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Complete Subsets of C(X, Y) with Respect to Hausdorff Distance

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Presented by P. Kenderov

Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be metric spaces, and let h_ρ denote Hausdorff distance in $X\times Y$, induced by the metrix ρ on $X\times Y$, given by $\rho[(x_1, y_1), (x_2, y_2)]=\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$. We may regard $C(X, Y)$ as a metric subspace of the closed subsets of $X\times Y$ with distance h_ρ. Here we study completeness criteria for closed subsets of the function space.

Let $C(X, Y)$ denote the continuous functions from a metric space $\langle X, d_X\rangle$ to a metric space $\langle Y, d_Y\rangle$. We denote uniform distance in $C(X, Y)$ by d_1; that is,

$$d_1(f, g)=\sup_{x\in X} d_Y[f(x), g(x)].$$

Here we consider a related metric on $C(X, Y)$, which is equivalent to d_1 whenever X is compact [9]. Let h_ρ denote Hausdorff distance (see, e.g., [2] or [7]) between closed nonempty subsets of $X\times Y$, induced by the box metric ρ on $X\times Y$, defined by the formula

$$\rho[(x_1, y_1), (x_2, y_2)]=\max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

If we identify members of $C(X, Y)$ with their graphs, then h_ρ defines an infinite valued metric on $C(X, Y)$, which we denote by d_2. Of course, any other metric compatible with the product uniformity on $X\times Y$ yields a Hausdorff metric uniformly equivalent to d_2 on $C(X, Y)$. From the point of view of constructive approximation theory, such spaces have been thoroughly studied by B. Penkov, V. Popov, Bl. Sendov, V. Veselinov and their associates in Sofia (see, e.g., [10], [12] and [14]). This author's interest in Hausdorff distance as applied to functions is quite different (see, e.g., [3], [4], [5], and [6]). In the sequel we retain the notation of [6]: (i) $CL(X)$ denotes the collection of closed nonempty subsets of a metric space X (ii), $S_\varepsilon[K]$ denotes the union of all open ε-balls whose centers run over a set K in a metric space (iii), LsC_n denotes the set of points each neighborhood of which meets infinitely many terms of a sequence $\{C_n\}$ of sets in a metric space.

Although d_1 and d_2 are equivalent when X is compact, they fail except in degenerate situations to be uniformly equivalent. Most conspicuously, the completeness of $\langle C(X, Y), d_1\rangle$ for X compact and Y complete does not ensure completeness for $\langle C(X, Y), d_2\rangle$ (see Example 1 of [6]). In this setting it is not hard to characterize those d_2-closed subsets of $C(X, Y)$ which are d_2-complete [6]

Theorem 1. *Let $\langle X, d_X\rangle$ be a compact metric space, and let $\langle Y, d_Y\rangle$ be a complete metric space. Suppose $\Omega\subset C(X, Y)$ is d_2-closed. The following are equivalent:*

(1) *Ω is d_2-complete*

(2) *Each d_2-Cauchy sequence in Ω is d_1-Cauchy*

(3) *Whenever $\{f_n\}$ is a sequence in Ω h_ρ-convergent to a closed set E, then E is the graph of a function from X to Y.*

The purpose of this note is to determine if conditions (2) and (3) serve more generally as completeness criteria for d_2-closed subsets of $C(X, Y)$. We first dispense with condition (2).

Theorem 2. *Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be metric spaces. Then condition (2) of Theorem 1 characterrizes those d_2-closed subsets of $C(X, Y)$ that are complete if and only if Y is complete and d_1 and d_2 are equivalent metrics on $C(X, Y)$.*

Proof. First, suppose Y is complete, and d_1 and d_2 are equivalent metrics. Let Ω be a d_2-closed subset of $C(X, Y)$ for which condition (2) of Theorem 1 holds. To see that Ω is d_2-complete, let $\{f_n\}$ be a d_2-Cauchy sequence in Ω. By condition (2) and the completeness of Y, $\{f_n\}$ must be d_1-convergent to a continuous function f. Since d_1 and d_2 are equivalent and Ω is d_2-closed, $\{f_n\}$ d_2-converges to an element of Ω. On the other hand, if Ω is d_2-complete, each d_2-Cauchy sequence in Ω must be d_1-convergent, ergo, d_1-Cauchy. Thus, condition (2) holds for Ω.

