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Quantifierfree Boolean Formulas and Their Relation to Modal Logic S5

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A modal propositional formula $\sigma\varphi$ is assigned to each quantifierfree Boolean formula φ so that the following holds : φ is true in all Boolean algebras if and only if $\sigma\varphi$ is a theorem of modal calculus $S5$.

1. Introduction and notation

In this paper we consider quantifierfree Boolean formulas in connection with modal propositional calculus $S5$. The Theorem 3.6. shows that if the Boolean operations are naturally identified with propositional connectivies, then the Boolean ordering $\leqq$ may be considered as the strict implication $⥽$, and the equality sign $=$ as the strict equivalence $\equiv$ in $S5$. This result may add a new feature to a semantical theory of modal logic. As a by-product a decision procedures for quantifierfree Boolean formulas[1] and modal calculus $S5$ are obtained[2].

In a sense, this paper continues the work in [7a], where we used the method of Kripke's tableaux, and, thus it is not a surprise that a connection between Boolean formulas and modal calculus exists.

Throughout we use the following terminology and notation. The logical signs are denoted by $\wedge$, $\vee$, $\neg$, $\Rightarrow$ $\leftrightarrow$, and the modal possibility operator by M. The language of Boolean algebras (sometimes we call them shortly BA) is $L_{BA}=\{+, \cdot, {}', \leqq, 0, 1\}$, where the displayed signs are interpreted in BA's in the obvious way. $x\rightarrow y$ stands for $x'+y$, and $x\leftrightarrow y$ for $(x\rightarrow y)\cdot(y\rightarrow x)$.

The language L^*_{BA} is $L_{BA}\cup\{*, \mathrm{o}\}$, where $*$, o are unary operation symbols. Any standard (equational) axiomatization in L_{BA} of BA's is denoted also by BA. The theory BA^* is BA plus the following (definition) axioms for $*$, o ;

$$\mathrm{o}^*=\mathrm{o}, \quad x>0\Rightarrow x^*=1, \quad x^0=x^{|*|}.$$

Models of the theory BA^* are of the form $\mathsf{B}^*=(\mathsf{B}, *, o)$, where B is a BA, and they are called Boolean algebras with operators or simply BA^*-algebras.

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[1] Of course, in the light of Tarski's result that the first order theory of Boolean algebras is decidable, it adds not much to the subject, but the proposed decision procedure is very efficient, and in particular cases can be done in few steps. For the earlier results on the matter one may consult [3], [7a].

[2] There are many decision procedures for $S5$. The latest one is probably of M. Sato [8]. In fact, we give here a new proof of a decision procedure for $S5$, which is due to Carnap essentially, see [2], p. 116.

The sum (supremum) and the product (infimum) of a set x are denoted respectively by $\Sigma_{x\in X}\, x$, $\Pi_{x\in X} x$. Two-element Boolean algebra is denoted by **2**, and its domain by 2.

We assume any standard axiomatization of modal calculus $S5$[3].

2. Σ_0 – logic

As we are dealing only with quantifierfree formulas, it is a natural question is there an appropriate "logic" for them. We show that the quantifierfree fragment of the predicate calculus is axiomatizable, and has its own model theory.

Let L be a first order language. By $Term_L$ the set of all terms over L is denoted. By $For_L \cap \Sigma_0$ the set of quantifierfree formulas over L is denoted. Each $\varphi \in For_L \cap \Sigma_0$ is called shortly Σ_0-formula. If α is a term (or Σ_0-formula) of L, w a term of L, and z a variable, then $\alpha_z(w)$ denotes the term (formula) obtained from α by replacing all occurrences of z in α by w.

