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Electromagnetic Scattering Theory for a Dielectric with a Perfect Conductor Core in Low-Frequencies

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In this work we present a systematic and integrated theory for the scattering of an electromagnetic wave by a dielectric scatterer which contains a perfect conductor core. We construct the integral representation for the electric field, and we express the normalized scattering amplitude in a closed form. Using the low-frequency expansions the scattering problem is reduced to a sequence of potential problems. We evaluate the leading terms of the normalized scattering amplitude and the scattering cross-section in the low-frequency region which are proportional to the third power and the fourth power of the wave number, respectively.

1. Introduction

The scattering problem in low-freguency region arises when a time harmonic wave is scattered by a bounded obstacle and the wavelength of the incident radiation is large compared with the characteristic dimension of the scatterer.

The method of reducing electromagnetic scattering problems to a series of problems in potential theory was first investigated by Lord Rayleigh. In his classic paper [8] he examined the scattering of both acoustical and electromagnetic waves. A significant contribution to this subject also came from Stevenson who described a method for finding the general term of the electromagnetic field in low-frequency series expansion and a special technique for evaluating the first three terms of the series [9, 10]. P. Werner in his papers [13, 14] has established the validity of the series expansions for the electric and the magnetic field. A main contribution to electromagnetic scattering in low-frequencies is made by R. Kleinman. In his paper [6] he rectified some weaknesses to Stevenson's procedure. Among other theoretical results for the far-field behaviour of the electromagnetic field, he derived expressions for the far field coefficient for any radiating electromagnetic field in terms of the near field [7] and with Asvestas presented a new method for obtaining solutions to the exterior boundary value problem [1]. Kleinman also studied in cooperation with other scientists the low-frequency electromagnetic scattering for scatterers of special shapes. A posteriori bounds to the error and various properties of the potential solution are derived by D. Jones [5]. Some important theorems for multiple scattering of electromagnetic waves are proved by V. Twersky [12]. The low-frequency scattering of sound waves is examined in [2], and the corresponding theory of elastic wave scattering is studied in [3].

The purpose of this paper is to present a systematic and integrated theory for the scattering of an electromagnetic wave by a dielectric scatterer which contains a perfect conductor core. We conclude to a well-posed scattering problem for the electric field only, that is, we derive the equation which describes the phenomenon, the boundary conditions and the radiation condition. In this way we achieve to decouple the electromagnetic scattering problem. We observe that Maxwell's equations and the boundary conditions for the dielectric are invariant under the substitution

$$\mathbf{E}\to\mathbf{H},\quad \mathbf{H}\to-\mathbf{E},\quad \varepsilon\leftrightarrow\mu.$$

This property does not hold for the boundary condition on the perfect conductor. Hence, the magnetic field can be evaluated, in a similar way, by solving a well-posed scattering problem for the magnetic field using the above described substitution and suitable boundary condition on the core. Thus, we have the complete solution of the electromagnetic scattering problem.

We give the fundamental dyadic solution, and we construct the integral representation for the total, incident plus scattered, electric field. Based on the asymptotic form of the integral representation for the total field, we express the normalized scattering amplitude in a closed form relation. Using the definition of the scattering cross-section we derive a relation connecting the cross-section with the normalized scattering amplitude.

By expanding all the field quantities in power series of the wave number, we succeed to reduce the scattering problem to a sequence of potential problems which can be solved recursively by means of appropriate harmonic functions. Integral representations for every coefficient of the low-frequency expansion, as well as their asymptotic form, far away from the scatterer, are found. Besides, particular solutions of the corresponding potential problems, in low-frequency region, are derived, based on the asymptotic form of the integral representation.

We prove that the leading term approximation, in the low-frequency region, of the normalized scattering amplitude is proportional to the third power of the wave number, and that the leading term of the scattering cross-section is proportional to the fourth power of the wave number.

2. General consideration of the problem

(a) Equations of electrodynamics and the fundamental dyadic solution. We consider the propagation of electromagnetic waves in a medium. As it is well known, the electric field $\mathscr{E}(\mathbf{r}, t)$ and the magnetic field $\mathscr{H}(\mathbf{r}, t)$ are governed by the Maxwell's equations:

$$(1)\qquad \nabla\times\mathscr{E}(\mathbf{r}, t)=-\mu\frac{\partial\mathscr{H}(\mathbf{r}, t)}{\partial t},\quad \nabla.\mathscr{E}(\mathbf{r}, t)=0$$

$$(2)\qquad \nabla\times\mathscr{H}(\mathbf{r}, t)=\varepsilon\frac{\partial\mathscr{E}(\mathbf{r}, t)}{\partial t}+\sigma\mathscr{E}(\mathbf{r}, t),\quad \nabla.\mathscr{H}(\mathbf{r}, t)=0,$$

where ε is the dielectric constant, μ the permeability, and σ the conductivity of the medium.
Assuming, without any loss of generality, harmonic time dependence for the electric and the magnetic field, we have

$$\mathscr{E}(\mathbf{r},\ t)=\mathbf{E}(\mathbf{r})e^{-i\omega t}\quad \mathscr{H}(\mathbf{r},\ t)=\mathbf{H}(\mathbf{r})e^{-i\omega t},$$

where ω is the angular frequency. In what follows, we can suppress the time dependence from all field quantities and then, for steady-state waves in not conductive media ($\sigma=0$), the equations corresponding to Eqs. (1, 2) are

$$\nabla\times\mathbf{E}(\mathbf{r})=\mu i\omega\ \mathbf{H}(\mathbf{r}),\quad \nabla.\mathbf{E}(\mathbf{r})=0 \tag{3}$$

$$\nabla\times\mathbf{H}(\mathbf{r})=-\varepsilon i\omega\ \mathbf{E}(\mathbf{r}),\quad \nabla.\mathbf{H}(\mathbf{r})=0. \tag{4}$$

Elimination of the $\mathbf{H}(\mathbf{r})$ field in Eq. (3) by substitution of Eq. (4) gives us the following equations for the electric field

