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On the Controllability of Control Constrained Linear Systems

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The notion of controllability index is extended to control constrained linear systems. An exact estimate of the growth of the reachable set for small time intervals is proved. The robustness of the instant local controllability is discussed.

1. Introduction

Consider a linear time-invariant control system

$$(1) \qquad \dot{x}=Ax+u,$$

where $x\in \mathsf{R}^n$ is the state, $u\in U$ is the control parameter, U is a given subset of R^n. Denote by $R(x, T)$ the reachable set of (1) on $[0, T]$, starting from the point x at $t=0$, namely,

$$R(x\,;T)=\{x(T)\,;\ x(\cdot) \text{ solves (1) for some } u(\cdot)\in L_1[0,T],$$
$$u(t)\in U, \text{ and } x(0)=x\}.$$

Definition. *System* (1) *is U-instantly locally controllable* (*ILC*) *iff* $0\in \operatorname{int} R(0\,;T)$ *for every* $T>0$.

This crucial property of system (1) is very well studied. Together with the classical result of Kalman concerning the case when U is a subspace, we shall mention here the paper of R. Brammer [1], which is the first one studying the controllability in the control constrained case, and the paper of R. Bianchini [2]. The latter gives necessary and suffcient conditions for ILC for an arbitrary constraining set U.

In the case when U is a subspace of R^n a quantitative characterization of the controllability of the system (1) is given by the so-called controllability index

$$\sigma=\min\{k\,;U+AU+\ldots+A^{k-1}U=\mathsf{R}^n\}.$$

In this case it is proved by V. Korobov [3] that $R(0\,;t)$ contains a ball centered at the origin with a radius proportional to t^σ and, moreover, the exponent of this estimate is exact. The estimations of the radius of the maximal ball contained in $R(0\,;t)$ as a function of t are closely related to the local Hölder property of

Bellman's function $T(x)=\inf\{t\ ;\ x\in R(0\ ;\ t)\}$ at $t=0$ (see the paper by N. Petrov [4]) and, more generally, to the sensitivity of the time-optimal control problem.

Using polynomial approximations G. Stefani [5] proves an estimate of the Hölder exponent for a nonlinear system which is affine in the control. There the constraining set U is not supposed to be a subspace. However, this important result does not imply, at least directly, an estimate of the Hölder exponent in case of a control constrained system, because of the assumption, that the approximating system (which is constructed independently of U) is locally controllable by means of controls with values in U.

In the present paper we extend the notion of controllability index to the system (1) with the constraining set U which locally is a cone. Let $\rho(t)=\max\{\alpha\ ;\ |x|\leqq\alpha$ implies $x\in R(0\ ;\ t)\}$. In this case we obtain the exact estimate

$$mt^{\sigma}\leqq\rho(t)\leqq Mt^{\sigma}$$

of the radius of the maximal ball contained in the reachable set. Here m and M are positive constants and σ is the U-controllability index defined in Section 2. Thus the number σ is a quantitative characterization of the controllability property of the system (1) by means of U. The definition of the U-controllability index is based on a necessary and sufficient controllability condition in the form of the Kalman rank condition, which is also presented in Section 2.

In Section 3 we discuss the robustness of the ILC property of the systems (1) with constraints, i. e. the problem whether ICL will be preserved in case of perturbations. It turns out that this property is in general not stable even with respect to perturbations only in the matrix A (with a fixed set U). A sufficient condition for robust controllability is presented.

2. Main result

In this section we shall prove an estimation of the growth of the reachable set of (1) with the constraint $u\in U$, which turns out to be exact and can be considered as a quantitative measure of the controllability of the system (1) with control constraints. First we shall introduce some notations.