The Σ_0-logic over L is given by the following set of axioms and inference rules :

Axioms: (1) *Each instance of tautologies of proportional calculus.*
(2) $x=x,\ \ x=y \Rightarrow y=x,\ \ x=y \wedge y=z \Rightarrow x=z$
(3) $x=y \Rightarrow u_z(x)=u_z(y),\quad u\in Term_L$
$x=y \Rightarrow \varphi_z(x) \leftrightarrow \varphi_z(y),\quad \varphi\in For_L\cap\Sigma_0.$

Rules of inferences:

$$\frac{\varphi,\ \varphi\Rightarrow\psi}{\psi},\quad \frac{\varphi}{\varphi_x(w)},\quad \varphi,\ \psi\in For_L\cap\Sigma_0,\quad w\in Term_L$$

Models (in Σ_0-logic) for a language L are ordinary first order structures for L. If $\mathfrak{A}$ is a model of L, and $\varphi\in For_L\cap\Sigma_0$ then, by definition, φ holds in $\mathfrak{A}$(in Σ_0-logic) iff the universal closure of φ holds in $\mathfrak{A}$. We write $\mathfrak{A}\models\varphi$ if φ folds in $\mathfrak{A}$.

The set of all sentences (formulas without variables) of L is denoted by $Sent_L$. Any set $T\subseteq For_L\cap\Sigma_0$ is called Σ_0 theory in L. $T\vdash_{\Sigma_0}\varphi$ means that φ is deducible from T in Σ_0-logic. T is consistent if a contradiction is not deducible from T. $T\models\varphi$ means that φ is true in all models of T.

As it is expected, the completeness theorem holds :

Theorem 2.1. (Completeness theorem for Σ_0-logic). *If T is a Σ_0-theory in L then $T\models\varphi$ iff $T\vdash_{\Sigma_0}\varphi$, $\varphi\in For_L\cap\Sigma_0$.*

The proof of the theorem can be carried out through the following assertions. Proofs of the most claims are straightforward, so they are omitted or just indicated. Here, T denotes a Σ_0-theory in L, and $\varphi\in For_L\cap\Sigma_0$.

Claim 1 (Deduction theorem). *If $\psi\in Sent_L$ then $T,\psi\vdash_{\Sigma_0}\varphi$ implies $T\vdash_{\Sigma_0}\psi\Rightarrow\varphi$.*

[3] E. g., besides axioms of propositional calculus, also $L\varphi\Rightarrow\varphi\ L(\varphi\Rightarrow\psi)\Rightarrow(L\varphi\Rightarrow L\psi)$, $M\varphi\Rightarrow LM\varphi$, and the rule of necessitation : if $\vdash\varphi$ then $\vdash L\varphi$.

Claim 2. *If c is a constant symbol which does not occur in T, and if* $T \vdash_{\Sigma_0} \varphi_x(c)$, *then* $T \vdash_{\Sigma_0} \varphi_x(w)$ *for any* $w \in Term_L$.

Claim 3. *Assume not* $T \vdash \varphi(x_1, \ldots, x_n)$. *If* $c_1, \ldots, c_n$ *are new constant symbols* (i.e., $L \cap \{c_1, \ldots, c_n\} = \emptyset$), *then* $T \cup \{\neg\varphi(c_1, \ldots, c_n)\}$ *is a consistent theory in* $L \cup \{c_1, \ldots, c_n\}$.

Claim 4. *If T is a consistent set of sentences of L, then there is a complete, consistent set of sentences T' in L such that* $T \subseteq T'$.

Claim 5. *Let* $C = \{c_0, c_1, \ldots\}$ *be a denumerable set of new constant symbols (i.e.* $L \cap C = \emptyset$), *T a consistent* Σ_0*-theory in L, and* $T(C) = \{\varphi(w_1, \ldots, w_n) | \varphi(x_1, \ldots, x_n) \in T$, $w_1, \ldots, w_n$ *are closed terms in* $L \cup C\}$.
Then $T(C)$ *is a consistent theory of* Σ_0*-sentences.*