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r})-k^2\,\mathbf{E}(\mathbf{r})=\mathbf{0},\quad \nabla.\mathbf{E}(\mathbf{r})=0, \tag{5}$$

where k is the propagation constant $k=\omega/c$ and c is the phase velocity $c=(\mu\varepsilon)^{-1/2}$. Similarly, the elimination of the $\mathbf{E}(\mathbf{r})$ field in Eq. (4) gives us an equation of the same type as Eq. (5) for the magnetic field only.
The fundamental dyadic solution $\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')$ satisfies the equation

$$\nabla\times\nabla\times\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')-k^2\,\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')=-4\pi\tilde{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}'), \tag{6}$$

where $\mathbf{r}$ is the position of the observation point, $\mathbf{r}'$ is the position of the source point, $\tilde{\mathbf{I}}$ is the identity dyadic, and $\delta(\mathbf{r},\ \mathbf{r}')$ is the three dimensional delta function. Eq. (6) can be written in the form

$$(\nabla^2+k^2)\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')=4\pi\left(\tilde{\mathbf{I}}+\frac{\nabla\nabla}{k^2}\right)\delta(\mathbf{r}-\mathbf{r}'). \tag{7}$$

Setting

$$\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')=-\left(\tilde{\mathbf{I}}+\frac{\nabla\nabla}{k^2}\right)G(\mathbf{r},\ \mathbf{r}'), \tag{8}$$

Eq. (7) takes the form $(\tilde{\mathbf{I}}+\frac{\nabla\nabla}{k^2})[(\nabla^2+k^2)\,G(\mathbf{r},\ \mathbf{r}')+4\pi\delta(\mathbf{r},\ \mathbf{r}')]=\tilde{\mathbf{0}}$. Considering that $G(\mathbf{r},\ \mathbf{r}')$ must satisfy the following equation

$$(\nabla^2+k^2)\,G(\mathbf{r},\ \mathbf{r}')=-4\pi\delta(\mathbf{r}-\mathbf{r}'), \tag{9}$$

the tensor $\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')$ satisfies Eq. (6).
From Eqs. (7, 8, 9) we conclude that

The symbol "$\sim$" on the top of a capital letter denotes a dyadic.

$$\tilde{\Gamma}(\mathbf{r},\ \mathbf{r}')=-(\tilde{\mathbf{I}}+\frac{\nabla\nabla}{k^2})\ \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\ .$$

Finally, the fundamental dyadic solution is given by

$$\tilde{\Gamma}(\mathbf{r},\ \mathbf{r}')=\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{k^2|\mathbf{r}-\mathbf{r}'|^3}\{k^2(\mathbf{r}-\mathbf{r}')\otimes(\mathbf{r}-\mathbf{r}')+(1-ik|\mathbf{r}-\mathbf{r}'|)\,(\tilde{\mathbf{I}}-3\frac{(\mathbf{r}-\mathbf{r}')\otimes(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2})\}$$

$$-\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{\mathbf{I}}.$$

(b) The scattering problem. We consider a scatterer, which is a bounded, convex and closed subset of R^3, with a smooth boundary S_1. We assume that the scatterer is a dielectric with dielectric constant ε_2 and permeability μ_2 which lies in an infinite homogeneous isotropic medium V_1 with dielectric constant ε_1 and permeability μ_1. We also consider that entirely within the scatterer lies a perfect conductor with smooth boundary S_0, that is, we consider a scatterer with a core. We call V_2 the region between the surfaces S_1 and S_0.

We assume that a plane electromagnetic wave $\mathbf{E}^{in}$, $\mathbf{H}^{in}$ is incident upon the obstacle. If $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ are the scattered electric and magnetic waves, respectively, and $\mathbf{E}_i(\mathbf{r})$, $\mathbf{H}_i(\mathbf{r})$ the total waves for the spaces V_i, the vector fields $\mathbf{E}^{in}(\mathbf{r})$, $\mathbf{E}(\mathbf{r})$, $\mathbf{E}_i(\mathbf{r})$, $\mathbf{H}^{in}(\mathbf{r})$, $\mathbf{H}_i(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ must satisfy the equations

$$\nabla\times\nabla\times\mathbf{w}(\mathbf{r})-K_i^2\mathbf{w}(\mathbf{r})=\mathbf{0},\quad \mathbf{r}\in V_i,\quad i=1,\ 2 \tag{10}$$
$$\nabla\,.\,\mathbf{w}(\mathbf{r})=0,$$

where

$$K_i^2=\omega^2\mu_i\,\varepsilon_i. \tag{11}$$

The boundary conditions for the electric field on the surface of the dielectric S_1 are given by the equations

$$\left.\begin{aligned}\hat{\mathbf{n}}\times\mathbf{E}_1(\mathbf{r}')&=\hat{\mathbf{n}}\times\mathbf{E}_2(\mathbf{r}')\\ \hat{\mathbf{n}}\times(\nabla\times\mathbf{E}_1(\mathbf{r}'))&=\frac{\mu_1}{\mu_2}\hat{\mathbf{n}}\times(\nabla\times\mathbf{E}_2(\mathbf{r}'))\end{aligned}\right\},\quad \mathbf{r}'\in S_1. \tag{12}$$

For the magnetic field the boundary conditions on S_1 can be derived by Eqs. (12) substituting $\mathbf{E}_i$ with $\mathbf{H}_i$ and μ_i with ε_i. On the surface of the perfect conductor S_0 the following equations must be satisfied:

$$\left.\begin{aligned}\hat{\mathbf{n}}\times\mathbf{E}_2(\mathbf{r}')&=\mathbf{0}\\ \hat{\mathbf{n}}\times\nabla\times\mathbf{H}_2(\mathbf{r}')&=\mathbf{0}\end{aligned}\right\},\quad \mathbf{r}'\in S_0. \tag{13}$$

On the surface S_1 of the dielectric the boundary condition $\hat{\mathbf{n}}\ .\mathbf{E}_1(\mathbf{r}')$

$=\frac{\varepsilon_2}{\varepsilon_1}\hat{\mathbf{n}}.\mathbf{E}_2(\mathbf{r}')$, $\mathbf{r}'\in S_1$ must also be satisfied. This condition is a consequence of the integral relation

$$\int_S \hat{\mathbf{n}}.\mathbf{E}(\mathbf{r}')\,dS(\mathbf{r}')=0, \tag{14}$$

where S is any closed surface surrounding a free charges region. Equation (14) can easily be derived by Maxwell's equation [11].

