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Approximation by Szász-Mirakijan Operators in $L_p[0, \infty]$

Snejana Tomova

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In the present paper we consider the approximation of a function $f \in L_p[0, \infty]$ $(1 \leqq p \leqq \infty)$ by Szász-Mirakijan operators $M_n(f;\cdot)$. We obtain the following estimate of the L_p-distance $(1 \leqq p \leqq \infty)$ between a function f and its image by means of the operators $M_n(f;\cdot)$ by the averaged moduli of smoothness $\tau_k(f;\delta)_p$: of $n \in \mathbb{N}$, $f \in L_p[0, \infty)$, then $\| M_n f - f \|_{L_p[0,\infty)} \leqq c \cdot \tau_2(f;(x/n)^{1/2} + (1/n))_p$, where c is an absolute constant.

1. Introduction

Let us consider functions belonging to the spaces $L_p[0, \infty)$ $(1 \leqq p \leqq \infty)$ with the norm $\|f\|_{L_p[0,\infty)} = \{\int_0^\infty |f(x)|^p \, dx\}^{1/p}$ for Szász-Mirakijan operators ([4, 5]) for functions, which are defined in $[0, \infty)$, are given by

$$(1) \qquad M_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} f(k/n)(nx)^k/k!, \quad \text{where } n \in \mathbb{N}.$$

In this paper our purpose is to find an estimate of

$$\| M_n(f;\cdot) - f \|_{L_p[0,\infty)} = \{\int_0^\infty |M_n(f;x) - f(x)|^p \, dx\}^{1/p}$$

by the so-called averaged moduli of smoothness $\tau_k(f;\delta)_p$ (or τ-moduli) of the function f ([1, 15, 16])

$$(1.1) \qquad \tau_k(f;\delta)_p = \| \omega_k(f, \cdot\, ; \delta(\cdot)) \|_{L_p},$$

where $\omega_k(f, x; \delta(x)) = \sup \{ |\Delta_h^k f(t)|,\ t,\ t + kh \in [x - k\delta/2,\ x + k\delta/2] \}$,

$$\Delta_h^k f(t) = \sum_{n=0}^{k} (-1)^{k-n} \binom{k}{n} f(t + nh).$$

In (1.1) δ is a positive function of $x \in [0, \infty)$.

For the first time these moduli are used by Bl. Sendov [2] and P. Korovkin [3] in the case of $k=1$, $p=1$. They have many important applications ([6 – 12]).

The most essential properties of $\tau_k(f;\delta)_p$ are given in [1, 15, 16]. We shall use the following of them :

(1.2) $$\tau_k(f;\delta_1)\leqq\tau_k(f;\delta_2)_p \quad \text{for } \delta_1(x)\leqq\delta_2(x),\ x\in[0,\infty)\,;$$

(1.3) $$\omega_k(f;\delta)\leqq\tau_k(f;\delta)_p,$$

where $\omega_k(f;\delta)_p=\sup\{\|\Delta_h^k f\|_{L_p}:|h|\leqq\delta\}$ are the so-called L_p-moduli ;

(1.4) $$\tau_k(f;\lambda\delta)_p\leqq c\,.\,\tau_k(f;\delta)_p \quad \text{for } \delta(x)=(x/m^2)^{1/2}+m^{-2},\quad c=c(\lambda),\ m\in\mathrm{N}\,;$$

(1.5) $$\tau_k(f;\delta)_p\leqq k\,.\,\delta^k\,\|f^{(k)}\|_{L_p},$$ where δ is an absolute constant.

As we already mentioned, we shall obtain an estimate of the L_p-distance $(1\leqq p\leqq\infty)$ between a function f and its image by means of the operators $M_n(f;x)$.

In view of (1) we shall separate the bounded function f from its class of equivalence in L_p and we shall assume that every function is given by its values at every point $x\in[0,\infty)$.

A. S. Andreev and V. A. Popov have obtained the following result for the Szász-Mirakijan operators (see [14]) : if f is a bounded p-integrable function in $[0,\infty)$, $f(x)=0$ for $x\geqq a$, then

(1.6) $$\|M_n(f;\cdot)-f\|_{L_p[0,a]}\leqq c(a)\,.\,\tau_2(f;n^{-1/2})_p,$$

where the constant $c(a)$ depends only on a.

In the continuous case ($f\in C[0,\infty)$) V. Totik shows in [17] that for $0<\alpha<1$ the statements

$$M_n(f)-f=0(n^{-\alpha})$$

and $\|x^\alpha\Delta_h^2(f;x)\|_{C[0,\infty)}=0(h^{2\alpha})$ are equivalent.