Proof of claim 5. We prove the following: If $\varphi \in For_L \cap \Sigma_0$ and if $T \cup T(C) \vdash \varphi$, then $T \vdash_{\Sigma_0} \varphi$. For that, assume $T \cup T(C) \vdash \varphi$.
Thus, there are $\psi_1(w_{11}(\vec{c}), \ldots, w_{1n}(\vec{c})), \ldots, \psi_m(w_{m1}(\vec{c}), \ldots, w_{mn}(\vec{c})) \in T(C)$[4], where $w_{ij}(x) \in Term_L$, so that
$T \cup \{\psi_1(w_{11}(\vec{c}), \ldots, w_{1n}(\vec{c})), \ldots, \psi_m(w_{m1}(\vec{c}), \ldots, w_{mn}(\vec{c}))\} \vdash_{\Sigma_0} \varphi$. By deduction theorem, and claim 2 we have

$$T \vdash_{\Sigma_0} \psi_1(w_{11}(\vec{x}), \ldots, w_{1n}(\vec{x})), \ldots, \psi_m(w_{m1}(\vec{x}), \quad \ldots, \quad w_{mn}(\vec{x})) \Rightarrow \varphi.$$

As $\psi_1(\vec{x}), \ldots, \psi_m(\vec{x}) \in T$, by the substitution rule,

$$T \vdash_{\Sigma_0} \psi_1(w_{11}(\vec{x}), \ldots, w_{1n}(\vec{x})) \wedge \ldots \wedge t_m(w_{m1}(\vec{x}), \ldots, w_{mn}(\vec{x})), \text{ hence } T \vdash_{\Sigma_0} \varphi.$$

Now, assume $T(C)$ is inconsistent. Then $T(C) \vdash_{\Sigma_0} x_1 \neq x_1$, thus $T \cup T(C) \vdash_{\Sigma_0} x_1 \neq x_1$, and therefore $T \vdash_{\Sigma_0} x_1 \neq x_1$, a contradiction.

Claim 6. *Let T be as in claim 5 and S a complete set of* Σ_0*-sentences in* $L \cup C$ *so that* $T(C) \subseteq S$. *Further, let* $\mathfrak{A}$ *be the canonical model of* S[5]. *Then* $\mathfrak{A}$ *is a model of T.*

Now, the proof of the theorem is as follows (the nontrivial part). Assume $T \models \varphi(x_1, \ldots, x_n)$, and suppose not $T \vdash_{\Sigma_0} \varphi(x_1, \ldots, x_n)$. By Claim 3. for new constant symbols $c_1, \ldots, c_n$ $T \cup \{\neg\varphi(c_1, \ldots, c_n)\}$ is a consistent theory. Therefore, there is a model $(\mathfrak{A}, a_1, \ldots, a_n)$ of $T \cup \{\neg\varphi(c_1, \ldots, c_n)\}$, so $\mathfrak{A} \models T$ and $\mathfrak{A} \models \neg\varphi(a_1, \ldots, a_n)$, a contradiction.

As an immediate consequence we have that Σ_0-logic is a real fragment of predicate calculus, i.e. that predicate calculus is a conservative extension of Σ_0-logic. Thus, we shall write $T \vdash \varphi$ instead of $T \vdash_{\Sigma_0} \varphi$. Specifically we have

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[4] $\vec{c}$, $\vec{x}$, etc. stand for $c_1, \ldots, c_n$ and $x_1, \ldots, x_n$, respectively.

[5] Domain A of $\mathfrak{A}$ is $\{w/\sim | w$ is a closed term in $L \cup C\}$, where for closed terms u, v in $L \cup C$ $u \sim v$ iff $S \vdash_{\Sigma_0} u = v$. For n-ary function symbol f and $w_1/\sim, \ldots, w_n/\sim \in A$, $f^{\mathfrak{A}}(w_1/\sim, \ldots, w_n/\sim) = f(w_1, \ldots, w_n)/\sim$, and similarly for relation symbols.

Corollary 2.2. *For any Σ_0-Boolean formula φ $BA \models \varphi$ iff $BA \vdash \varphi$.*
The following assertion will be of an interest.