Conversely, we assume condition (2) serves as a completeness criterion for d_2-closed subsets of $C(X, Y)$. Suppose Y fails to complete. Let $\{y_n\}$ be a nonconvergent Cauchy sequence in Y. For each $n\in Z^+$ let $f_n : X\to Y$ map each x in X to y_n. Then, $\Omega=\{f_n : n\in Z^+\}$ is d_2-closed, satisfies condition (2) of Theorem 1, but fails to be d_2-complete, a contradiction. To show d_1 and d_2 define the same topology on Ω, we need only show that d_2 is at least as strong as d_1. Let $\{f_n\}$ d_2-converge to f. Since $\Omega=\{f\}\cup\{f_n : n\in Z^+\}$ is d_2-complete and (1) implies (2), $\{f_n\}$ is d_1-Cauchy. We know now that Y is complete; so, there exists $g\in C(X, Y)$ for which $\lim_{n\to\infty} d_1(f_n, g)=0$. Since $d_2(f_n, g)\leqq d_1(f_n, g)$, we conclude that $\{f_n\}$ d_2-converges to g, and by the uniqueness of d_2-limits, we obtain $f=g$. Thus, $\{f_n\}$ d_1-converges to f.

Theorem 2 raises as many questions as it answers. Are there noncompact X for which d_1 and d_2 are equivalent? If so, can we describe them concretely? The answer to both questions is yes. If Y has at least one nontrivial path component, then d_1 and d_2 are equivalent on $C(X, Y)$ if and only if (a) the set X' of limit points of X is compact, and (b) for each $\varepsilon>0$ the set of points in X whose distance from X' exceeds ε is uniformly discrete [5]. Such spaces, first studied by M. Atsuji [1], have many other interesting characterizations. Most importantly, they are the spaces on which each continuous function is uniformly continuous. Called UC or

Lebesgue spaces in the literature, such spaces are complete, but need not be locally compact. Further results on these spaces can be found in J. Rainwater [11], W. Waterhouse [15], and Gh. Toader [13].

Evidently, condition (1) of Theorem 1 always implies condition (3), with no assumptions on X or Y whatsoever. What about the converse? The situation is hopeless if there exists a d_2-Cauchy sequence $\{f_n\}$ in $C(X, Y)$ that fails to h_ρ-converge to anything in $CL(X \times Y)$: $\Omega = \{f_n : n \in Z^+\}$ would be a d_2-closed noncomplete subset of $C(X, Y)$ satisfying condition (3). To ensure that no such sequence exists, we can require that X and Y be complete. As a result, $\langle X \times Y, \rho \rangle$ will be complete whence $\langle CL(X \times Y, h_\rho \rangle$ is complete [7]. Actually, there is very little loss of generality in making this assumption, a technical point that we shall return to at the end of this note.

In such a setting, condition (3) is equivalent to the following usually stronger condition: each d_2-Cauchy sequence in Ω h_ρ-converges to a function with a closed graph. Thus, for X and Y complete condition (3) will guarantee completeness if and only if $C(X, Y)$ is a closed subset of the functions from X to Y with closed graph (when topologized by Hausdorff distance). Such functions are the subject of a recent monograph of T. Hamlett and L. Herrington [8].