In the case of $f\in L_p[0,\infty)$, there are some results for the L_p-distance $(1\leqq p\leqq\infty)$ between the function f and its image by means of the Kantorovich variant of Szász-Mirakijan operators

$$S_n(f;x)=\sum_{k=0}^{\infty}\Big(n\int_{k/n}^{(k+1)/n}f(u)\,du\Big)e^{-nx}(nx)^k/k!$$

The following result was obtained by V. Totik in [13]. Let $f\in L_p[0,\infty)$ $(1\leqq p\leqq\infty)$, $0<\alpha<1$. Then

$$\| S_nf-f\|_{L_p[0,\infty)}=0(n^{-\alpha}) \text{ iff } \|\Delta^2_{h\varphi}f\|_{L_p(h^2,\infty)}=0(h^{2\alpha}), \quad h\to 0.$$

Actually, there was found an estimate with an integral modulus in L_p (or L_p-modulus). But we could not expect a similar result about the classical Szász-Mirakijan operators because the alteration of a function at different points in $[0,\infty)$ does not effect on the L_p-module.

Obviously, (1.6) treats a rather general case. But in that estimate averaged moduli of smoothness with $\delta=\text{const}$ are used. This is the reason why the function considered is actualy defined in a finite interval.

Now let us represent our result.

Theorem. *If f is locally bounded, $n\in\mathbf{N}$, $f\in L_p[0,\infty)$ $(1\leqq p\leqq\infty)$, then $\|M_nf-f\|_{L_p[0,\infty)}\leqq c\,.\,\tau_2(f;(x/n)^{1/2}+1/n)_p$, where C is an absolute constant.*

We may consider this Theorem as a generalization of the results mentioned above.

Here is the structure of our paper : in the next paragraph some definitions and notations are given ; § 3 and § 4 contain auxiliary statements and their proofs respectively. Our main result is proved in § 5.

2. Definitions and Notations

Let $f\in L[a,b]$ and $h>0$. The modified Steklov function (see [1]) is

$$\tilde{f}_{h,k}=(-1)^{k-1}h^{-k}\int_0^h\int_0^h\ldots\int_0^h\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}f(x+(r/k)(t_1+\ldots+t_k))dt_1\ldots dt_k.$$

It is easy to see that for $k=2$ the following equation hold.

$$(2.1)\quad \tilde{f}_{h,2}(x)=(2/h^2)\int_0^h\int_0^h f(x+(1/2)(t_1+t_2))dt_1dt_2-(1/h^2)\int_0^h\int_0^h f(x+t_1+t_2)dt_1dt_2,$$

$$(2.2)\qquad (\tilde{f}_{h,2}(x))''=(\delta/h^2)\Delta^2_{h/2}f(x)-(1/h^2)\Delta^2_hf(x)\ ;\ \text{a.e.}$$

$$(2.3)\qquad f(x)-\tilde{f}_{h,2}(x)=(1/h^2)\int_0^h\int_0^h\Delta^2(t_1+t_2)/2f(x)dt_1\,dt_2.$$

We denote

$$W^2_p[0,\infty)=\{g:g,g'\in AC[0,\infty) \text{ and } g,g''\in L_p[0,\infty)\}\,;$$

$$L_p=L_p[0,\infty)$$

$$\varphi(x)=(x+t^2)^{1/2},\quad 0<t\leqq t_0\,;$$

$$\Delta(t,x)=t(x)^{1/2}+t^2\,;$$

$$\Delta_m(x)=\Delta(m^{-1},x),\quad \text{where } m\in\mathbf{N}.$$

For our function $\varphi(x)$ and $f\in L_p[0,\infty)$ we shall consider the following module of smoothness (defined by V. Totik in [13], p. 447-451) for $x\in[0,\infty)$:

$$(2.4)\qquad \omega(f,t)=\sup_{0<h\leq t}\{\int_{(c_1h)^2+c_1ht}^{\infty}|\Delta^2_{h(x+t^2)^{1/2}}f|^p dx\}^{1/p}$$

$$+\{\int_0^a[f(x)-(1/a)\int_0^a f(u)du-(12(x-a/2)/a^3)\int_0^a f(u)(u-a/2)du]^p dx\}^{1/p},$$

where $a=2[(c_1t)^2+c_1t^2]$.

At the end of this paragraph let us define the Peetre K functional

$$K(t^2,f)=\inf_{g\in W^2_p[0,\infty)}(\|f-g\|_{L_p}+t^2\|\varphi^2g''\|_{L_p}),$$

where $f\in L_p$, $0<t\leq t_0$.

3. Auxiliary statements

By analogy with Theorem 1 in [13] it is possible to prove the following

Theorem A. *Let $\varphi, K(t^2,f)$ and ω be as above. There is a constant $L>0$ independent of $f\in L_p[0,\infty)$ $(1\leq p\leq\infty)$ and $0<t\leq t_0$ such that $(1/L)\,\omega(f,t)\leq K(t^2,f)\leq L.\omega(f,t)$ holds.*

Moreover, we shall use the next auxiliary statements, which will be proved in § 4.