The incident wave has the form

$$\mathbf{E}^{in}(\mathbf{r})=\hat{\mathbf{b}}\,e^{ik_1\hat{\mathbf{k}}.\mathbf{r}},\quad \mathbf{H}^{in}(\mathbf{r})=(\hat{\mathbf{k}}\times\hat{\mathbf{b}})\left(\frac{\varepsilon_1}{\mu_1}\right)^{1/2}e^{ik_1\hat{\mathbf{k}}.\mathbf{r}},$$

where $\hat{\mathbf{k}}$ is the unit vector in the direction of propagation, $\hat{\mathbf{b}}$ is the unit polarization vector for the electric field, $\hat{\mathbf{b}}.\ \hat{\mathbf{k}}=0$, and $\hat{\mathbf{k}}\times\hat{\mathbf{b}}$ is the unit polarization vector for the magnetic field.

The scattered fields $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$ satisfy the radiation condition due to Müller-Silver

$$\lim_{r\to\infty}\mathbf{r}\times\{\nabla\times\begin{pmatrix}\mathbf{E}(\mathbf{r})\\ \mathbf{H}(\mathbf{r})\end{pmatrix}\}+ik_1 r\begin{pmatrix}\mathbf{E}(\mathbf{r})\\ \mathbf{H}(\mathbf{r})\end{pmatrix}=\mathbf{0} \tag{15}$$

uniformly over all directions.

So we achieve to define two well-posed scattering problems one for the electric field and one for the magnetic field, which can be solved independently.

3. Integral representation

In this section we will construct the integral representation of the electric field for the region outside the scatterer. As it is well known, the integral representations of the scattering problems contain all the informations about the boundary conditions, the radiation conditions, the source conditions, and the P. D. E. which govern the phenomena. We examine the electric field only because as we have already seen the determination of the electric field suffices for the evalution of the total electromagnetic field.

The scattered field $\mathbf{E}(\mathbf{r})$ admits the following representation in terms of the electric field only [see Appendix]:

$$\mathbf{E}(\mathbf{r})=\frac{1}{4\pi}\int_{S_1}\{\nabla\times\mathbf{E}(\mathbf{r}').(\hat{\mathbf{n}}\times\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}'))-(\hat{\mathbf{n}}\times\mathbf{E}(\mathbf{r}')).\nabla_{r'}\times\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')\}\,dS(\mathbf{r}'),$$

where the radiation condition given by Eq. (15) is included. Since the incident electric field belongs to the kernel of the differential operator of the problem, we have that

$$\int_{S_1}\{\nabla\times\mathbf{E}^{in}(\mathbf{r}').(\hat{\mathbf{n}}\times\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}'))-(\hat{\mathbf{n}}\times\mathbf{E}^{in}(\mathbf{r}')).\nabla_{r'}\times\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')\}\,dS(\mathbf{r}')=\mathbf{0}.$$

Using the last two equations and the fact that the total field $\mathbf{E}_1(\mathbf{r})$ for the region V_1 is the superposition of the incident and the scattered field we conclude that

$$\mathbf{E}_1(\mathbf{r}) = \mathrm{E}^{in}(\mathbf{r}) + \frac{1}{4\pi}\int_{S_1} \{\nabla \times \mathbf{E}_1(\mathbf{r}') . (\hat{\mathbf{n}} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')) - (\hat{\mathbf{n}} \times \mathbf{E}_1(\mathbf{r}')) . \nabla_{r'} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')\}\, dS(\mathbf{r}').$$

Introducing the boundary conditions given by Eqs. (12), we have

$$\mathbf{E}_1(\mathbf{r}) = \mathbf{E}^{in}(\mathbf{r}) + \frac{1}{4\pi}\int_{S_1} \{\frac{\mu_1}{\mu_2}\nabla \times \mathbf{E}_2(\mathbf{r}') . (\hat{\mathbf{n}} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')) - (\hat{\mathbf{n}} \times \mathbf{E}_2(\mathbf{r}')) . \nabla_{r'} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')\}\, dS(\mathbf{r}'). \tag{16}$$

In order to incorporate the boundary condition which is satisfied on the surface of the core we work as follows.

First, applying the dyadic form of Green identity [11] on $\mathbf{E}_2(\mathbf{r})$, $\tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')$ in V_2 and using that $\mathbf{E}_2(\mathbf{r})$ and $\tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')$ are solutions of Eq. (10) in V_2 and V_1, respectively, we conclude that

$$\int_{S_1 - S_0} \{\nabla \times \mathbf{E}_2(\mathbf{r}') . (\hat{\mathbf{n}} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')) - (\hat{\mathbf{n}} \times \mathbf{E}_2(\mathbf{r}')) . \nabla_{r'} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')\}\, dS(\mathbf{r}') = (k_1^2 - k_2^2)\int_{V_2} \mathbf{E}_2(\mathbf{r}') . \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')\, dU(\mathbf{r}'). \tag{17}$$

Second, using the dyadic form of Gauss theorem, we have the relation

$$\int_{S_1 - S_0} (\hat{\mathbf{n}} \times \mathbf{E}_2(\mathbf{r}')) . \nabla_{r'} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')\, dS(\mathbf{r}') = \int_{V_2} \nabla \times \mathbf{E}_2(\mathbf{r}') . \nabla_{r'} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')\, dU(\mathbf{r}') - k_1^2 \int_{V_2} \mathbf{E}_2(\mathbf{r}') . \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')\, dU(\mathbf{r}'). \tag{18}$$

Third, we introduce the boundary condition satisfied on the surface of the perfect conductor S_0 given by Eq. (13) in Eqs. (17, 18).