Given a convex set $V\subset \mathsf{R}^n$, we denote by $F(V)$ the facial space of V at the origin :

$$F(V)=\{v\in\mathsf{R}^n\ ;\quad \delta v\in V \text{ for all sufficiently small } |\delta|\}.$$

We denote by con V the minimal closed convex cone with vertex at the origin, generated by V. Define a sequence of spaces

$$H_1=F(\text{con } U),\quad H_{k+1}=F(H_k+AH_k+\text{con } U),\quad k=1,\dots,n-1.$$

Theorem 1. *System* (1) *with the constraint $u\in U$ is instantly locally controllable if and only if $H_n=\mathsf{R}^n$.*

Proof. The assertion of this theorem is another formulation of the results given in theorems 2.1 and 3.1 in [2]. Let the system (1) be U-ILC, but dim $H_n<n$. Since $H_k=H_{k-1}$ for some k implies $H_{k+1}=H_k$, the space H_n is A-invariant and $F(H_n+\text{con } U)=H_n$. In particular, if S is the orthogonal complement of H_n and $P_s(\text{con } U)$ is the projection of con U over S, then $F(P_s \text{con } U)=\{0\}$. That

contradicts Theorem 2.1 [2], which claims that $F(P_s \operatorname{con} U) \neq \{0\}$ for every A^*-invariant subspace S (A^* is the transposed matrix of A). On the other hand, if there is an A^*-invariant subspace S such that $P_s(\operatorname{con} U)=\{0\}$, then $S^\perp$ contains H_n. Hence, using again Theorem 2.1 [2], we conclude that $H_n=\mathrm{R}^n$ implies U-ILC. The proof is complete.

Since we are interested in the local behaviour of the function $\rho(t)$ defined in Section 1, further we shall consider a compact set U, which guarantees that $\rho(t)$ is bounded when $t\in[0, 1]$. Moreover, we shall suppose that U looks locally like a cone :

Assumption A. U is compact and there exist a convex cone V and $\nu>0$ such that $\operatorname{co} U\cap B_\nu=V\cap B_\nu$, where $B_\nu=\{x\in \mathrm{R}^n \,;\, |x|\leqq \nu\}$.

If system (1) is ILC, we can define the integer

$$\sigma=\min\{k \,;\, H_k=\mathrm{R}^n\}.$$

We shall refer to σ as to U-controllability index of (1). Obviously in the unconstrained case the above notion coincides with the well-known index of controllability. The following result could be a motivation for our extension of this notion.

Theorem 2. *Let system (1) be ILC and condition A be fulfilled. If σ is the U-controllability index of (1), then there exist constants $m>0$ and M such that*

(2) $$mt^\sigma \leqq \rho(t) \leqq Mt^\sigma \text{ for every } t\in[0, 1].$$

Proof. If $Y(\cdot):\mathrm{R}^n\to\mathrm{R}^n$ is an analytic function, then we shall denote by $\exp(tY)x$ the solution of the equation $\dot{x}=Y(x)$, $x(0)=x$, at t. For two analytic vector fields Y and Z, the following Campbell-Baker-Hasdorff formula holds :

(3) $$\exp(-t_1 Y)\exp(t_2 Z)\exp(t_1 Y)=\exp(t_2 Z-t_1 t_2[Y,Z]+\ldots),$$

where $[Y,Z]=(dZ/dx)Y-(dY/dx)Z$ is the Lie bracket of Y and Z, and the sum on the right-hand side is convergent to an analytic vector field for all sufficiently small t_1 and t_2. Since we shall use further only affine vector fields, we can assume that (3) holds for t_1, $t_2\in[0, 1]$.

We shall denote by $o(s)$ any vector or scalar function with the property that $|o(s)|/s$ tends to zero together with s.

Now, we start with the proof. First we shall prove the left estimate in (2). Denote by G_k the set of all vectors $g\in H_k$ for each of which there exist $g_i\in A^i H_k$, $i=1,\ldots n-k$, and positive numbers c and d such that for every $p\geqq k$ and $\alpha\in[0, 1]$

$$\exp\left(\alpha t^p g+\alpha \sum_{i=1}^{n-p} t^{p+1} g_i+o(t^n)\right)\exp(ctA)x\in R(x, dt),$$

for every $x\in\mathrm{R}^n$ and $t\in[0, 1]$. Obviously $g\in G_k$ implies $\alpha g\in G_k$ for $\alpha\in[0, 1]$. We shall prove that $h, g\in G_k$ implies $h+g\in G_k$, which yields that G_k is a convex cone in R^n. Let for $p\geqq k$

$$K_1(t)x=\exp\left(\alpha t^p h+\alpha \sum_{i=1}^{n-p} t^{p+i} h_i+o(t^n)\right)\exp(c_1 tA)x\in R(x\,;\, d_1 t),$$

and

$$\exp(\alpha t^p g+\alpha \sum_{i=1}^{n-p} t^{p+i} g_i+o(t^n)) \exp(c_2 tA) x\in R(x\,; d_2 t),$$