Lemma 1. *If $f\in L_p[0,\infty)$, $t=1/m$, then $\omega(f,t)\leq c_2\tau_2(f;\Delta_m(x))_p$, where c_2 is an absolute constant.*

Lemma 2. *Let $m\in\mathrm{N}$, $f\in L_p$, $t=1/m$. Then there exists a function $g\in W^2_p[0,\infty)$ such that*

$$(3.1)\qquad \|f-g\|_{L_p}\leq c_3\tau_2(f;\ \Delta_m(x))_p \text{ and}$$

$$(3.2)\qquad \|(1/m^2)\varphi^2g''\|_{L_p}\leq c_3\tau_2(f;\ \Delta_m(x))_p,$$

where c_3 is an absolute constant.

We denote

$$u_n(x;\theta)=\begin{cases}M_n((t-\theta)_+;x) & \text{if}\quad \theta\geq x,\\ M_n((\theta-t)_+;x) & \text{if}\quad \theta<x.\end{cases}$$

Lemma 3. *If $n\in\mathrm{N}$, $g,g'\in AC[0,\infty)$, then*

$$Mn(g;x)-g(x)=\int_0^\infty u_n(x;\theta)g''(\theta)d\theta.$$

Lemma 4. *Let $n \in \mathrm{N}$. Then*

$$u_n(x\,;\theta) \geqq 0\,; \tag{3.3}$$

$$\int_0^\infty u_n(x\,;\theta)\theta^{-1}\,d\theta \leqq c_4 . n^{-1}\,; \tag{3.4}$$

$$\int_0^\infty u_n(x\,;\theta)dx \leqq c_4^* . \theta . n^{-1}, \tag{3.5}$$

where c_4 and c_4^* are absolute constants.

Lemma 5. *If $n \in \mathrm{N}, g \in W_p^2\,[0, \infty), \psi(x) = x$, then $\| M_n g - g \|_{L_p} \leqq c_5 . n^{-1} \| \psi . g'' \|_{L_p}$, where c_5 is an absolute constant.*

Lemma 6. *Let $f \in L_p$ and g is the function from Lemma 2.* Then $\| M_n f - M_n g \|_{L_p} \leqq c_6 \cdot \tau_2(f\,; (x/n)^{1/2} + 1/n)_p$, where c_6 is an absolute constant.

4. Proofs of the auxiliary statements

Proof of Lemma 1. Because of $h \leqq t$, we have

$$\omega(f, t) = \sup_{0 < h \leqq t} \{ \int_{a/2} | \Delta^2_{h(x+t^2)^{1/2}} f(x)|^p dx\}^{1/p}$$

$$+ \{ \int_0^a [f(x) - (1/a) \int_0^a f(u) du - 12(x - a/2)/a^3 \int_0^a f(u)(u - a/2) du]^p dx\}^{1/p}.$$

Setting

$$P = \sup_{0 < h \leqq t} \{ \int_{a/2}^\infty | \Delta^2_{h(x+t^2)^{1/2}} f(x)|^p dx\}^{1/p} \quad \text{and}$$

$$Q = \{ \int_0^a [f(x) - (1/a) \int_0^a f(u) du - (12(x - a/2)/a^3) \int_0^a f(u)(u - a/2) du]^p dx\}^{1/p},$$

we shall show that

$$P \leqq \tau_2(f\,; \Delta_m(x))_p \quad \text{and} \tag{4.1}$$

$$Q \leqq B . \tau_2(f\,; \Delta_m(x))_p, \tag{4.2}$$

where B is an absolute constant.

For $h \leqq t$ and $h_1 = h(x + t^2)^{1/2}$ the inequality $h_1 \leqq \Delta(t, x)$ holds. Then

$$P \leqq \sup_{h \leqq t} \{ \int_0^\infty | \Delta^2_{h(x+t^2)^{1/2}} f(x)|^p dx\}^{1/p}$$

$$\leqq \sup_{h_1 \leqq \Delta(t,x)} \{\int_0^\infty |\Delta_{h_1}^2 f(x)|^p dx\}^{1/p} = \omega_2(f;\Delta(t,x))_p \leqq \tau_2(f;\Delta(t,x))_p.$$

We set $t=1/m$ and the proof of (4.1) is completed.

To prove (4.2) we shall use the Hölder inequality

$$(4.3) \qquad \| \int_a^b f(x,\cdot)dx \|_{L_p[c,d]} \leqq \int_a^b \| f(x,\cdot) \|_{L_p[c,d]} dx.$$

Let $L(f)=f(x)-(1/a)\int_0^a f(u)du-(12(x-a/2)/a^3)\int_0^a f(u)(u-a/2)du$ and $g\in W_p^2[0,\infty)$. The Taylor formula for $t,x\in[0,\infty)$ gives

$$(4.4) \qquad g(x)=g(0)+x.g'(0)+\int_0^x g''(t)(x-t)dt.$$

Using (4.4) we obtain

$$L(g)=\int_0^x g''(t)(x-t)dt+(6x/a^2-4/a)\int_0^a[\int_0^u g''(t)(u-t)dt]du$$

$$-(12x/a^3-6/a^2)\int_0^a u[\int_0^u g''(t)(u-t)dt]du.$$

A change of the integration order gives

$$L(g)=\int_0^x g''(t)(x-t)dt+\int_0^a g''(t).A(a,x,t)dt,$$

where $A(a,x,t)=t-x+3xt^2/a^2-2t^2/a+t^3/a^2-2xt^3/a^3$.