Finally, using the above-described substitution, we can replace the surface integrals on S_1 in Eq. (16) with surface integrals on S_0 and volume integrals in V_2. Taking into account the relation given by Eq. (11) in Eq. (16), we conclude that

$$\mathbf{E}_1(\mathbf{r}) = \mathbf{E}^{in}(\mathbf{r}) + \frac{1}{4\pi}\frac{\mu_1}{\mu_2}\int_{S_0} \nabla \times \mathbf{E}_2(\mathbf{r}') . (\hat{\mathbf{n}} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}'))\, dS(\mathbf{r}') - \frac{1}{4\pi}\int_{V_2} \{(\frac{\varepsilon_2}{\varepsilon_1} - 1) k_1^2 \mathbf{E}_2(\mathbf{r}') . \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}') + (1 - \frac{\mu_1}{\mu_2}) \nabla \times \mathbf{E}_2(\mathbf{r}') . \nabla_{r'} \times \tilde{\mathbf{\Gamma}}(\mathbf{r}, \mathbf{r}')\}\, dU(\mathbf{r}'). \tag{19}$$

So, we achieve to express the electric field $\mathbf{E}_1(\mathbf{r})$ in terms of a surface integral on S_0 and a volume integral of the region contained between S_1 and S_0 of the electric field $\mathbf{E}_2(\mathbf{r})$.

4. Far field behaviour

The scattering theory deals with the influence of the discontinuities of the medium to the wave propagation. The effect of the scatterer on the propagation of the incident wave is measured by the amount of energy that the scatterer receives from the incident wave and reradiates in all directions. The knowledge of the scattering cross-section is of specific interest as a measure of the energy disturbance caused by the scatterer. Another measure which also is of interest is the normalized scattering amplitude because it describes the behaviour of the scattered wave in the region of radiation and through it we also succeed to evaluate the scattering cross-section. Using the above magnitudes, to a certain extend, we can explain the mechanism of energy transfered during the scattering process.

a) Normalized scattering amplitude. We define as normalized or dimensionless scattering amplitude the coefficient of the zeroth order spherical Hankel function of the first kind in the asymptotic form of the scattered field. So we have that it satisfies the relation

$$\mathbf{E}(\mathbf{r})=\mathbf{g}(\hat{\mathbf{r}},\ \hat{\mathbf{k}})\,h(k_1 r)+0\left(\frac{1}{r^2}\right),\quad r\to\infty \tag{20}$$

where $\hat{\mathbf{r}}$, $\hat{\mathbf{k}}$ are the unit vectors in the directions of observation and propagation, respectively.
In order to conclude in closed form for $\mathbf{g}(\hat{\mathbf{r}},\hat{\mathbf{k}})$ we use the asymptotic form of the integral representation for the scattered electric field.
Using the asymptotic relations $|\mathbf{r}-\mathbf{r}'|=\mathbf{r}-\hat{\mathbf{r}}.\mathbf{r}'+0(1/r)$, $r\to\infty$, $(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|=\hat{\mathbf{r}}+0(1/r)$, $r\to\infty$, we conclude to the following asymptotic forms for the dyadic $\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')$ and $\nabla_{r'}\times\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')$, as $r\to\infty$:

$$\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')=-(\tilde{\mathbf{I}}-\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\,e^{-ik_1\hat{\mathbf{r}}.\mathbf{r}'}\frac{e^{ik_1 r}}{r}+0\left(\frac{1}{r^2}\right)$$

$$\nabla_{r'}\times\tilde{\mathbf{\Gamma}}(\mathbf{r},\ \mathbf{r}')=\tilde{\mathbf{I}}\times\hat{\mathbf{r}}\,e^{-ik_1\hat{\mathbf{r}}.\mathbf{r}'}\frac{ik_1 e^{ik_1 r}}{r}+0\left(\frac{1}{r^2}\right).$$

Thus, introducing the two last equations in Eq. (19) we have that the scattered field admits the following representation, as $r\to\infty$:

$$\begin{aligned}\mathbf{E}(\mathbf{r})=\mathbf{E}_1(\mathbf{r})-\mathbf{E}^{in}(\mathbf{r})=\frac{1}{4\pi}\Big\{&-ik_1\frac{\mu_1}{\mu_2}\int_{S_0}\nabla\times\mathbf{E}_2(\mathbf{r}').(\hat{\mathbf{n}}\times(\tilde{\mathbf{I}}-\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}))\,e^{-ik_1\hat{\mathbf{r}}.\mathbf{r}'}dS(\mathbf{r}')\\ &+\left(\frac{\varepsilon_2}{\varepsilon_1}-1\right)ik_1^3\int_{V_2}\mathbf{E}_2(\mathbf{r}').(\tilde{\mathbf{I}}-\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\,e^{-ik_1\hat{\mathbf{r}}.\mathbf{r}'}dU(\mathbf{r}')\\ &+\left(1-\frac{\mu_1}{\mu_2}\right)k_1^2\int_{V_2}\nabla\times\mathbf{E}_2(\mathbf{r}').(\tilde{\mathbf{I}}\times\hat{\mathbf{r}})\,e^{-ik_1\hat{\mathbf{r}}.\mathbf{r}'}dU(\mathbf{r}')\Big\}\cdot\frac{e^{ik_1 r}}{ik_1 r}+0\left(\frac{1}{r^2}\right).\end{aligned} \tag{21}$$

Thus, from Eqs. (20, 21) the normalized scattering amplitude is given by

$$\mathbf{g}(\hat{\mathbf{r}}, \hat{\mathbf{k}})=\frac{1}{4\pi}\{-ik_1\frac{\mu_1}{\mu_2}\int_{S_0}\nabla\times\mathbf{E}_2(\mathbf{r}').(\hat{\mathbf{n}}\times(\tilde{\mathbf{I}}-\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}))e^{-ik_1\hat{\mathbf{r}}.\mathbf{r}'}dS(\mathbf{r}')$$

(22)

$$+(\frac{\varepsilon_2}{\varepsilon_1}-1)ik_1^3\int_{V_2}\mathbf{E}_2(\mathbf{r}').(\tilde{\mathbf{I}}-\hat{\mathbf{r}}\otimes\hat{\mathbf{r}})e^{-ik_1\hat{\mathbf{r}}.\mathbf{r}'}dU(\mathbf{r}')$$

$$+(1-\frac{\mu_1}{\mu_2})k_1^2\int_{V_2}\nabla\times\mathbf{E}_2(\mathbf{r}').(\tilde{\mathbf{I}}\times\mathbf{r})e^{-ik_1\mathbf{r}.\mathbf{r}'}dU(\mathbf{r}')\}.$$

(b) Scattering cross-section. We define as scattering cross-section the ratio of the time average rate (over a period) at which energy is scattered by the body to the corresponding time average rate at which the energy of the incident wave crosses a unit area normal to the direction of propagation. Because of the particular normalization of the scattering cross-section, as total scattered energy measured in incident energy per unit area, it has the dimensions of area and this fact justifies the characterization "cross-section".