$x\in R^n$, $t\in[0,\ 1]$. From (3) it easily follows that

$$K_2(t)x=\exp(c_2 tA) \exp(\alpha t^p g+\alpha \sum_{i=1}^{n-p} t^{p+i} \tilde{g}_i+o(t^n)) x\in R(x, d_2 t),$$

for some $\tilde{g}_i\in A^i H_k$. Hence $K(t)x=K_2(t)K_1(t)x\in R(x\,;(d_1+d_2)t)$. On the other hand,

$$K(t)=\exp(c_2 tA) \exp(\alpha t^p(h+g)+\alpha \sum_{i=1}^{n-p} t^{p+i}(h_i+\tilde{g}_i)+o(t^n)) \exp(c_1 tA)$$

$$=\exp(\alpha t^p(h+g)+\alpha \sum_{i=1}^{n-p} t^{p+i} f_i+o(t^n)) \exp((c_1+c_2)tA)$$

for some $f_i\in A^i H_k$, which gives us $g+h\in G_k$.

Now, we shall prove by induction that $G_k=H_k$ for $k=1,\dots,n$. Let $k=1$ and $u\in \operatorname{con}(\operatorname{co} U)$. For some integer l we have $v=u/l\in V\cap B_v$ and hence $\alpha v\in U$ for every $\alpha\in[0,\ 1]$. Then we can use the control function $\alpha t^{p-1} v$, $p\geqq 1$ in (1) to obtain that

$$K(t)x=\exp(t(Ax+\alpha t^{p-1} v)) x\in R(x\,; t), \quad x\in R^n,\ t\in[0,\ 1].$$

On the other hand, we have

$$K(t)=\exp(t(Ax+\alpha t^{p-1} v)) \exp(-tA) \exp(tA)$$

$$=\exp(\alpha t^p v+0.5\, t^{p+1} Av+\dots) \exp(tA),$$

which yields $v\in G_1$ if $v\in H_1$. From the additive invariance of G_1 we obtain $u\in G_1$ if only $u\in H_1$.

Suppose that $H_k=G_k$ and take an arbitrary $g=Ah\in AH_k$. From $h\in G_k$ we have

$$K_1(t)x=\exp(\alpha t^p h+\alpha \sum_{i=1}^{n-p} t^{p+i} h_i+o(t^n)) \exp(c_1 tA) x\in R(x\,; d_1 t),$$

and since $-h\in H_k=G_k$,

$$K_2(t)x=\exp(c_2 tA) \exp(-\alpha t^p h+\alpha \sum_{i=1}^{n-p} t^{p+i} \tilde{h}_i+o(t^n)) x\in R(x, d_2 t)$$

for some $h_i, \tilde{h}_i\in A^i H_k, c_1, c_2, d_1, d_2>0$ and for every $t, \alpha\in[0,\ 1]$, $x\in R^n$. Then for every $\beta>0$

(4) $$K(t)x=K_2(t)\exp(\beta tA)K_1(t)x\in R(x;(d_1+d_2+\beta)t).$$

On the other hand, using (3) we obtain that

$$K(t)\exp(-(c_1+c_2+\beta)tA)=K_2(t)\exp(\alpha t^p h+\alpha\beta t^{p+1}Ah+\alpha t^{p+1}h_1$$

$$+\alpha\sum_{i=1}^{n-p-1}t^{p+i+1}g_i+o(t^n))\exp(-c_2tA)=\exp(c_2tA)\exp(\alpha t^{p+1}(\beta Ah+\tilde{h}_1+h_1)$$

$$+\alpha\sum_{i=1}^{n-p-1}t^{p+i+1}f_i+o(t^n))\exp(-c_2tA)=\exp(\alpha t^{p+1}(\beta Ah+\tilde{h}_1+h_1)$$

$$+\alpha\sum_{i=1}^{n-p-1}t^{p+i+1}\tilde{f}_i+o(t^n)),$$

where $g_i, f_i, \tilde{f}_i\in A^2H_k\subset AH_{k+1}$. This together with (4) implies that $\beta Ah+h_1+\tilde{h}_1\in G_{k+1}$ for every $\beta>0$. Since G_{k+1} is a cone and $h_1, \tilde{h}_1\in AH_k$, we can conclude that $g=Ah\in cl(G_{k+1}\cap AH_k)$. Thus $AH_k\subset cl(G_{k+1}\cap AH_k)$ and from the convexity of G_{k+1} it follows that $AH_k\subset G_{k+1}$.