Obviously, for every $x\in[0,a]$, we have $\int_0^a|A(a,x,t)|dt\leqq c.a^2$, where c is an absolute constant.

Let g be the modified Steklov function $\tilde{f}_{h,2}$. Then from (2.1), (2.2), (2.3) and the equality $L(f)=L(f-\tilde{f}_{h,2})+L(\tilde{f}_{h,2})$ we get

$$L(f)=(1/h^2)\int_0^h\int_0^h \Delta_{(t_1+t_2)/2}^2 f(x)dt_1\,dt_2-(1/a)\int_0^a[(1/h^2)\int_0^h\int_0^h \Delta_{(t_1+t_2)/2}^2 f(u)dt_1\,dt_2]du$$

$$-(12/a^3)(x-a/2)\int_0^a[(u-a/2)(1/h^2)\int_0^h\int_0^h \Delta_{(t_1+t_2)/2}^2 f(u)dt_1\,dt_2]du$$

$$+\int_0^x[(8/h^2)\Delta_{h/2}^2 f(t)-(1/h^2)\Delta_h^2 f(t)](x-t)dt+\int_0^a[(8/h^2)\Delta_{h/2}^2 f(t)-(1/h^2)\Delta_h^2 f(t)]\,A dt.$$

The triangle inequality and (4.3) give

$$\| L(f) \|_{L_p[0,2a]} \leqq \{\int_0^{2a} |\Delta_h^2 f(x)|^p dx\}^{1/p}+\{\int_0^{2a}[\int_0^a (1/a)|\Delta_h^2 f(u)|du]^p dx\}^{1/p}$$

$$+\{\int_0^{2a}[\int_0^a(9/a)|\Delta_h^2 f(u)|du]^p dx\}^{1/p}+\{\int_0^{2a}[\int_0^x(9/h^2)2a|\Delta_h^2 f(t)|dt]^p dx\}^{1/p}$$

$$+C.\{\int_0^{2a}[\int_0^a(9/h^2)|\Delta_h^2 f(t)|.\,adt]^p dx\}^{1/p}$$

$$\leqq\omega_2(f,h)_{L_p[0,2a]}+\int_0^a\{\int_0^{2a}a^{-p}|\Delta_h^2 f(u)|^p dx\}^{1/p}du$$

$$+\int_0^a\{\int_0^{2a}9^p a^{-p}|\Delta_h^2 f(u)|^p dx\}^{1/p}du+\int_0^{2a}\{\int_0^{2a}(18a/h^2)^p|\Delta_h^2 f(t)|^p dx\}^{1/p}dt$$

$$+\int_0^a\{\int_0^{2a}(9ca/h^2)^p|\Delta_h^2 f(t)|^p dx\}^{1/p}dt.$$

Since $h\leqq a$ we have

$$L(f)_{L_p[0,2a]}\leqq\omega_2(f,a)_{L_p[0,2a]}+(2a)^{1/p}.\,a^{-1}\int_0^a|\Delta_a^2 f(u)|du$$

$$+9a^{-1}(2a)^{1/p}\int_0^a|\Delta_a^2 f(u)|du+18.\,a^{-1}(2a)^{1/p}\int_0^a|\Delta_a^2 f(t)|dt+ca^{-1}(2a)^{1/p}\int_0^a|\Delta_a^2 f(t)|dt$$

$$\leqq\omega_2(f,a)_{L_p[0,2a]}+B_1(2a)^{1/p-1}.\,\omega_2(f,a)_{L_1[0,2a]}.$$

It is easy to see that the Hölder inequality

$$|\int_a^b f(x)g(x)dx|\leqq\{\int_a^b|f(x)|^p dx\}^{1/p}.\{\int_a^b|g(x)|^{p/(p-1)}dx\}^{1-1/p}$$

for $f(x)=\Delta_h^2 f(x)$ and $g(x)=1$ yields

$$(2a)^{1/p-1}.\,\omega_2(f,a)_{L_1[0,2a]}\leqq\omega_2(f,a)_{L_1[0,2a]}.$$

Therefore $\|L(f)\|_{L_p[0,2a]}\leqq B_2\,\omega_2(f,a)_{L_p[0,a]}$, where B_2 is an independent of a and p constant. Because of $Q=\|L(f)\|_{L_p[0,a]}$, from (1.2) and (1.3) we have $Q\leqq B_2.\,\tau_2(f;a/2)_{L_p[0,a]}$.