In order to evaluate the scattering cross-section in electrodynamics we use the Poynting vector which is defined as the vector flux Re $\mathbf{E}(\mathbf{r})e^{-i\omega t}\times$ Re $\mathbf{H}(\mathbf{r})e^{-i\omega t}$. Using the above definition for the scattering cross-section, we have that the time averaged Poynting vector is given by

$$\frac{1}{T}\int_0^T \mathrm{Re}\ \mathbf{E}(\mathbf{r})e^{-i\omega t}\times \mathrm{Re}\mathbf{H}(\mathbf{r})e^{-i\omega t}dt=\frac{1}{2}\mathrm{Re}\{\frac{\mathbf{E}^*(\mathbf{r})\times\nabla\times\mathbf{E}(\mathbf{r})}{i\omega\mu_1}\},$$

where $\mathbf{E}^*(\mathbf{r})$ is the complex conjugate of $\mathbf{E}(\mathbf{r})$.

Thus, the total energy scattered by the scatterer is given by a surface integral of the scattered vector flux over a large sphere S_r, $r\gg 1$ enclosing the scatterer. We can do this substitution because the space between the surface of the scatterer and the surface of the sphere has no singularities. So, we can push the sphere so far that the asymptotic field can be used. Besides, the radial symmetry of S_r is also exploited. Finally, we have

$$\frac{1}{2}Re\int_{S_r}\{\frac{\mathbf{E}^*(\mathbf{r})\times\nabla\times\mathbf{E}(\mathbf{r})}{i\omega\mu_1}\}.\hat{\mathbf{n}}\,dS(\mathbf{r})$$

(23)

$$=\frac{1}{2}\mathrm{Re}\int_{|\hat{\mathbf{r}}|=1}\{(\frac{\mathbf{g}^*(\hat{\mathbf{r}},\hat{\mathbf{k}})e^{-ik_1r}}{-ik_1r}+0(\frac{1}{r^2}))\times(\nabla\times\frac{\mathbf{g}(\hat{\mathbf{r}},\hat{\mathbf{k}})e^{ik_1r}}{ik_1r}+0(\frac{1}{r^2}))\}$$

$$\cdot\frac{\hat{\mathbf{r}}}{i\mu_1\omega}r^2d\Omega(\hat{\mathbf{r}})=\frac{1}{2\omega\mu_1k_1}\int_{|\hat{\mathbf{r}}|=1}|\mathbf{g}(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2d\Omega(\hat{\mathbf{r}}).$$

Similarly, the incident energy which crosses the unit area, can be derived from the incident vector flux, and we take

$$\frac{1}{2}\left|\operatorname{Re}\frac{(\mathbf{E}^{in}(\mathbf{r}))^{*}\times\nabla\times\mathbf{E}^{in}(\mathbf{r})}{i\omega\mu_1}\right|=\frac{k_1}{2\omega\mu_1}. \tag{24}$$

So, by definition, using Eqs. (23, 24), we conclude that the scattering cross-section is equal to

$$\sigma=\frac{1}{k_1^2}\int_{|\hat{\mathbf{r}}|=1}|\mathbf{g}(\hat{\mathbf{r}},\hat{\mathbf{k}})|^2\,d\Omega(\hat{\mathbf{r}}). \tag{25}$$

For the normalized scattering amplitude the reciprocity and the scattering theorem have been proved by V. Twersky [12]. Also, for the scattering cross-section holds the well-known optical theorem [4, 12].

5. Derivation of the approximations

(a) The electric field. Since the solutions of the vector Helmholtz equation considered as functions of the wavenumber are analytic in a neighborhood of zero [13], we can expand them in a convergent power series. So we have for the electric fields $\mathbf{E}_i(\mathbf{r})$, $i=1, 2$ the expansions

$$\mathbf{E}_i(\mathbf{r})=\sum_{n=0}^{\infty}\frac{(ik_1)^n}{n!}\Phi_n^{(i)}(\mathbf{r}),\ \mathbf{r}\in V_i,\quad i=1,\ 2. \tag{26}$$

Inserting the expansions of Eq. (26) into Eq. (10) and equating equal powers of k_1, the following sequence of partial differential equations is obtained:

$$\begin{gathered}\nabla\times\nabla\times\Phi_n^{(i)}(\mathbf{r})+n(n-1)m_i\Phi_{n-2}^{(i)}(\mathbf{r})=\mathbf{0}\\ \nabla\cdot\Phi_n^{(i)}(\mathbf{r})=0,\quad n=0,\ 1,\ 2,\ldots,\quad \mathbf{r}\in V_i,\quad i=1,\ 2,\end{gathered} \tag{27}$$

where

$$m_i=\begin{cases}1, & \text{for } i=1\\ \dfrac{\mu_2\varepsilon_2}{\mu_1\varepsilon_1}, & \text{for } i=2.\end{cases}$$

The boundary conditions given by Eqs. (12, 13) are transformed into the boundary conditions

$$\hat{\mathbf{n}}\times\Phi_n^{(1)}(\mathbf{r}')=\hat{\mathbf{n}}\times\Phi_n^{(2)}(\mathbf{r}'),\quad \hat{\mathbf{n}}\times(\nabla\times\Phi_n^{(1)}(\mathbf{r}'))=\frac{\mu_1}{\mu_2}\hat{\mathbf{n}}\times(\nabla\times\Phi_n^{(2)}(\mathbf{r}')),\ \mathbf{r}'\in S_1$$

on the surface of the dielectric and

$$\hat{\mathbf{n}} \times \Phi_n^{(2)}(\mathbf{r}') = \mathbf{0}, \quad \mathbf{r}' \in S_0$$

on the surface of the perfect conductor.