Example 1. There exist complete metric spaces X and Y for which $C(X, Y)$ is not an h_ρ-closed subset of the functions from X to Y with closed graph. Let L be the Banach space of bounded real sequences with norm $\|\alpha_i \xi\| = \sup\{|\alpha_i| : i \in Z^+\}$. For each $i \in Z^+$ let $e_i \in L$ be defined by $e_i(i) = 1$ and $e_i(j) = 0$ for each $j \neq i$. For each i let W_i denote the line segment in L joining e_i to the origin, and let $X = Y = \bigcup_{i=1}^{\infty} W_i$. We define $f : X \to Y$ as follows: on each segment W_i let the graph of f consist of the two line segments connecting (0, 0), $(\frac{1}{i} e_i, e_i)$, and (e_i, e_i) is succession. Clearly, $\lim_{i\to\infty} f(\frac{1}{i} e_i) \neq f(0)$; so, $f \notin C(X, Y)$. However, if $x = 0$ then f is continuous at x, and if $\lim_{k\to\infty} x_k = 0$, then either $\lim_{k\to\infty} f(x_k) = 0$ or $\lim_{k\to\infty} f(x_k)$ does not exist. Thus, f has a closed graph. Now for each $n \in Z^+$ let $f_n \in C(X, Y)$ be the function that agrees with f on W_i for $i \leq n$ and whose graph when restricted to W_i for $i > n$ consists of the two line segments connecting (0, 0), $(\frac{1}{n} e_i, e_i)$, and (e_i, e_i) in succession. We leave to the reader to verify that for each n $h_\rho(f_n, f) = 1/n$.

We now proceed with some positive results.

Theorem 3. Let X and Y be complete metric spaces. Then condition (3) of Theorem 1 characterizes d_2-completeness for d_2-closed subsets of $C(X, Y)$ if any of the following conditions hold:

(a) Y is compact

(b) Y is locally compact and X is locally connected

(c) X is locally compact.

Proof. By the previous remarks we need to show that each condition implies that $C(X, Y)$ is an h_ρ-closed subset of the functions from X to Y with closed graph. Condition (a) works because a function into a compact space is continuous if and only if its graph is closed (see, e. g., [2]). To see that condition (b) is sufficient, suppose $\{f_n\}\subset C(X, Y)$ h_ρ-converges to a function with closed graph f and f is discontinuous at $x=x_0$. Choose $\varepsilon>0$ and a sequence $\{x_n\}$ convergent to x_0 such that for each n $d_Y(f(x_n), f(x_0))>2\varepsilon$. Since Y is locally compact, there exists $\delta<\varepsilon$ such that $\{y : d_Y(y, f(x_0))=\delta\}$ is compact. For each $n\in Z^+$ choose $\delta_n<\delta$ such that each two points in $S_{\delta_n}[x_0]$ are connected in $S_{1/n}[x_0]$. We claim (*): for each $k\in Z^+$ there exists $n>k$ and $p_n\in S_{1/k}[x_0]$ such that $d_Y(f_n(p_n), f(x_0))=\delta$. Pick $n>k$ so large that both $d_X(x_n, x_0)<1/2\,\delta_k$ and $h_\rho(f_n, f)<1/2\,\delta_k$. By the second condition we can find $\{w_n, k_n\}\subset X$ satisfying

$$\rho[(z_n, f_n(z_n)), \quad (x_n, f(x_n))]<\frac{1}{2}\delta_k$$

and

$$\rho[(w_n, f_n(w_n)), \quad (x_0, f(x_0))<\frac{1}{2}\delta_k.$$

Since $\{z_n, w_n\}\subset S_{\delta_k}[x_0]$, there exists a connected subset C of $S_{1/k}[x_0]$ containing $\{z_n, w_n\}$. Since $\delta_k<\delta<\varepsilon$, we have $d_Y(f_n(z_n), f(x_0))>\delta$ and $d_Y(f_n(w_n), f(x_0))<\delta$. Therefore, the connected set $f_n(C)$ must meet $\{y : d_Y(f(x_0), y)=\delta\}$; so, there exists $p_n\in C\subset S_{1/k}[x_0]$ such that $d_Y(f_n(p_n), f(x_0))=\delta$. This establishes (*). Hence, we can find an increasing sequence of integers $\{n_k\}$ and a sequence $\{p_{n_k}\}$ convergent to x_0 such that $d_Y(f_{n_k}(p_{n_k}), f(x_0))=\delta$ for each positive integer k. Since $\{y : d_Y(y, f(x_0))=\delta\}$ is a compact set, there exists y_0 in this set such that $(x_0, y_0)\in Lsf_n$. We conclude that $f=Lsf_n$ is not the graph of a function, a contradiction.

