Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

Mathematica Balkanica

New Series Vol. 2, 1988, Fasc. 2-3

Constancy Levels of Increasing Mappings of Some Kinds of Ordered Sets

Duro R. Kurepa

0. At some occasions I had the opportunity to ask whether there exists a strictly increasing function from the tree σQ of some bounded well-ordered non empty subsets of $(Q, \leqq)$ into $(Q, \leqq)$ (see D. Kurepa, References $\leqq$ 1951).

0 : 1. Finally, I succeeded to answer the question by the negative. The proof was quite complicated and was backed on the proof that the tree σQ was the union of Al_1 antichains but not of Al_0 antichains (see [8 T. 3.1; 9 TR. 1, 10 Th. 3.1].

0 : 2. A remarkable statement. Afterwards, I found a nice statement that whenever

(0 : 3) $(E, \leqq)$ be an ordered set, then the tree

(0 : 4) $(w(E, =), <_k)$ of all well-ordered subsets of (0 : 3) (including the empty set v as well) is such that there is no strictly increasing mapping of (0 : 4) into the starting set (0 : 3). The proof of this fundamental fact is unthinkingly simple (see D. Kurepa [11]).

0 : 5. In this note we shall exhibit a proof of a statement published in D. Kurepa [10, Théorème 3.1][1] in a generalized version (see Theorem 1 : 0).

0 : 6. Number $\Gamma(E, \leqq)$. For an ordered set $(E, \leqq)$ we denote by $\Gamma(E, \leqq)$ the least ordinal number n which is strictly greater than the order type of every well-ordered subset of (0 : 3).

0 : 7. We denote by $w(E, \leqq)$ (resp. $\sigma(E, \leqq)$) the system of all well-ordered (and bounded and non empty) subsets of (0 : 3) ordered by the relation $\leqq_k$ "to be an initial segment of".

0.8. In $n^0 2$ we shall prove an interesting theorem concerning strictly increasing functions on union of Al_γ antichains into a subchain of $(P\omega_\nu, \subset)$ (see Theorem 2 : 1).

[1] Todorčević Stevo reminded me that no proof of this theorem was published.

1. Statement of theorem

1:0. Theorem. *Let α be any ordinal number and $(E, \leqq)$ any subset of the power set $(P\omega_\alpha, \subset)$ such that*

$$(1:1) \qquad \Gamma(E, \leqq) = \Gamma P\omega_\alpha = \omega_{\alpha+1}.$$

If f is any increasing mapping of $(\sigma E, \leqq)$ into $(E, \leqq)$, then the system S of all pairs $\{x, y\}_{\neq}$ of points of $\sigma(E, \leqq)$ for which $fx = fy$ has a power $> Al_\alpha$ (Al stands for the Hebrew "alef").

P r o o f. Assume, contrarily, that the statement 1 :0 does not hold and that for some ordinal α and some subset $(E, \leqq)$ of $P\omega_\alpha$ satisfying (1 : 1) the system S of all 2-point-subsets of $(E, \leqq)$ in each of which an increasing function $f:\sigma(E, \leqq)$ into $(E, \leqq)$ is constant, has a power $\leqq Al_\alpha$. Because of the condition (1 : 1) we infer that the tree $\sigma(E, \leqq)$ has a rank (=height) equal to $\omega_{\alpha+1}$. Therefore, there would be an ordinal n of second kind such that

$$(1:2) \qquad \cup S \subset (\cdot, n]_\sigma : = \cup R_i \quad (i \leqq n)$$

and that, consequently, in the remaining right section

$$(1:3) \qquad (n, \cdot)_\sigma : = \cup R_i \sigma, \quad (n < i < \omega_{\alpha+1})$$

the mapping f is strictly increasing.
Obviously, $\gamma(1:3) = \omega_{\alpha+1} = \Gamma(E, \leqq)$; therefore there exists an ordinal r between n and $\omega_{\alpha+1}$ and such that $n + r = r$. If we consider a point $a \in R_r$ (1 : 3), then we have well-defined r-sequence of points $b_i = fa_i$ $(i < r)$, where $a_i <_k a$ and $a_i \in R_i$ (1 : 3). The points b_i constitute a well-ordered subset of f (1 : 3). Now let us consider a following function g on