For the incident electric wave we have the convergent power series of k_1:

$$\mathbf{E}^{in}(\mathbf{r}) = \hat{\mathbf{b}} \sum_{n=0}^{\infty} \frac{(ik_1)^n}{n!} (\hat{\mathbf{k}} . \mathbf{r})^n. \tag{28}$$

The fundamental dyadic $\tilde{\Gamma}(\mathbf{r}, \mathbf{r}')$ admits the expansion

$$\tilde{\Gamma}(\mathbf{r}, \mathbf{r}') = \sum_{n=0}^{\infty} \frac{(ik_1)^n}{n!} \tilde{\gamma}_n(\mathbf{r}, \mathbf{r}'), \tag{29}$$

where $\tilde{\gamma}_n(\mathbf{r}, \mathbf{r}') = -\frac{|\mathbf{r}-\mathbf{r}'|^{n-1}}{n+2} \{(n+1)\tilde{\mathbf{I}} - (n-1)\frac{(\mathbf{r}-\mathbf{r}')\otimes(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\}$ and the dyadic $\nabla_{\mathbf{r}'} \times \tilde{\Gamma}(\mathbf{r}, \mathbf{r}')$ has the expansion

$$\nabla_{\mathbf{r}'} \times \tilde{\Gamma}(\mathbf{r}, \mathbf{r}') = \sum_{n=0}^{\infty} \frac{(ik_1)^n}{n!} \tilde{\delta}_n(\mathbf{r}, \mathbf{r}'), \tag{30}$$

where $\tilde{\delta}_n(\mathbf{r}, \mathbf{r}') = (n-1)\,|(\mathbf{r}-\mathbf{r}'|^{n-3}(\mathbf{r}-\mathbf{r}') \times \tilde{\mathbf{I}}$. Substituting the series expansions given by the relations (26, 28, 29, 30) into the integral formula given by Eq. (19), reordering terms, interchanging summation, and integration, and equating like powers of ik_1, we have the following result:

$$\Phi_n^{(1)}(\mathbf{r}) = \hat{\mathbf{b}}(\hat{\mathbf{k}} . \mathbf{r})^n + \frac{1}{4\pi}\frac{\mu_1}{\mu_2} \sum_{p=0}^{n} \binom{n}{p} \int_{S_0} (\nabla \times \Phi_p^{(2)}(\mathbf{r}'))$$

$$. (\hat{\mathbf{n}} \times \tilde{\gamma}_{n-p}(\mathbf{r}, \mathbf{r}'))\, dS(\mathbf{r}') + \frac{1}{4\pi} \sum_{p=0}^{n} \binom{n}{p} \int_{V_2} \{(\frac{\varepsilon_2}{\varepsilon_1} - 1)\, p(p-1)\Phi_{p-2}^{(2)}(\mathbf{r}') \tag{31}$$

$$. \tilde{\gamma}_{n-p}(\mathbf{r}, \mathbf{r}') - (1 - \frac{\mu_1}{\mu_2}) \nabla \times \Phi_p^{(2)}(\mathbf{r}') . \tilde{\delta}_{n-p}(\mathbf{r}, \mathbf{r}')\}\, dU(\mathbf{r}').$$

In order to have the asymptotic integral representation for the n-th coefficient we use the fact that $\tilde{\gamma}_0(\mathbf{r}, \mathbf{r}') = 0(1/r)$, $\tilde{\delta}_0(\mathbf{r}, \mathbf{r}') = 0(1/r)$, $r \to \infty$. Thus, the asymptotic representation can be derived from the integral relation (31) if we omit the n-th term in the right hand side, which is of the order of $1/r$. By substitution of the nonvanishing part of the asymptotic representation in Eq. (27), as $r \to \infty$, we conclude that it provide a particular solution of the nonhomogeneous equation. So the solution of Eq. (27) can be written as the sum of this particular solution plus a solution of the homogeneous equation, which must also be of the order of $1/r$, as $r \to \infty$.

In low-frequencies the magnetic field assumes the series expansions [14]:

$$\mathbf{H}_i(\mathbf{r}) = \sum_{n=0}^{\infty} \frac{(ik_1)^n}{n!} \Psi_n^{(i)}(\mathbf{r}), \quad \mathbf{r} \in V_i, \quad i=1,\ 2.$$

In order to evaluate the coefficient $\Psi_n^{(i)}(\mathbf{r})$ from Maxwell's equations and the series expansions given by Eq. (26) we conclude that

$$\nabla \times \Phi_n^{(i)}(\mathbf{r}) = \left(m_i \frac{\mu_i}{\varepsilon_i}\right)^{1/2} n \Psi_{n-1}^{(i)}(\mathbf{r}), \quad \mathbf{r} \in V_i, \quad i=1,\ 2. \tag{32}$$

(b) The scattering amplitude. In order to derive the low-frequency expansion for the scattering amplitude the expansion

$$e^{-ik_1\hat{\mathbf{r}}.\mathbf{r}'} = \sum_{n=0}^{\infty} (-1)^n (ik_1)^n (\hat{r}\ .r')^n/(n\ !)$$

and the series given by Eq. (26) are substituted into Eq. (22). So, we obtain

$$\mathbf{g}(\hat{\mathbf{r}},\ \hat{\mathbf{k}}) = -\frac{1}{4\pi}\frac{\mu_1}{\mu_2} \sum_{n=0}^{\infty} \frac{(ik_1)^{n+1}}{n!} \sum_{p=0}^{n} \binom{n}{p} (-1)^p \int_{S_0} \nabla \times \Phi_{n-p}^{(2)}(\mathbf{r}')$$

$$.(\hat{\mathbf{n}} \times (\tilde{\mathbf{I}} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}))\ (\hat{\mathbf{r}}\,.\,\mathbf{r}')^p\, dS(\mathbf{r}') - \frac{1}{4\pi}\left(\frac{\varepsilon_2}{\varepsilon_1} - 1\right) \sum_{n=0}^{\infty} \frac{(ik_1)^{n+3}}{n!}$$

$$\sum_{p=0}^{n} \binom{n}{p} (-1)^p \int_{V_2} \Phi_{n-p}^{(2)}(\mathbf{r}').(\tilde{\mathbf{I}} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}})\ (\hat{\mathbf{r}}\,.\,\mathbf{r}')^p\, dU(\mathbf{r}')$$

$$-\frac{1}{4\pi}\left(1 - \frac{\mu_1}{\mu_2}\right) \sum_{n=0}^{\infty} \frac{(ik_1)^{n+2}}{n!} \sum_{p=0}^{n} \binom{n}{p} (-1)^p \int_{V_2} \nabla \times \Phi_{n-p}^{(2)}(\mathbf{r}').(\tilde{\mathbf{I}} \times \hat{\mathbf{r}})\ (\hat{\mathbf{r}}\,.\,\mathbf{r}')^p\, dU(\mathbf{r}').$$