Since obviously $H_k\subset G_k\subset G_{k+1}$ and as above $V\cap H_{k+1}\subset G_{k+1}$, we obtain that $H_{k+1}=G_{k+1}$. Thus we proved by induction that $G_\sigma=H_\sigma=\mathrm{R}^n$.

Let $h_1,\dots,h_{2n}$ be a nonnegative generating set of R^n. Then there exist positive numbers c_i and d_i, $i=1,\dots,2n$, such that

$$\exp(t^\sigma h_i+o(t^n))\exp(c_itA)x\in R(x;d_it),\quad x\in\mathrm{R}^n, t\in[0,\ 1].$$

Applied to $x=0$ this gives us

$$t^\sigma h_i+o(t^\sigma)\in R(0;d_it),\quad i=1,\dots,2n,$$

which by a standard argument implies the desired left estimate in (2) for some positive constant m.

Now, we shall prove the existence of the constant M in (2). Since U is compact, the case $\sigma=1$ is trivial. Therefore, we suppose that $\sigma>1$. Denote $S_k=H_{k-1}^\perp$, $k=2,\dots\sigma$ and let P_k be the projection operator over S_k. From the definition of H_k it follows that $F(P_kV)=\{0\}$ for $k=1,\dots,\sigma$. Moreover, if $\tilde{P}_k$ is the projection operator over $S_k\cap H_k$, then $F(\tilde{P}_kV)=\{0\}$ for $k<\sigma$. Hence there exists a vector $l_\sigma\in S_\sigma$ and vectors $\tilde{l}_k\in S_k\cap H_k=S_k\cap S_{k+1}^\perp$ $(k<\sigma)$ such that

(5) $$\langle l_\sigma, v\rangle>0 \text{ for every } v\in P_\sigma V, v\neq 0,$$

(6) $$\langle \tilde{l}_k, v\rangle>0 \text{ for every } v\in\tilde{P}_kV,\ v\neq 0.$$

Define successively $l_k=(l_{k+1}+\tilde{l}_k)/|l_{k+1}+\tilde{l}_k|$, $k=\sigma-1,\dots,2$. We shall prove that the following properties hold true :

(i) $l_k\in S_k$, $|l_k|=1$ and there exists $c_k>0$ such that $\langle l_k, v\rangle\geqq c_k|P_k v|$ for every $v\in V$, $k=2,\dots,\sigma$;

(ii) $(l_{k+1})^0\cap S_{k+1}\subset(l_k^0)$ for $k=2,\dots,\sigma-1$.

Since $l_{k+1}\in S_{k+1}\subset S_k$ and $\tilde{l}_k\in S_k$ we have $l_k\in S_k$. Taking an arbitrary $v\in V$, we have $\langle l_k, v\rangle=\langle l_k, P_k v\rangle$. From (5) and from the closedness of $P_k V$ we conclude that there exists $c_k>0$ such that (1) is fulfilled for $k=\sigma$.

Now, suppose that (i) is fulfilled for $k+1$. Then setting $\eta=1/|l_{k+1}+\tilde{l}_k|>0$ we obtain that

$$\langle l_k, v\rangle=\langle l_k, P_k v\rangle=\eta\langle l_{k+1}+\tilde{l}_k, P_{k+1}v+\tilde{P}_k v\rangle$$

$$=\eta\langle l_{k+1}, P_{k+1}v\rangle+\eta\langle\tilde{l}_k, \tilde{P}_k v\rangle\geqq\eta c_{k+1}|P_{k+1}v|+\eta d_k|\tilde{P}_k v|$$

$$\geqq\eta\ \min\{c_{k+1}, d_k\}|P_k v|=c_k|P_k v|,$$

where $d_k>0$ exist according to (6), thanks to the closedness of $\tilde{P}_k V$. Thus (i) is proved.