$$(1:4) \qquad \{v\} \cup \sigma(E, \leqq):$$

we set $gv : = b_0$ and $\{gR_i(\cdot, n)]_\sigma\} = b_{i+1}$ $(i \leqq n)$; then g would be strictly increasing from $\{v\} \cup \sigma$ to $(E, \leqq)$. In such circumstances, $(E, \leqq)$ has no last member, because in the opposite case the tree (1 : 4) would be identical to the tree $w(E, \leqq)$ and g would be a strictly increasing mapping of $w(E, \leqq)$ into $(E, \leqq)$, contrarily to our statement 0 : 2.

Thus $(E, \leqq)$ has no last element ; in this case let us consider a new point p and let us join p to $(E, \leqq)$ and convene that p follows all points of $(E, \leqq)$; let $(F, \leqq) : = (E, \leqq) + \{p\}$; then we would be able to exhibit a function $j | w(F, \leqq) \rightarrow (F, \leqq)$, which would be strictly increasing, in contradiction with the statement 0 : 2. As a matter of fact, it would be sufficient to set, by definition,

$$jx = gx \quad \text{for } x = v \text{ and for } x \in \sigma(E, \leqq)$$

$$jx = p \quad \text{for every well-ordered subset}$$

x of $(F, \leqq)$ such that p be the last point of X. Q. E. D. 1 :5. Since every linear set $(E, \leqq)$ is order-imbeddable into $(PQ, \subset)$, it is sufficient to consider the embedding

$$e \in E \rightarrow (Q, \leqq)(\cdot, e),$$

$(Q, \leqq)$ denoting the ordered set of rational numbers, the theorem 1 :0 implies

1:6 Theorem. *Let* $(E, \leqq)$ be any set of real numbers such that $\Gamma(E, \leqq)=\omega_1$; *if f is any increasing mapping of the tree*

$$(1:7) \qquad (\sigma(E, \leqq)\leqq_k) \text{ into } (E, \leqq),$$

then the system S of all 2-point-subsets $\{x, y\}$ of (1 :7) *such that $fx=fy$ has a power* $pS > Al_0$ (see D. Kurepa [10 Theor. 3]) ; *pS : =power of S.*

1:8. Theorem. *The tree*

$$(1:9) \qquad (w(Q, \leqq)\leqq_k)$$

is the union of Al_1 of its antichains, but is not a union of $=Al_0$ of its antichains (see D. Kurepa [8 Theorem 2 :1,9 Theorem 2 :1]).

Proof. If (1 :9) were a union of Al_0 of its antichains, then there would exist a strictly increasing mapping f of (1 :9) into $(Q, \leqq)$ (Theor. 1 p. 837 in D. Kurepa [4]), contradicting the above main theorem 1 :0 for the case $E=Q$ (the condition $\Gamma(Q, \leqq)=\omega_1$ is satisfied).

2. A theorem concerning strictly increasing mappings of unions of antichains

2 :0 Every ordered set is union of various systems of antichains ; e. g., every tree T is union of rows $R_i T (i<\gamma T)$ and each point $t \in T$ belongs to a unique row $R_{\gamma t} T$; the mapping $t \in T \rightarrow \gamma t$ is strictly increasing on T and its range is the chain of all ordinals $<\gamma T$ (=the height of T). It is rather surprising that a very general theorem holds like the following one.

2:1. Theorem. *If an ordered set* (1) $(E, \leqq)$ *is union of Al_ν antichains* ; (2) $A_j (j<\omega_\nu)$, *then there is a strictly increasing mapping g of* (1) *onto a subchain of* $(P\omega_\nu, \subset)$; *the mapping*

$$(3) \qquad g:(E, \leqq)\rightarrow(P\omega_\nu, \subset)$$

is strictly increasing and the range gE is a chain; shortly speaking, g is a chain embedding of $(3)_1$ *into* $(3)_2$.

First one has

2:2. Lemma. *Any ordered set* $(E, \leqq)$ *of power* $<Al_\nu$ *is orderrisomorphic with a subset of* $(P\omega_\nu, \subset)$.