Evidently, there exists a constant M such that $a/2\leqq M\,\Delta_m(x)$ for $t=1/m$. That is why using (1.2) and (1.4) we get (4.2). But we have already obtained (4.1). Those two statements prove the Lemma.

P r o o f o f L e m m a 2. Applying Theorem A and Lemma 1 for $t=1/m$ we get

$$K(1/m^2,f)\leqq L.\,\omega(f,1/m)\leqq L.\,c_2.\,\tau_2(f;\Delta_m(x))_p,$$

i. e.

$$\inf_{g\in W_p^2[0,\infty)}\{\|f-g\|_{L_p}+(1/m^2)\,\|g''\varphi^2\|_{L_p}\}\leqq c_3\,\tau_2(f;\Delta_m(x))_p.$$

Hence there exists a function $g\in W_p^2[0,\infty)$ for which (3.1) and (3.2) hold true. So our Lemma is proved.

Proof of Lemma 3. From the definition of $u_n(x;\theta)$ we obtain

$$\int_0^\infty u_n(x;\theta)g''(\theta)d\theta=\int_0^x M_n((\theta-t)_+;x)g''(\theta)d\theta+\int_x^\infty M_n((t-\theta)_+;x)g''(\theta)d\theta.$$

We shall consider two cases.

a) $k\leqq n\theta$. Then

$$\int_{k/n}^x (\theta-k/n)g''(\theta)d\theta=\int_{k/n}^x (\theta-k/n)dg'(\theta)=(x-k/n)g'(x)-g(x)+g(k/n)$$

b) $k\geqq n\theta$. Then

$$\int_x^{k/n} (k/n-\theta)g''(\theta)d\theta=\int_x^{k/n} (k/n-\theta)dg'(\theta)=(x-k/n)g'(x)-g(x)+g(k/n).$$

Therefore

$$\int_0^\infty u_n(x;\theta)g''(\theta)d\theta=e^{-nx}\sum_{k=0}^\infty ((nx)^k/k!)[(x-k/n)g'(x)-g(x)+g(k/n)].$$

Using the obvious equalities $M_n(1;x)=1$ and $M_n(t;x)=x$ we have $\int_0^\infty u_n(x;\theta)g''(\theta)d\theta=M_n(g;x)-g(x)$, which was to be demonstrated.

Proof of Lemma 4. The inequality (3.3) is an immediate consequence of the definition of $M_n(f)$.

By analogy with the proof of Lemma 3 we have

$$\int_0^\infty u_n(x;\theta)\theta^{-1}d\theta=\int_x^\infty M_n((t-\theta)_+;x)\theta^{-1}d\theta+\int_0^x M_n((\theta-t)_+;x)\theta^{-1}d\theta$$

a) $k\leqq n\theta$. Integrating by parts we get

$$\int_{k/n}^x (\theta-k/n)\theta^{-1}d\theta=\int_{k/n}^x (\theta-k/n)d\ln\theta=(k/n)(\ln(k/n)-\ln x)+x-k/n.$$

b) $k\geqq n\theta$. By analogy with a) :

$$\int_x^{k/n} (k/n-\theta)\theta^{-1}d\theta=(k/n)(\ln(k/n)-\ln x)+x-k/n.$$

Hence

$$\int_0^\infty u_n(x;\theta)\theta^{-1}d\theta=e^{-nx}\sum_{k=0}^\infty ((nx)^k/k!)[x+(k/n)(\ln(k/n)-\ln x-1)]$$

$$=M_n(t\ln t;x)-x\ln x.$$

Let us set $h(x)=x\ln x$. Obviously, $h'(x)=1(x)$. But if $|xh''(x)|\leqq c$ for $h\in C[0,\infty)$ (where $c>0$ is a constant), then $M_n(h)-h=0(1/n)$ (see [18]). This proves (3.4).

Before proving (3.5) we shall give some notations and do some calculations. We denote $q_{n,k}(x)=l^{-nx}(nx)^k/k!$;

$$r_k(\theta)=(1/n)\sum_{i=0}^{k}(i/n)-\theta)_+,\quad r_{-1}=0\,;$$

$$s_k(\theta)=(1/n)\sum_{i=k}^{\infty}(\theta-i/n)_+.$$

Integrating by parts we see that

$$\int_0^\theta q_{n,k}(x)dx=(1/n)\sum_{i=k+1}^{\infty}q_{n,i}(\theta)\,;$$

$$\int_\theta^\infty q_{n,k}(x)dx=(1/n)\sum_{i=0}^{k}q_{n,i}(\theta).$$

Evidently $r_k(\theta)=0$ if $k\leqq n\theta$ and $s_k(\theta)=0$ if $k\geqq n\theta$. We shall show that