In order to find the leading term for the scattering amplitude we observe that the coefficient of (ik_1) is a surface integral on S_0 which contains the term $\nabla \times \Phi_0^{(2)}(\mathbf{r})$. Because $\Phi_0^{(2)}(\mathbf{r})$ is solution of the electrostatic problem it is the gradient of a function. So, we have that $\nabla \times \Phi_0^{(2)}(\mathbf{r})$ is equal to zero. Thus, the coefficient of (ik_1) is equal to zero. The coefficient of $(ik_1)^2$ is the surface integral $1/4\pi \int_{S_0} \nabla \times \Phi_1^{(2)}(\mathbf{r}')$ $.(\hat{\mathbf{n}} \times (\tilde{\mathbf{I}} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}))\, dS(\mathbf{r}')$ (another volume integral in V_2 containing $\nabla \times \Phi_0^{(2)}(\mathbf{r})$ is equal to zero). From Maxwell's Eq. (3) we have that the rotation of the electric field is proportional to the magnetic field. So, the integral which includes $\nabla \times \Phi_1^{(2)}(\mathbf{r})$ is proportional to a surface integral of the form $\int_{S_0} \Psi_0^{(2)}(\mathbf{r}').\hat{\mathbf{n}}\, dS(\mathbf{r}')$. But for any closed surface surrounding a free charges region integrals of the electric or the magnetic field of this type are equal to zero [11]. So, the coefficient of the second order term in low-frequency expansion for the scattering amplitude is equal to zero. Thus, the normalized scattering amplitude is of $0(k_1^3)$, as $k_1 \to 0$. In particular, the leading term approximation as $k_1 \to 0$ is

$$\mathbf{g}(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = (ik_1)^3 \{ -\frac{1}{8\pi}\frac{\mu_1}{\mu_2}\int_{S_0} \nabla \times \Phi_2^{(2)}(\mathbf{r}').(\hat{\mathbf{n}} \times (\tilde{\mathbf{I}} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}))dS(\mathbf{r}')$$

$$+\frac{1}{4\pi}\frac{\mu_1}{\mu_2}\int_{S_0} \nabla \times \Phi_0^{(2)}(\mathbf{r}').(\hat{\mathbf{n}} \times (\tilde{\mathbf{I}} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}})).(\hat{\mathbf{r}}.\mathbf{r}')dS(\mathbf{r}')$$

$$-\frac{1}{4\pi}(\frac{\varepsilon_2}{\varepsilon_1} - 1)\int_{V_2} \Phi_0^{(2)}(\mathbf{r}').(\tilde{\mathbf{I}} - \hat{\mathbf{r}}\otimes\hat{\mathbf{r}})\,dU(\mathbf{r}')$$

$$-\frac{1}{4\pi}(1 - \frac{\mu_1}{\mu_2})\int_{V_2} \nabla \times \Phi_1^{(2)}(\mathbf{r}').(\tilde{\mathbf{I}} \times \hat{\mathbf{r}})\,dU(\mathbf{r}')\} + 0(k_1^4), \quad k_1 \to 0.$$

(c) The scattering cross-section. In order to evaluate the leading term approximation for the scattering cross-section we introduce in Eq. (25) the form of $\mathbf{g}(\hat{\mathbf{r}}, \mathbf{k})$, and so we have that the leading term approximation for the scattering cross-section in low-frequency region is given by the equation

$$\sigma = k_1^4\{\frac{1}{24\pi}\frac{\mu_1^2}{\mu_2^2}|\int_{S_0} \nabla \times \Phi_2^{(2)}(\mathbf{r}') \times \hat{\mathbf{n}}dS(\mathbf{r}')|^2$$

$$+\frac{1}{60\pi}\frac{\mu_1^2}{\mu_2^2}\|\int_{S_0} \mathbf{r}'\otimes(\nabla \times \Phi_1^{(2)}(\mathbf{r}') \times \hat{\mathbf{n}})dS(\mathbf{r}')\|^2 + \frac{7}{60\pi}\frac{\mu_1^2}{\mu_2^2}|\int_{S_0} (\nabla \times \Phi_1^{(2)}(\mathbf{r}') \times \hat{\mathbf{n}}) \,.\, \mathbf{r}'dS(\mathbf{r}')|^2$$

$$(33) \quad +\frac{1}{6\pi}(\frac{\varepsilon_2}{\varepsilon_1} - 1)^2|\int_{V_2} \Phi_0^{(2)}(\mathbf{r}')dU(\mathbf{r}')|^2 + \frac{1}{6\pi}(1 - \frac{\mu_1}{\mu_2})^2|\int_{V_2} \nabla \times \Phi_1^{(2)}(\mathbf{r}')dU(\mathbf{r}')|^2$$

$$+\frac{1}{6\pi}\frac{\mu_1}{\mu_2}(\frac{\varepsilon_2}{\varepsilon_1} - 1)\int_{S_0} \nabla \times \Phi_2^{(2)}(\mathbf{r}') \times \hat{\mathbf{n}}dS(\mathbf{r}') \,.\, \int_{V_2} \Phi_0^{(2)}(\mathbf{r}')dU(\mathbf{r}')$$

$$+\frac{1}{6\pi}\frac{\mu_1}{\mu_2}(1 - \frac{\mu_1}{\mu_2})\int_{S_0} (\nabla \times \Phi_1^{(2)}(\mathbf{r}') \times \hat{\mathbf{n}}) \times \mathbf{r}'dS(\mathbf{r}') \,.\, \int_{V_2} \nabla \times \Phi_1^{(2)}(\mathbf{r}')dU(\mathbf{r}')\} + 0(k_1^6),$$

where the norm of a dyadic is defined as $\|\mathbf{a}\otimes\mathbf{b}\|^2 = \Sigma_{i,j=1}^3(a_ib_j)^2$.