Theorem 2.3. *Let $\mathcal{M}$ be a class of models of a language L which contains only function and constant symbols*[6]*, and assume that $\mathcal{M}$ contains free model $\mathfrak{A}$ with an infinite set of free generators $p_0, p_1 \dots$.*
Then:
1° If $u, v \in Term_L$ and $\mathfrak{A} \models u(p_1, \dots, p_n) = v(p_1, \dots, p_n)$ then $\mathcal{M} \models \mathfrak{U} = v$.
2° If φ, ψ are Σ_0-formulas over L which are built only from atomic formulas and logic connectivies $\wedge$, $\vee$, then $\mathcal{M} \models \varphi \vee t$ implies $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \psi$.

Proof. 1° Let $\mathfrak{B} \in \mathfrak{M}$ be any model, $b_1, \dots, b_n \in B$ and $h: \mathfrak{A} \to \mathfrak{B}$ a homomorphism so that $h(p_0) = b_0, \dots, h(p_n) = b_n$. Then $\mathfrak{B} \models u(b_0, \dots, b_n) = v(b_0, \dots, b_n)$. As $b_0, \dots, b_n \in B$ were arbitrary, it follows $\mathfrak{B} \models u = v$, hence $\mathcal{M} \models u = v$.

2° There are formulas $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_m$ so that φ is $\varphi_1 \vee \dots \vee \varphi_n$ and ψ is $\psi_1 \vee \dots \vee \psi_m$, where φ_i, ψ_j are conjunctions of atomic formulas. As $\mathcal{M} \models \varphi \vee \psi$ it follows $\mathcal{M} \models \varphi \vee \psi$, so for some i $\mathcal{M} \models \varphi_i(p_1, \dots, p_n)$ or $\mathfrak{A} \models \psi_i(p_1, \dots, p_n)$, and according to the previous part $\mathcal{M} \models \varphi_i$ or $\mathcal{M} \models \psi_i$, hence $\mathcal{M} \models \varphi$ or $\mathcal{M} \models \psi$.

Corollary 2.4. *Let $\mathcal{M}$ be an equational class of models (e.g. BA) of an algebraic language L, and $A_1, \dots, A_n$, $B_1, \dots, B_n \in Term_L$. Then $\mathcal{M} \models A_1 = B_1 \vee \dots \vee A_n = B_n$ iff there is $i \leqq n$ so that $\mathcal{M} \models A_i = B_i$.*

3. Σ_0-Boolean formulas interpreted as modal formulas

We define three mappings of syntactical character, and investigate their properties. We assume that the set P of variables of Σ_0-logic is equal to the set of propositional letters and consists of $p_0, p_1, \dots$

Definition 3.1. The map σ from the set of Σ_0-formulas of L^*_{BA} into the set $Sent$[7]$_{S5}$ is defined as follows. First, we define σu, $u \in Term_{L_{BA}}$ by induction on complexity of u:

If u is a variable, then σu is u. Also, $\sigma 0$ is "false" and $\sigma 1$ is "true"[7½].

If u is v', then σu is $\neg\ \sigma v$.

If u is $v \cdot w$, then σu is $(\sigma v) \wedge (\sigma w)$.

If u is v^*, then σu is $M \sigma v$.

Now, we define $\sigma\varphi$, $\varphi \in For_{L^*_{BA}} \cap \Sigma_0$, by induction on complexity of φ.

If φ is an atomic formula $u = v$ (or $u \leqq v$), $u, v \leqq Term_{L_{BA}}$, then $\sigma\varphi$ is $\sigma u \mathrel{-\!\!3} \sigma v$ (i.e. $\sigma u \mathrel{-\!\!3} \sigma v$).

If φ is $\psi \wedge \theta$, then $\sigma\varphi$ is $\sigma\psi \wedge \sigma\theta$.

If φ is $\neg \psi$, then $\sigma\varphi$ is $\neg \sigma\psi$.

The map τ is defined in [7a]. For convenience we repeat the definition of τ.