If $l\in(l_{k+1})^0\cap S_{k+1}$, then $\langle l, l_{k+1}\rangle\leqq 0$. We have

$$\langle l, l_k\rangle=\langle l, l_{k+1}+\tilde{l}_k\rangle/|l_{k+1}+\tilde{l}_k|=\langle l, l_{k+1}\rangle/|l_{k+1}+l_k|\leqq 0,$$

which gives us $l\in(l_k)^0$.

Further we denote by c_1, $c_2,\dots$ appropriate positive constants. We shall prove the following assertion :

A_k. There is a constant M_k such that if $x(\cdot)$ is a trajectory of (1) on $[0, t]$ and $x(t)\in(l_k)^0\cap S_k$, then

$$|P_k x(s)|\leqq M_k t^k \text{ for every } s\in[0, t].$$

In particular, for $k=\sigma$ and $l=-l_\sigma$ we obtain that if $-\psi(t)l_\sigma\in R(0, t)$ with $\psi(t)>0$, then $\psi(t)\leqq M_\sigma t^\sigma$, which implies the upper estimate in (2).

Let us prove the assertion A_k by induction. From the compactness of U we have $|x(s)|\leqq M_1 t$, $s\in[0, t]$, for some constant M_1 and an arbitrary trajectory $x(\cdot)$ of (1). We shall refer to this fact as to the assertion A_1.

Let A_k hold for some $k\geqq 1$ and let $x(\cdot)$ be a trajectory of (1) on $[0, t]$ corresponding to an admissible control $u(\cdot)$, such that $x(t)\in(l_{k+1})^0\cap S_{k+1}$. If $k>1$, we may represent $x(s)=y(s)+z(s)$, where $y(s)\in H_{k-1}$ and $z(s)\in S_k$. According to (ii) we have $x(t)\in(l_k)^0\cap S_k$, and by the inductive supposition

$$|z(s)|=|P_k x(s)|\leqq M_k t^k,\quad s\in[0, t] \tag{7}$$

(in case $k=1$ we denote $z(s)=x(s)$, $y(s)=0$ and (7) holds by A_1). Thus

$$\langle l_{k+1}, x(t)\rangle = \int_0^t (\langle l_{k+1}, Ax(s)\rangle + \langle l_{k+1}, u(s)\rangle)ds$$

$$\geqq \int_0^t \langle l_{k+1}, A(y(s)+z(s))ds + d_{k+1} \| P_{k+1} u(\cdot)\|_{L_1}.$$

From $Ay(s)\in AH_{k-1}\subset H_k$ we get $\langle l_{k+1}, Ay(s)\rangle = 0$. Then using (7) and the relation $x(t)\in(l_{k+1})^0$ we obtain that

$$0 \geqq \langle l_{k+1}, x(t)\rangle \geqq -c_1 t^{k+1} + d_{k+1} \| P_{k+1} u(\cdot)\|_{L_1}.$$

Hence

(8) $$\| P_{k+1} u(\cdot)\|_{L_1} \leqq c_2 t^{k+1}.$$

Now, consider

$$P_{k+1} x(s) = \int_0^s (P_{k+1} Ax(\theta) + P_{k+1} u(\theta))d\theta.$$

Since $P_{k+1} Ax(\theta) = P_{k+1} Az(\theta)$, using (7) and (8) we obtain

$$|P_{k+1} x(s)| \leqq (c_3 + c_2) M_k t^{k+1},$$

which completes the proof of the theorem.

The assumption that the set U is locally a cone is essential for the lower estimation in (2). This can be seen from the next example.

Example. Let $U=\{(u,v)\in \mathrm{R}^2 ; u \geqq v\}$ and consider the system

(9) $$\dot{x} = y + u \qquad \dot{y} = v \quad .$$

In this case $V=\mathrm{con}(\mathrm{co}\, U)=\{(u,v)\in\mathrm{R}^2 ; u\geqq 0\}$, $H_1=\{(0,v) ; v\in\mathrm{R}\}$, $H_2=\mathrm{R}^2$, thus the system is controllable. The V-controllability index of (9) is equal to 2, but it is easy to calculate that $\rho(t)\leqq ct^3$. The reason is that (in the notations of the proof of Theorem 2) the vector (0, 1) does not belong to G_1 but $\pm(0,1)\in G_2$. We may see this by using the control $(u(s), v(s)) = 0.5(t^2, \pm t)$ for $s\in[0, t]$. Then $G_3=\mathrm{R}^2$ and we obtain $mt^3 \leqq \rho(t)$.