To show that condition (c) is sufficient, we need only show that if $\{f_n\}\subset C(X, Y)$ converges to a function f with closed graph, then for each x_0 in X there exists $\delta>0$ such that the restriction of f to $\{x : d_X(x, x_0)\leqq\delta\}$ has compact graph. This, of course, is a stronger requirement than that f should have a locally compact graph, a condition that does not ensure continuity. Choose $\delta>0$ such that $\{x : d_X(x, x_0)\leqq 2\delta\}$ is compact. Since $E=\{(x, f(x)) : d_X(x, x_0)\leqq\delta\}$ is a closed subset of the complete metric space $\{x : d_X(x, x_0)\leqq\delta\}\times Y$, it is complete. To show E is totally bounded, let ε in $(0,2\delta)$ be arbitrary and choose $n\in Z^+$ for which $h_\rho(f_n, f)<\frac{1}{2}\varepsilon$. Since $F=\{(x, f_n(x)) : d_X(x, x_0)\leqq 2\delta\}$ is totally bounded, there is a finite subset F^* of F satisfying $F\subset S_{\varepsilon/2}[F^*]$. Since both $\frac{1}{2}\varepsilon<\delta$ and $f\subset S_{\varepsilon/2}[f_n]$ hold, we have $E\subset S_{\varepsilon/2}[E]\subset S_\varepsilon[F^*]$.

Example 1 shows that the local compactness requirements in conditions (b) and (c) connot be omitted. The construction of an example showing that "locally connected" cannot be replaced by "connected" in condition (b) provided the author with a minor headache, which he will now share with the reader.

Example 2. Let L and $\{e_i : i \in Z^+\}$ be as described in Example 1. For each $i \in Z^+$ let M_i be the starshaped set consisting of all line segments joining the points of $\{0, e_{2i}, \frac{1}{2}e_{2i}, \frac{1}{3}e_{2i}, \ldots\}$ to e_{2i-1}. Set $X = \cup_{i=1}^{\infty} M_i$; it is clear that X is a complete connected subspace of L. We will produce a sequence $\{f_n\} \subset C(X, R)$ h_ρ-convergent to a function f with closed graph that is not continuous. We describe how each f_n is defined on M_i, $i = 1, 2, 3, \ldots$ We will always let $f_n(0) = f_n(e_{2i-1}) = 0$, and on $\{\frac{1}{k}e_{2i} : k \in Z^+\}$ we set

$$f_n(\frac{1}{k}e_{2i}) = \begin{cases} i \text{ if } k \leqq \min\{n, i\} \\ 0 \text{ otherwise} \end{cases}$$

We extend f_n linearly on each of the segments that make up M_i; since f_n is zero in a neighborhood of the origin, the result is a globally defined continuous function. We describe f similarly: on M_i let $f(0) = f(e_{2i-1}) = 0$, and set

$$f(\frac{1}{k}e_{2i}) = \begin{cases} i \text{ if } k \leqq i \\ 0 \text{ otherwise} \end{cases}$$

As with each f_n the values of f at remaining points of M_1 are obtained through linear extension. Clearly, f is continuous on M_i, and since M_i contains no limit points of $X - M_i$ other that the origin, f is continuous on $X - \{0\}$. Now if i and j are arbitrary positive integers, it is easy to check that the point on the line segment from $\frac{1}{j}e_{2i}$ to e_{2i-1} nearest the origin has norm $\frac{1}{j+1} \geqq \frac{1}{2j}$. Hence, if $k \in Z^+$ and $\|x\| < \frac{1}{2k}$, then either $f(x) = 0$ or for some integers i and j with $i \geqq j \geqq k$ the point x lies on the line segment joining $\frac{1}{j}e_{2i}$ to e_{2i-1}. Since this line segment last intercepts $\{z : \|z\| = \frac{1}{2k}\}$ at the point

$$z_0 = \frac{2k-1}{2k}(\frac{1}{j}e_{2i}) + \frac{1}{2k}e_{2i-1},$$

we conclude that x is between $\frac{1}{j}e_{2i}$ and z_0, so that

$$f(x) \geqq f(z_0) = \frac{2k-1}{2k} i \geqq k - \frac{1}{2}$$

Thus, if $\{x_j\}$ is a sequence in X convergent to the origin such that $\lim_{j\to\infty} f(x_j) \neq 0$, then the limit does not exist. Hence, f has a closed graph. It is left to the reader to convince himself that $\lim_{n\to\infty} h_\rho(f, f_n) = 0$.