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[6] We call such language an algebraic language.
[7] the set of all sentences of $S5$.
[7½] "true" is any theorem in $S5$, and "false" any contradiction in $S5$.

Definition 3.2. *If u is a term of L^*_{BA}, then τu is u.*
If u,v are terms of L^*_{BA}, then $\tau(u=v)$ is $(u\leftrightarrow v)^0$ and $\tau(u\leqq v)$ is $(u\rightarrow v)^0$.
If $\alpha,\beta\in For_{L^*_{BA}}\cap\Sigma_0$, then $\tau(\neg\alpha)$ is $(\tau\alpha)'$ and $\tau(\alpha\wedge\beta)$ is $\tau(\alpha)\cdot\tau(\beta)$.

The following is proved in [7a].

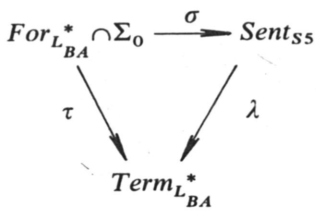
Proposition 3.3. *For any Σ_0-formula α of L^*_{BA} the following hold:*
1° $BA^*\models\tau\alpha=0\vee\tau\alpha=1$
2° $BA^*\models\alpha\leftrightarrow(\tau\alpha=1)$.

Corollary 3.4. *For any $\varphi\in For_{L^*_{BA}}\cap\Sigma_0\ BA^*\models\varphi$ iff $BA^*\models\tau\varphi=1$.*

The map λ assigns to each modal propositional formula φ a term $\lambda\varphi$ of L^*_{BA} in the natural way[8].

Obviously, the following holds:

Proposition 3.5. $\tau=\lambda\circ\sigma$



Now, we prove the main theorem.

Theorem 3.6. 1° *Let φ be any Σ_0-formula of L^*_{BA}. Then $BA^*\models\varphi$ iff $\vdash_{S5}\sigma\varphi$*
2° *Let φ be any Σ_0-formula of L_{BA}. Then $BA\models\varphi$ iff $\vdash_{S5}\sigma\varphi$.*

Proof. In the proof we use the following well-known fact:

(1) If $\psi\in Sent_{S5}$, then $\vdash_{S5}\psi$ iff $BA^*\models\lambda\psi=1$.

Now we prove 1°. Let $\varphi\in For_{L^*_{BA}}\cap\Sigma_0$. Then

by Corollary 3.4. $BA^*\models\varphi$ iff $BA^*\models\tau\varphi=1$,
using $\tau=\lambda\circ\sigma$ $BA^*\models\varphi$ iff $BA^*\models\lambda\sigma\varphi=1$
by (1) $BA^*\models\varphi$ iff $\vdash_{S5}\sigma\varphi$.

For 2° observe that BA^* is a conservative extension of BA.

By completeness theorem for Σ_0-logic (Theorem 2.1.) it follows that for all Σ_0-Boolean formulas φ $BA\vdash_{\Sigma_0}\varphi$ iff $\vdash_{S5}\sigma\varphi$.

If we observe that $\sigma\varphi$ is a modal propositional formula in which every propositional letter occurs under the scope of exactly one modal operator, we may

[8] All occurrences of signs $\neg$, $\wedge$, $\vee$, L, M in φ are replaced respectively by $'$, $\cdot$, $+$, 0, $*$.

consider the theory BA in Σ_0-logic as a modal propositional calculus of first-degree modal formulas[9].

The same result can be obtained by use of Kripke's tableaux, and somewhat finer analysis may be carried out. We are going to indicate this method, too. Let B be a finite BA and A the set of all atoms of B. A will be called the set of conditions.

Further, let $S: A \to \{x \mid x \subseteq P\}$.

