{6!}{61}=11.803\dots$	$\frac{315}{17}=18.529\dots$	$\frac{8!}{1387}=29.069\dots$
$(n+1)n^n$	108	1280	18750	326592	6588344	150994944

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