Finally, we justify confining our search for metric spaces X and Y for which condition (3) of Theorem 1 characterizes d_2-completeness for d_2-closed subsets of $C(X, Y)$ to complete metric spaces.

Theorem 4. *Let* $\langle X, d_X\rangle$ *and* $\langle Y, d_Y\rangle$ *be metric spaces for which condition* (3) *of Theorem* 1 *serves as a completeness criterion for* d_2*-closed subsets of* $C(X, Y)$; *then* Y *is complete; and if* $C([0, 1], Y)$ *is nontrivial, then* X *is also complete.*

Proof. Suppose Y is not complete. Let $\{y_n\}$ be a Cauchy sequence in Y that fails to converge. If for each n we let $f_n : X\to Y$ be the constant function with each value equal to y_n, then $\Omega=\{f_n : n\in Z^+\}$ is a d_2-closed subset of $C(X, Y)$ satisfying condition (3) vacuously that is not d_2-complete. This violates the hypotheses of the theorem.

If X is not complete and Y has some nontrivial path component, we again seek to produce a d_2-Cauchy sequence $\{f_n\}$ in $C(X, Y)$ that is not h_ρ-convergent to any set; thus, $\Omega=\{f_n;\ n\in Z^+\}$ will be a d_2-closed subset of $C(X, Y)$ satisfying condition (3) vacuously, but will not be d_2-complete. Let $\{x_n\}$ be a nonconvergent Cauchy sequence in X with distinct terms. Since $\{x_n : n\in Z^+\}$ has no limit points in X, we can find for each n an $\varepsilon_n>0$ such that the collection $\{S_{\varepsilon_n}[x_n] : n\in Z^+\}$ is pairwise disjoint. Since $C([0, 1], Y)$ is nontrivial, there exist distinct points a and b in Y and $\theta\in C(0, 1, Y)$ such that $\theta(0)=a$ and $\theta(1)=b$. We now distinguish two cases: (1) infinitely many x_n are isolated points of X; (2) finitely many x_n are isolated points of X. In the first case, by passing to a subsequence we can assume each x_n is isolated; without loss of generality we can also assume for each n that $S_{\varepsilon_n}[x_n]=\{x_n\}$. If for each n we let $f_n(x)=a$ if $x=x_n$ and $f_n(x)=b$ otherwise, then $\{f_n\}$ is d_2-Cauchy. However, $\{f_n\}$ h_ρ-converges to no closed subset of $X\times Y$; if it did, the limit set would necessarily be $Lsf_n=X\times\{b\}$, but $h_\rho(f_n, X\times\{b\})=d_Y(a, b)$ for each n. In the second case, by passing to a subsequence, we can assume no x_n is an isolated point of X. The second lemma of [5] allows us to construct for each $n\in Z^+$ a continuous $f_n : X\to Y$ satisfying:

(a) $$f_n(S_{\varepsilon_n}[x_n])\subset\theta([0, 1])\subset S_{\varepsilon_n}[f_n(S_{\varepsilon_n}[x_n])]$$

(b) $$f_n(x)=b \text{ whenever } d_X(x, x_n)\geqq\varepsilon_n.$$

Since for each n and j we have $\varepsilon_n<d_X(x_n, x_j)$, we conclude that $d_2(f_n, f_j)\leqq 3d_X(x_n, x_j)$. Thus, $\{f_n\}$ is d_2-Cauchy, and the previous argument indicates that $\{f_n\}$ fails to h_ρ-converge to any closed subset of $X\times Y$.

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