(4.5) $$r_k(\theta)\leqq(1/2)((k+1)/n-\theta)^2+\theta/2n \text{ for } k>n\theta \text{ and}$$

(4.6) $$s_k(\theta)\leqq(1/2)(k/n-\theta)^2+\theta/n \text{ for } k<n\theta.$$

If y is a positive we shall write $\{y\}$ for $y-[y]$, where $[y]$ is the greatest integer number less than or equal to y. It is easy to see that

$$r_k(\theta)=(1/n)\sum_{i=[n\theta]+1}^{k}(i/n-\theta)$$

$$=\pm(1/2)(k+1)/n-\theta)^2+(1/n^2)[k(k+1)/2-(n\theta-\{n\theta\})(n\theta-\{n\theta\}+1)/2]$$

$$-(\theta/n)(k-n\theta+\{n\theta\})$$

$$=(1/2)(k+1)/n-\theta)^2+\theta/2n+\{n\theta\}/2n^2-\{n\theta\}^2/2n^2-(k+1)/2n^2$$

$$<(1/2)(k+1)/n-\theta)^2+\theta/2n+k/2n^2-\{n\theta\}^2/2n^2-k/2n^2-1/2n^2$$

$$\leqq(1/2)((k+1)/n-\theta)^2+\theta/2n.$$

Thus (4.5) is proved.

By analogy with the proof of (4.5) we have :

$$S_k(\theta)=(1/n)\sum_{i=k}^{[n\theta]}(\theta-i/n)=\pm(1/2)((k/n)-\theta)^2+(\theta/n)(n\theta-\{n\theta\}-k+1)$$

$$-(1/n^2)[(n\theta-\{n\theta\})(n\theta-\{n\theta\}+1)-k(k+1)]/2=(1/2)((k/n)-\theta)^2+\theta/2n$$

$$+\{n\theta\}/2n^2-\{n\theta\}^2/2n^2-k/2n^2\leqq(1/2)((k/n)-\theta)^2+\theta/n.$$

Therefore (4.6) also holds true.

To prove (3.5) we shall use Abel transformation

$$(4.7)\qquad \sum_{k=1}^{m} a_k(b_k-b_{k-1})=a_{m+1}b_m-a_0b_0-\sum_{k=1}^{m+1} b_{k-1}(a_k-a_{k-1})$$

for the sequences $a_0, a_1, \ldots, a_{m+1}$ and $b_0, b_1, \ldots, b_m$. Obviously,

$$\int_0^\infty u_n(x\,;\theta)dx=\int_0^\theta M_n((t-\theta)_+\,;x)dx+\int_\theta^\infty M_n((\theta-t)_+\,;x)dx$$

$$=\sum_{k=0}^{\infty}\{((k/n)-\theta)_+\int_0^\theta q_{n,k}(x)dx+(\theta-(k/n))_+\int_\theta^\infty q_{n,k}(x)dx\}$$

$$=(1/n)\sum_{k=0}^{\infty}\{((k/n)-\theta)_+\sum_{i=k+1}^{\infty} q_{n,i}(\theta)+(\theta-k/n)_+\sum_{i=0}^{k} q_{n,i}(\theta)\}.$$

From the definition of $r_k(\theta)$ and $s_k(\theta)$ and (4.7) we get

$$\int_0^\infty u_n(x\,;\theta)dx=\sum_{k=0}^{\infty}\{(r_k(\theta)-r_{k-1}(\theta))\sum_{i=k+1}^{\infty} q_{n,i}(\theta)+(s_k(\theta)-s_{k+1}(\theta))\sum_{i=0}^{k} q_{n,i}(\theta)\}$$

$$=\sum_{k=0}^{\infty} r_k(\theta)q_{n,k+1}(\theta)+\sum_{k=0}^{\infty} s_k(\theta)q_{n,k}(\theta)$$

$$\leqq\sum_{k=0}^{\infty}(r_{k-1}(\theta)+s_k(\theta))q_{n,k}(\theta).$$

The inequalities (4.5), (4.6) and $M_n(t^2\,;x)=x^2+x/n$ give

$$\int_0^\infty u_n(x\,;\theta)dx\leqq\sum_{k=0}^{\infty}\{(1/2)(k/n-\theta)^2+\theta/n\}q_{n,k}(\theta)$$

$$=(1/2)(\theta^2+\theta/n)-\theta^2/2+\theta/n=(3/2)(\theta/n)\leqq c_4^*.\theta.n^{-1}$$

which proves Lemma 4.

P r o o f o f L e m m a 5. Let T be a linear operator mapping the space $L_p[a,b]$ to $L_q[a,b]$. If there exists a constant $C>0$ such that $\|Tf\|_{L_q[a,b]}\leqq c\|f\|_{L_p[a,b]}$ for every function $f\in L_p[a,b]$, then we say that the operator T is of type (p,q). The least C with this property we call (p,q)-norm of the operator.