In fact, since $pE < Al_\nu$, there is a bijection b of E onto a subset F of numbers $<\omega_\nu$: The mapping $e \in E \to he := (E, \leqq)(\cdot, e]$ is such an isomorphism ; in particular, $hE = F$ and obviously, b carries PE onto PF ; consequently, the composed mapping bh carries isomorphically (1) onto a subset of $(3)_2$.

2 : 3. Proof of Theorem 2 : 1. For a representative of Al_ν we shall take any ω_ν-sequence of pairwise disjoint sets M_i, each of power Al_ν ; let

(4) $$M := \cup_j M_j, M^j := \cup_i M_i \quad (i \leqq j), M^{<j} := \cup_i M_i (i < j < \omega_\nu).$$

Let

(5) $w_j (j < \omega_\beta, \beta := (P\omega_\nu)$ be a normal well-order of PM.

Of course, since $pM_j = pM$, the power set PM_j is a cofinal subset in (5). Let us now prove the theorem; the proof will be carried out by transfinite induction argument. Put

(6) $$A^j := \cup_i A_i (0 \leqq i \leqq j < \omega_\nu), A^{<j} := \cup_i A_i (i < j).$$

Let us define also an ω_ν-sequence

(7) $$s_j \leqq j \, (j < \omega_\nu)$$

of ordinals $<\omega_\nu$ and an ω_ν-sequence

(8) $$f_j (j < \omega_\nu)$$

of strictly $\subset$-increasing mappings with strictly increasing domains. To start with, let $s_0 = 0$ and let $f_0 A_0 := \{w_{i_0}\}$, i. e. for every $x \in A_{s_0}$ let $fx := w_i$, where i_0 is the first member of PM_{s_0} in the well-order (5). Suppose that $0 < j < \omega_\nu$ and that the left j-segment $s_i (i < j)$ of (7) is defined such that

(a) for every $i < j$ the set A^i is mapped by a strictly increasing mapping f_i onto a subchain $f_i A^i$ of $(PM^i, \subset)$;

(b) if $e < i < j$, then $f_e \subset \neq f_i$;

(c) every pseudo-cut in $f_i A^i$ is of a power $< Al_\nu$.

2:4. Definition. *A pseudo-cut in a chain L is an ordered pair (A, B) of subsets of L such that $\cup_{a \in A} L(\cdot, a] | \cup_{b \in B} L[b, \cdot)$ is a cut of L. A (pseudo-) cut is said to be of power $< Al_\nu$ if each of its components A, B is of power Al_ν.*

Let us define f_j : On $A^{<j}$ let f_j be $f^{<j} := \Sigma_i f_i (i < j)$; $f_j | A_j$ will be defined in the following way. Every $x \in A_j$ induces a cut

(9) $$A(x) | B(x)$$

in the set

(10) $$f_j A^{<j};$$

by induction hypothesis (c) the cut (9) is equivalent to a pseudo-cut $C(x)|D(x)$ of power $<Al_\nu$; we define $f_j x$ as the first number of (5) which belongs to $P(\cup_\alpha M^i)(i<j)$ such that $C(x)\subset\neq f_j x\subset\neq D(x)$, thus also $A(x)\subset\neq f_j x\subset\neq B(x)$; if such a $f_j x$ exists for all $x\in A_j$, we put $s_j := \sup s_i(i<j)$; we do so also if j is limit. If there is some $x\in A_j$ such that $f_j x$ does not exist as was just described, we define s_j as sup $(s_i+1)(i<j)$ and $f_j x$ as the first member y of (5) which belongs to $PM^s j$ and such that $y\cap M_{s_j}$ be singleton.

The existence of such a y for $x\in A_j$ follows from the Lemma 2 : 2 and from the fact that the cut $A(x)|B(x)$ is equivalent to some pseudo-cut, which is of power $<Al_\nu$ (see condition (c)). Thus the induction procedure is going on for every $j<\omega_\gamma$ and the function $f|E := \cup_j f_j|A^j(j<\omega_\gamma)$ is a strictly increasing mapping of (1) onto a subchain of $(PM, \subset)$. If h is any isomorphism of $(PM, \subset)$ onto $(P\omega_\nu, \subset)$, then hf is a requested strictly increasing mapping of (1) into a subchain of $(3)_2$. Q. E. D.