The above example also shows how one can use the proof of Theorem 2 to obtain estimations of the reachable set in case of sets U being more complicated than cones.

3. Robust controllability

It is well known that the set of ILC systems with a fixed dimension is open (and dense) in the natural topology in the set of all systems with the same dimension. Unfortunately, this important property does not hold in case of constrained control. It is obvious, that arbitrarily small perturbations in the set U may destroy the facial space $F(U)$ and thus the ILC property. The following example shows that an ILC system may lose this property as a result of arbitrarily small perturbations in the matrix A (even when the set U is fixed).

Example. Consider the system

$$\dot{x}=Ax+b_1u_1+b_2u_2+b_3u_3, \quad x\in \mathrm{R}^4,$$

where

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & p & 0 & p \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad b_1=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad b_2=\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad b_3=\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

with the constraints $u_2\geqq 0$ and $u_3\geqq 0$. Here p is a parameter. Checking the necessary and sufficient condition given by Theorem 1, it is easy to verify that the above system is ILC just when $p=1$. The reason for this effect is, that for $p=1$ the projection of con U over the orthogonal complement of the space H_1+AH_1 contains a line, and dim $H_2>$ dim H_1, while for $p\neq 1$ this is not true. The above discontinuity of the facial space of the projection of a convex cone over a subspace is possible only when the dimension of the subspace in question is greater than 1 (equal to two in our example). This observation leads to the sufficient condition for robust controllability given in Theorem 4. But first we shall reformulate Theorem 1 in a more convenient way, using the following notations:

$$V=cl\ \mathrm{con}(\mathrm{co}\ U) \text{ (assumption } A \text{ is not required below)};$$

$$K(L)=L+AL+\ldots+A^{n-1}L \quad \text{for } L\text{-subspace of } \mathrm{R}^n;$$

$$D_0=\{0\}, \quad D_k=K(F(D_{k+1}+V)), \quad k=1, 2,\ldots$$

By a standard linear algebra argument one can obtain from Theorem 1 the following result.

Theorem 3. *System* (1) *is* U-ILC *if and only if*

$$D_{[\frac{n+1}{2}]}=\mathrm{R}^n$$

(*here* $[k]$ *means the largest integer not greater than* k).

Now, we can formulate a theorem for robust controllability. Together with system (1) we shall consider the perturbed system

$$\dot{x}=\tilde{A}x+u \tag{10}$$

with the same constraining set U.

Theorem 4. *Let system* (1) *be U-ILC and let* $\dim D_1 \geqq n-1$. *Then there exists* $\varepsilon > 0$ *such that for every* $n * n$*-matrix* $\tilde{A}$ *for which* $|A-\tilde{A}| < \varepsilon$, *system* (10) *is also U-ILC.*

Proof. Since (1) is $U-$ILC and $\dim D_1 \geqq n-1$, then $D_2 = \mathrm{R}^n$ and $\dim F(P_2 V) = 1$, where P_2 is the projecting operator over $D_1^{\perp}$. The set $\tilde{D}_1$ corresponding to $\tilde{A}$ has its orthogonal subspace $\tilde{D}_1^{\perp}$ arbitrarily closed to D_1, if only ε is sufficiently small. Thanks to the fact that $F(P_2 V)$ is one-dimensional,we may conclude that $F(\tilde{P}_2 V)$ is also one-dimensional ($\tilde{P}_2$ is the projecting operator over $\tilde{D}_1^{\perp}$) and hence $\tilde{D}_2 = \mathrm{R}^n$, i.e. (10) is U-ILC.

The sufficient condition given in Theorem 4 is essential, as it can be seen from the example given at the beginning of this section. If $p=1$, then $D_2 = \mathrm{R}^4$, but $\dim D_1 = 2 < n-1$ and therefore arbitrarily small changes of p may destroy the ILC property.

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