To prove Lemma 5 we shall use the following Theorem (Riesz-Thorin) (see [1]). If T is a linear operator of type (p_i,q_i) with norm $N_i(i=0,\ 1)$, then T is an operator of type (p_α,q_α) with norm $N_\alpha\leqq N_0^{1-\alpha}.N_1^\alpha$, where

$$1\leqq p_i\leqq\infty\ ;\quad 1/p_\alpha=(1-\alpha)/p_0+\alpha/p_1\ ;\quad i=0,\ 1\ ;\quad 0<\alpha<1.$$

$$1\leqq q_i\leqq\infty\quad 1/q_\alpha=(1-\alpha)/q_0+\alpha/q_1$$

We shall apply this Theorem for the linear operator
$Tf(x)=\int_0^\infty u_n(x;\theta)f(\theta)\theta^{-1}d\theta$. From (3.5) and (3.4) we get

$$\| Tf\|_{L_1[0,\infty)} \leqq \int_0^\infty [\int_0^\infty u_n(x;\theta)dx]\theta^{-1}f(\theta)d\theta \leqq c_4 . n^{-1} \|f\|_{L_1[0,\infty)},$$

$$\| Tf\|_{L_\infty[0,\infty)} \leqq \|f\|_{L_\infty[0,\infty)} \sup_{x\in[0,\infty)} \{\int_0^\infty u_n(x;\theta)\theta^{-1}d\theta\} \leqq c_{4,1} . n^{-1} \|f\|_{L_\infty[0,\infty)}.$$

Therefore

$$(4.8) \qquad \| Tf\|_{L_p[0,\infty)} \leqq c_5 . n^{-1} \|f\|_{L_p[0,\infty)}.$$

Let $f=\psi . g''$. Lemma 3 gives

$$(4.9) \qquad \| Tf\|_{L_p[0,\infty)} = \| T(\psi g'')\|_{L_p} = \{\int_0^\infty [\int_0^\infty u_n(x;\theta)g''(\theta)d\theta]^p dx\}^{1/p}$$

$$= \| M_n g - g\|_{L_p[0,\infty)}.$$

As an immediate consequence of (4.8) and (4.9) we obtain

$$\| M_n g - g\|_{L_p[0,\infty)} \leqq c_5 . n^{-1} \|\psi g''\|_{L_p[0,\infty)},$$

which ends the proof of Lemma 5.

Proof of Lemma 6. By analogy with Theorem 4.4 from [1] (p. 123) the following theorem can be proved.

Theorem B. *Let f be a bounded and measurable function in* $[0, \infty)$, *Sf is an interpolating spline of degree 1 at the points i/n, where $i=0, 1, \dots, n$ and $n\in\mathbb{N}$. Then for $1\leqq p\leqq\infty$ $\| Sf-f\|_{L_p[0,\infty)} \leqq \tau_2(f;1/n)_{L_p[0,\infty)}$ holds.*

We shall prove the following

Lemma A. *If F is a linear function in* $[a,b]$ *and* $1\leqq p<\infty$, *then*

$$(b-a)^{1/p}(|F(a)|^p+|F(b)|^p)^{1/p} \leqq [2(p+1)]^{1/p}(\int_a^b |F(t)|^p dt)^{1/p}.$$

Proof. We shall consider two cases.

a) $F(a).F(b)\geqq 0$. Then for $t\in[a,b]$ $|F(t)|^p=|F(a).(b-t)/(b-a)+F(b) .(t-a)/(b-a)|^p \geqq [1/(b-a)^p][|F(a)|^p(b-t)^p+|F(b)|^p(t-a)^p]$.

Therefore

$$(4.10)\qquad \int_a^b |F(t)|^p dt \geqq [1/(b-a)^p](|F(a)|^p+|F(b)|^p)\int_0^{b-a} x^p\,dx$$

$$=((b-a)/(p+1))(|F(a)|^p+|F(b)|^p).$$

b) $F(a).F(b)<0$. Without loss of generality we may assume that $F(a)>0$ and $F(a)\geqq|F(b)|$. Then for $t\in[a,(a+b)/2]$ we have $F(t)\geqq(2F(a)/(b-a))((a+b)/2-t)$. Hence

$$(4.11)\quad \int_a^b |F(t)|^p dt \geqq \int_a^{(a+b)/2} |F(t)|^p dt \geqq (2^p|F(a)|^p/(b-a)^p \int_0^{(b-a)/2} x^p dx$$

$$=|F(a)|^p(b-a)/(p+1)\geqq((b-a)/(2(p+1)))(|F(a)|^p+|F(b)|^p).$$

From (4.10) and (4.11) we obtain

$$(b-a)^{1/p}(|F(a)|^p+|F(b)|^p)^{1/p}\leqq[2(p+1)]^{1/p}(\int_a^b|F(t)|^p dt)^{1/p},$$

which implies the statement of the Lemma A.