2 : 5. Historical remark. For the particular case when $\nu=0$, the theorem 2 : 1 was found in 1937 (see D. Kurepa [3 ; 4, Theorem]). To be sure, this result concerned $(Q, \leqq)$ instead of $(P\omega_0, \subset)$, but the present formulation of Theorem 2 : 1 for $\nu=0$ is easily implied from the fact that if $(E, \leqq)$ is union of Al_0 antichains, then there exists a strictly increasing mapping f of $(E, \leqq)$ into $(Q, \leqq)$ (see D. Kurepa [4, Theorem]). As a matter of fact there is an isomorphism s between $(Q, \leqq)$ and a subset of $(P\omega_0, \subset)$: it is sufficient to consider any well-order $q_0, q_1, \ldots, q_i, \ldots (i<\omega_0)$ of Q and for any $q\in Q$ to consider the infinite set $Q(\cdot, q)$ of all elements $q_i\leqq q$ and the corresponding set $sq\in P_{\omega_0}$ of all indeces i such that $q_i\leqq q$; the mapping $q\in Q\to sq\in P\omega_0$ is an isomorphism. Therefore, the compound mapping sf is a strictly increasing mapping of the countable union of antichains into the subchain sQ of $(P\omega_0, \subset)$.

2 : 6. Remark. We stress the fact that the range of the mapping g in the wording of Theorem 2 : 1 is a subchain in the lattice $(P\omega_\nu, \subset)$. As stated in n^0 3 : 4, if $\nu=0$, one can assume that the subchain $(gE, \subset)$ is a part of some subchain of type η in $(P\omega_0, \subset)$.

How is the matter if $\nu>0$ (e. g., if $\nu=1$? (cf. Problem 2 : 7 : 12).

2 : 7. Chain $(D_\nu, <)$, tree T_ν. Chain $(Q_\nu, \leqq)$.

2 : 7 : 0. We are going to define, for any ordinal ν, a chain $(Q_\gamma, \leqq)$ which for $\nu=0$ becomes a chain similar to $(Q, \leqq)$. We formulate 2 problems concerning $(Q_\nu, \leqq)$.

We denote by ν any ordinal number.

2 : 7 : 1. Ordered chain D_ν (ν ordinal).

Definition. $D_\nu := W(\omega_\nu)^* + W\omega_\nu$, where for any ordinal α we convene that $W\alpha := \{x : x \text{ is ordinal number } <\alpha\}$.

Thus $(D_0, \leqq)$ is a chain similar to the chain $(D, \leqq)$ of rational integers.

2 : 7 : 2. Tree T_ν.

Definition. The set of all non empty sequences of length $<\omega_\nu$, of terms of D_ν is denoted by T_ν; this set is supposed to be ordered by the relation $\leqq_k$ "to be initial part of"; we get a well-defined tree $(T_\nu, \leqq_k) := T_\nu$ of power $\Sigma_{\alpha<\omega_\nu} Al_\nu^{p\alpha} := Al_\nu^{Al_\nu} \leqq 2^{Al_\nu}$.

2:7:3. Chain $(Q_\nu, \leqq) := Q_\nu$ is the set T_γ ordered by a relation $\leqq$ which extends $\leqq_k$ as well as the total ordering of every left node of the tree T_ν; in other words, if $a, b \in T$, we put $a \leqq b$ if and only if either $a \leqq_k b$ or if $a \|_k b$ and $a_e < b_e$, where $e := e(a, b)$ is the least ordinal such that $a_e \neq b_e$ and $a_i = b_i$ for every $i < e$.

2:7:4. Lemma. $(Q_\nu, \leqq)$ is order-dense. In fact, if $a, b \in Q_\nu$ and $a <_k b$, then $a <_k c <_k b$ for any c such that $c_{\gamma a} < b_{\gamma a}$. If $a \|_k b$, then we have a determined first node N such that $a', b' \in N$, $a' \|_k b'$ and $a' \leqq_k a$, $b' \leqq_k b$. Then any c such that $a' <_k c$ is located between a', b' and a forterio between a, b.