We derive Eq. (32) using the formulae

$$\int_{|\hat{\mathbf{r}}|=1} \hat{\mathbf{r}}d\Omega(\hat{\mathbf{r}}) = \mathbf{0}, \qquad \int_{|\hat{\mathbf{r}}|=1} \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}d\Omega(\hat{\mathbf{r}}) = \frac{4\pi}{3}\tilde{\mathbf{I}}, \qquad \int_{|\hat{\mathbf{r}}|=1} \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}d\Omega(\hat{\mathbf{r}}) = \mathbf{0}\otimes\mathbf{0}\otimes\mathbf{0},$$

$$\int_{|\hat{\mathbf{r}}|=1} \hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}\otimes\hat{\mathbf{r}}d\Omega(\hat{\mathbf{r}}) = \frac{4\pi}{15}\{\tilde{\mathbf{I}}\otimes\tilde{\mathbf{I}} + \sum_{i,j=1}^{3} \hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j\otimes\hat{\mathbf{e}}_i\otimes\hat{\mathbf{e}}_j + \sum_{i=1}^{3} \hat{\mathbf{e}}_i\otimes\tilde{\mathbf{I}}\otimes\hat{\mathbf{e}}_i\},$$

performing the necessary contractions, and omitting all terms that have been evaluated equal to zero.

6. Conclusions

In the preceding work we construct the integral representation for the solution of the electromagnetic scattering problem when the scatterer is a dielectric containing a perfect conductor core, and we develop a systematic procedure for the formulation of the above scattering problem in the low-frequency region.

Decoupling the electric from the magnetic field we construct integral representation for the electric field in terms of the electric field only, and we also give closed form for the normalized scattering amplitude.

Expanding in series of the wave number, we reduce the scattering problem in a series of well-posed potential problems which can be solved recursively. We give the integral representation for the n-th order coefficient of the electric field. The leading term approximation of the scaterring amplitude is proportional to k_1^3, as $k_1 \to 0$ and depends on Φ_0, Φ_1, Φ_2. The leading term approximation of the scattering cross-section is of the fourth order of the wave number, which is in agreement with Rayleigh's law.

In order to evaluate the magnetic field a similar problem can be constructed and solved. But in low-frequency the magnetic field coefficients can be directly evaluated by using Eq. (32). Finally, we will enumerate the special problems which can be obtained as degenerate cases of the most general present problem when there is a particular relation between the material constants or in geometrical degenerate cases.

a) When we have equality of the dielectric constants and the permeabilities of the spaces V_1 and V_2, then no scattering occurs on S_1 and the scatterer is the core S_0. The results in this case are the same as in the case where the scatterer is a perfect conductor. Considering also $S_0 \equiv S_1$ and $\varepsilon_2 = \mu_2 = 0$, we conclude to the same degenerate case.

b) When the material constants ε, μ are not equal but the volume of the core is equal to zero, we can derive the case where the scatterer is a dielectric.

Appendix

In order to construct the integral representation of the solution $\mathbf{u}(\mathbf{r})$ for an infinite region we use the dyadic form of Green theorem:

$$\text{(A.1)} \quad \begin{aligned} &\int_V \{\nabla \times \nabla \times \mathbf{u}(\mathbf{r}') \cdot \tilde{\Gamma}(\mathbf{r}, \mathbf{r}') - \mathbf{u}(\mathbf{r}') \cdot \nabla \times \nabla \times \tilde{\Gamma}(\mathbf{r}, \mathbf{r}')\}\, dU(\mathbf{r}') \\ &= \int_S \{(\hat{\mathbf{n}} \times \mathbf{u}(\mathbf{r}')) \cdot \nabla_{r'} \times \tilde{\Gamma}(\mathbf{r}, \mathbf{r}') - \nabla \times \mathbf{u}(\mathbf{r}') \cdot (\hat{\mathbf{n}} \times \tilde{\Gamma}(\mathbf{r}, \mathbf{r}'))\}\, dS(\mathbf{r}'), \end{aligned}$$

where $\hat{\mathbf{n}}$ is the exterior unit normal vector on S. Applying the above formula for a volume V bounded by the surface S of the scatterer and the surface of a sphere S_r, $r \to \infty$ and using the defining equations of $\mathbf{u}(\mathbf{r})$ and $\tilde{\Gamma}$ $(\mathbf{r}, \mathbf{r}')$, we get

$$\text{(A2)} \quad \begin{aligned} 4\pi\tilde{\mathrm{I}} \cdot \int_V \mathbf{u}(\mathbf{r}')\delta(\mathbf{r} - \mathbf{r}')\, dU(\mathbf{r}') = \int_{S+S_\infty} &\{(\hat{\mathbf{n}} \times \mathbf{u}(\mathbf{r}')) \cdot \nabla_{r'} \times \tilde{\Gamma}(\mathbf{r}, \mathbf{r}') \\ &- \nabla \times \mathbf{u}(\mathbf{r}') \cdot (\hat{\mathbf{n}} \times \tilde{\Gamma}(\mathbf{r}, \mathbf{r}'))\}\, dS(\mathbf{r}'). \end{aligned}$$

But for the sphere S_∞ we have that the surface integral is equal to zero by the radiation condition. Changing the normal on S from $\hat{\mathbf{n}}$ to $-\hat{\mathbf{n}}$, we get the integral representation

$$\text{(A.3)} \qquad \mathbf{u}(\mathbf{r})=\frac{1}{4\pi}\int_S \{\nabla\times\mathbf{u}(\mathbf{r}') \cdot (\hat{\mathbf{n}}\times\tilde{\Gamma}(\mathbf{r},\ \mathbf{r}'))-(\hat{\mathbf{n}}\times\mathbf{u}(\mathbf{r}')) \cdot \nabla_{r'}\times\tilde{\Gamma}(\mathbf{r},\ \mathbf{r}')\}dS(\mathbf{r}').$$

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