Let Sf and Sg are interpolating splines of degree 1 at the points i/n, $i=0, 1,\dots$

Applying Lemma A with $[a,b]=[k/n,\ (k+1)/n]$ and the linear function $F=Sf-Sg$ we get

$$4^p\int_{k/n}^{(k+1)/n} |(Sf-Sg)(x)|^p dx \geqq (1/n)[|f(k/n)-g(k/n)|^p+|f((k+1)/n)-g((k+1)/n)|^p].$$

Hence

$$(4.12)\qquad \{(1/n)\sum_{k=0}^{\infty}|g(k/n)-f(k/n)|^p\}^{1/p}\leqq 4\,\|Sf-Sg\|_{L_p[0,\infty)}.$$

We shall use Jensen inequality

$$(4.13)\qquad (\sum_{k=0}^{\infty}\alpha_k a_k)^p \leqq \sum_{k=0}^{\infty}\alpha_k . a_k^p,\ \text{where } \alpha_k\geqq 0 \text{ and } \sum_{k=0}^{\infty}\alpha_k=1.$$

From the definition of $M_n(f)$ and (4.13) we obtain

$$\|M_n f-M_n g\|_{L_p[0,\infty)}=\{\int_0^{\infty}[\sum_{k=0}^{\infty} l^{-nx}((nx)^k/k!)|f(k/n)-g(k/n)|]^p dx\}^{1/p}$$

$$\leqq\{\int_0^{\infty}\sum_{k=0}^{\infty} l^{-nx}((nx)^k/k!)|f(k/n)-g(k/n)|^p dx\}^{1/p}.$$

Because of $\int_0^{\infty} l^{-nx}(nx)^k/k! = 1/n$ we have

$$\| M_n f - M_n g \|_{L_p[0,\infty)} = \{(1/n) \sum_{k=0}^{\infty} |f(k/n) - g(k/n)|^p\}^{1/p}.$$

The result of (4.12) gives the estimate

$$\| M_n f - M_n g \|_{L_p[0,\infty)} \leqq 4 \| Sf - Sg \|_{L_p[0,\infty)} \leqq 4\{\| Sf - f \|_{L_p} + \| f - g \|_{L_p} + \| Sg - g \|_{L_p}\}.$$

It is easy to see that from Theorem B, (3.1) and (1.5) we get

$$\| M_n f - M_n g \|_{L_p[0,\infty)} \leqq 4\{\tau_2(f; 1/n)_p + c_3\, \tau_2(f; \Delta_m(x))_p + (2/n^2) \| g'' \|_{L_p}\}.$$

Let $m = [n^{1/2}] + 1$. Then (1.2) yields

(4.14) $$\tau_2(f; 1/n)_p \leqq \tau_2(f;\ (x/n)^{1/2} + 1/n)_p \quad \text{and}$$

(4.15) $$\tau_2(f; \Delta_m(x))_p \leqq \tau_2(f; (x/n)^{1/2} + 1/n)_p.$$

From (3.2) and the inequality $1/n^2 \leqq 16/m^4$ we obtain

(4.16) $$(1/n^2) \| g'' \|_{L_p} \leqq 16 \| (1/m^4) g'' \|_{L_p} \leqq (1/m^2)(x + 1/m^2) g'' \|_{L_p} \leqq c_3 \cdot \tau_2(f; \Delta_m(x))_p.$$

Therefore (4.14), (4.15) and (4.16) give the estimate

$$\| M_n f - M_n g \|_{L_p[0,\infty)} \leqq c_6 \cdot \tau_2(f; (x/n)^{1/2} + 1/n)_p,$$

where c_6 is an absolute constant. Lemma 6 is proved.

5. Proof of the main Theorem

Let $m = [n^{1/2}] + 1$. Using the result from Lemma 5 we have

$$\| M_n g - g \|_{L_p} \leqq c_5 \| (x/n) g'' \|_{L_p} \leqq c_{5,1} \| (1/m^2) \varphi^2 g'' \|_{L_p} \leqq c_7 \cdot \tau_2(f; \Delta_m(x))_p.$$

The estimate, (3.1) and Lemma 6 give

$$\begin{aligned} \| M_n f - f \|_{L_p[0,\infty)} &\leqq \| f - g \|_{L_p} + \| M_n g - g \|_{L_p} + \| M_n f - M_n g \|_{L_p} \\ &\leqq c_3\, \tau_2(f; \Delta_m(x))_p + c_7 \cdot \tau_2(f; \Delta_m(x))_p + c_6 \cdot \tau_2(f; (x/n)^{1/2} + 1/n)_p \\ &\leqq c \cdot \tau_2(f; (x/n)^{1/2} + 1/n)_p, \end{aligned}$$

which proves our Theorem.

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Bulgarian Academy of Sciences
Institute of Mathematics
P.O. Box 373
Sofia 1113
BULGARIA

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