2:7:5. Lemma. Right character of any q is ω_ν^*, because the set $R_0 q$ of immediate $\leqq_k$-successors of q is coinitial with $(q, \cdot)_{Q_\nu}$ and thus is coinitial with ω_ν^*, irrespective whether q is a left limit point in a node of T or whether q is isolated in the node $Q|q|$ to which q belongs; e.g., $(-\omega)$ is right nodal limit of $(-n)$ $(n=1, 2, \ldots)$ but $R_0(-\omega, \cdot)_{T_\nu}$ is located between $(-\omega)$ and $(-n)$ for positive integers n.

2:7:6. Left character of $q \in Q$. If the original height γ_q in T_ν is a limit ordinal, so is q a limit point in Q_ν, of the same left character. If q is isolated and if the last component of q is isolated in D_ν, then $(Q_\nu, \leqq)(\cdot, q)$ is cofinal with ω_ν; if the last component of q is a left limit point of character ω_σ in D_ν, then so is q in $(Q_\nu, \leqq)$.

2:7:7. Let $s := s_0, s_1, \ldots, s_j, \ldots (j < \gamma s)$ be a maximal sequence of elements of D_γ, such that $(s_0, s_1, \ldots, s_i, \ldots)_{i<j} D_\nu$ intersects both A and $B := CA$ of a given cut $A|B$ of $(Q_\nu, \leqq)$. Then $0 < \gamma s \leqq \omega_\nu (\gamma s$ denotes the ordinal length or height of $s)$.

2:7:8. First case $\gamma s < \omega_\nu$. If γs is of the first kind, in particular if $\gamma s = 1$, i.e. $s = (s_0)$, then sD_ν intersects both A and B; this implies a cut $M|N$ of D_ν, where $M : \{x \in D_\nu, sxD_\nu \subset A$; thus $s(x+1)D_\nu \subset B$; sxD_ν and $s(x+1)D_\nu$ are contiguous; thus $A|B$ is a gap of character $(\omega_\nu, \omega_\nu^*)$.

If γs is of second kind, we have two subcases: First subcase: sD_ν intersects both A, B; then again $A|B$ is a gap of character $(\omega_\nu, \omega_\nu^*)$. Second subcase: sD_ν is either in A or in B.

If sD_ν is in A, then sD_ν is cofinal with A, thus A is cofinal with ω_ν; in this case we have $(s_0+1)D_\nu$, $(s_0, s_1+1)D_\nu, \ldots, (s_0, s_1, \ldots, s_j+1)D_\nu, \ldots \subset B$; this is a decreasing sequence of sets the union of which is coinitial with B; thus the character of $A|B$ is the gap of $(\omega_\nu, \tau s)$; τs denotes the type of s.

Dually, if sD_ν is in B, one has a gap of character $(\tau s, \omega_\nu^*)$ because sD_ν is coinitial with B and the set (1) $\{x : x \in D_\nu, x <_k s\}$ is cofinal with A.

2:7:9. Second case: $\gamma s = \omega_n$.

First subcase: s has no final part composed of 0's; thus $s \notin Q_\nu$; in this case, s represents the gap $A|B$, and the set (1) is well-ordered of type ω_ν and is cofinal with A. The sequence

$$h(s_0+1)D_\nu, \ldots, s_0, s_1 \ldots, h(s_j+1))D_\nu \quad (j < \omega_n)$$

of strictly decreasing parts of B is coinitial with B; thus the gap $A|B$ has a symetric character $(\omega_\nu, \omega_\nu^*)$; here $h(x+1)$ means : $x+1$ for $x \in W\omega_\nu$, and $1+x$ for $x \in W\omega_\nu^*$.

Second subcase : The sequence s terminates with a right constant section composed of 0's ; thus $s \in Q_\nu$ (let us remark that every member of Q_ν could be obtained in this way on varying cut $A|B$) ; the right and the left character of s were determined in n^0 2 : 6 : 5 and 2 : 6 : 6 respectively.

2 : 7 : 10. Briefly, every cut $A|B$ of $(Q_\nu, \leqq)$ has at least one component of character of power Al_ν.

2 : 7 : 11. Problem. Is the ordered chain $(Q_\nu, \leqq)$ similar to a subchain of $(P\omega_\nu), \subset)$?

The answer is in affirmative for $\nu = 0$, and for any strongly inaccessible ω_ν (a proof runs like the one for $\nu = 0$ in $n^0 2:5$) and similarly, for any regular ω_ν under the General Continuum Hypothesis.

2 : 7 : 12. Problem. If $(E, \leqq)$ is union of Al_ν antichains, does there exist a strictly increasing mapping of $(E, \leqq)$ into $(Q_\nu, \leqq)$.

The answer is in affirmative at least for $\nu = 0$ (see D. Kurepa [4]).

References

1. S. Ginsburg On mapping from the family of well ordered subsets of a set. *Pacific J. Math.* **6**, 1956, 583-589.

Đuro R. Kurepa

2. 1937 : 2 Ensembles linéaires et une classe de tableaux ramifiés. Tableaux ramifiés de M. Aronszajn. *Publ. Inst. Math.* Beograd, **6**, 1937, 129-160. (*JFM* **64** (1940) 37 (Aumann). *Zbl.* **20** (1940) 508 (Blumberg).)
3. 1937 : 4 Transformations monotones des ensembles partiellement ordonnés. *C. R. Acad. Sci. Paris*, **205**, 1937, 1033-1035. (*JEM* **64** (1940) 173 (Seebach). *Zbl.* **27** (1943), 300 (J. Novak)).
4. 1940 : 2 Transformations monotones des ensembles partiellement ordonnés. *Revista Cienc. Lima*, **42**, No 434, 1940, 827-846. (*M. R.* **3** (1942) 225 (J. W. Tuckey).).
5. 1941 : 1 (Continuation of 1940 : 2) **43**, No 437, 1941, 483-500. (*M. R.* **3** (1942) 225 (J. W. Tuckey).).
6. 1948 : 2 Sur les ensembles ordonnés dénombrables. O prebrojivim uredenim skupovima. *Glasnik Mat. Fiz. Astron.* II, **3**, 1948, 145-150, 151. (*M. R.* **10** (1948) 437 (J. Todd). *Zbl.* **39** (1951), 277 (Landolino Guiliano).).
7. 1951 : 14 Teorija skupova. The Theory of Sets. Zagreb, 1951, 443 p. (*M. R.* **12** (1951) 638 (W. Feller). *Zbl.* **44** (1952) (1952), 272 (E. Hewitt).).
8. 1953 : 2 v. 1954 : 14. O realnim funkcijama u obitelji uredjenich skupova recionalnih brojeva. *Rad. Jug. Akad. Znan. Umjet.*, **296**, 1953, 95-103. (*M. R.* **16** (1955) 1006. *Zbl.* **58** (1954) 43 Otto Haupt).).
9. 1954 : 2 Fonctions croissantes dans la famille des ensembles bien ordonnés. *Bull. Sci. Conseil Acad. Yugosl.*, **2**, 1954, 9. (*RZM* (1959) 6767, *Zbl.* **58** (1958), 43).
10. 1954 : 14 Sur les fonctions réelles dans la famille des ensembles bien ordonnés des nombres rationnels. *Bull. Int. Acad. Youg. cl. math., phys., tech.* **12**, livre 4, 1954 (*M. R.* **16** (1955) 1006 (S. Ginsburg). *RZM* (1956) 7965. *Zbl.* **58** (1958), 43 (Otto Haupt).).
11. 1956 : 3 Ensembles ordonnés et leurs sous-ensembles bien ordonnés. *C. R. Acad. Sci. Paris*, **242**, 1956, 2202-2203. (*M. R.* **17** (1956), 1065 (S. Ginsburg). *RZM* (1958) 8717. *Zbl.* **71** (1956), 50 (Otto Haupt).).
12. 1964 : 5 Monotone mappings between some kinds of ordered sets. Monotona preslikavanja nekim vrstama uredjenih skupova. *Glasnik Mat. Fiz. Astron. II*, **19**, No 3-4, 1964, 175-186.(*Zbl.* **134** (1967), 256 (T. Katřinak).).

Stevo Todorčević

13. 1983 Real functions on the family of all bounded well ordered subsets of a partially ordered set. *J. Symb. Logic*, **48**, 1983, 91-96.
14. 1985 Partition relations for partially ordered sets. *Acta math.*, **155**, 1985, 1-25.

Zagrebacka 7
11000 Beograd
JUGOSLAVIA

Received 23.08.1987