Definition 3.7. *The forcing relation*[10] $\Vdash_S$ *is given for the set* $Sent_{S5}$ *as follows :*
If $p \in P$ then $a \Vdash_S p$ iff $p \in S_a$ $(a \in A)$
$a \Vdash_S \varphi \wedge \psi$ iff $a \Vdash_S \varphi$ and $a \Vdash_S \psi$.
$a \Vdash_S \neg \varphi$ iff not $a \Vdash_S \varphi$.
$a \Vdash_S M\varphi$ iff there is $b \in A$ such that $b \Vdash_S \varphi$.
The following statements are obvious.

Proposition 3.8. 1° $a \Vdash_S L\varphi$ *iff for all* $b \in A$ $b \Vdash_S \varphi$
2° $a \Vdash_S \varphi \prec \psi$ *iff for all* $b \in A$ $b \Vdash_S \varphi \Rightarrow \psi$
3° $a \Vdash_S \varphi \equiv \psi$ *iff for all* $b \in A$ $b \Vdash_S \varphi \leftrightarrow \psi$.

Definition 3.9. 1° $A \Vdash_S \varphi$ *iff for all* $a \in A$ $a \Vdash_S \varphi$
2° $A \Vdash \varphi$ *iff for all* $S: A \to \{x \mid x \in P\}$ $A \Vdash_S \varphi$
3° $\Vdash \varphi$ *iff for all finite* B $A \Vdash \varphi$.

The following is a characterization theorem for modal $S5$ Calculus.

Theorem 3.10. (*S. Kripke*) *Let* $\varphi \in Sent_{S5}$. *Then* $\vdash_{S5} \varphi$ *iff* $\Vdash \varphi$. *Now, if S is given, define a Boolean assignment* $v \in B^P$ *by*

$$v(p) = \Sigma \{a \in A \mid a \Vdash p\}^{11}.$$

If $t \in Term_{L^*_{BA}}$, then $v(t)$ denotes the value of t for the assignment v, and for $\psi \in Sent_{S5}$ define $\|\psi\| = v(\lambda\psi)$.

The following can be verified in the standard way[12].

Theorem 3.11. 1° *If* $\psi \in Sent_{S5}$, *then* $\|\psi\| = \Sigma \{a \in A \mid a \Vdash_S \psi \|\}$.
2° *If* $\psi \in Sent_{S5}$ *then* $a \Vdash_S \psi$ *iff* $a \leqq \|\psi\|$

If, in the following, φ *denotes a* Σ_0*-formula of* L^*_{BA} *then*

3° $a \Vdash_S \sigma\varphi$ *iff* $a \leqq v(\tau\varphi)$
4° $A \Vdash_S \sigma\varphi$ *iff* $\mathsf{B} \models \varphi[v]$
5° $A \Vdash \sigma\varphi$ *iff* $\mathsf{B} \models \varphi$
6° $\Vdash \varphi$ *iff* $BA^* \models \varphi$.

We are not going to prove all the statements. Most of them can be easily deduced by translating proofs of lemmas 4.1 — 4.4 in [7a]. The equivalence 4°

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[9] c. f [2], p.p. 50-51.

[10] It should be written $\Vdash_{(\mathsf{B}, A, S)}$ but we use the shorter form if the ambiguity may not arise.

[11] c. f. E. J. Lemmon [5]

[12] By induction on complexity of ψ, c.f. also [7a].

should have a special attention as it connects the ordinary satisfaction relation with relation $\Vdash_S$, and we prove it.

Proof of 4°. We have $A \Vdash_S \sigma\varphi$ iff for all $a \in A$ $v(\tau\varphi) \geqq a$
as B is atomic and iff $v(\tau\varphi) \geqq \Sigma_{a\in A}\, a$.
A is a set of atoms of iff $v(\tau\varphi) = 1$
by the Proposition 3.3.2° iff $\mathsf{B} \models \varphi[v]$.

If it is assumed that φ is a Boolean Σ_0-formula, all the relations 3°-6° hold, provided that B is a Boolean algebra, and if BA^* is replaced by BA. Also, we observe that relation 3° and 4° show that the truth of φ in B for the assignment v may be "partitioned" thus, the previous propositions may provide a new meaning of the truth of Σ_0-Boolean formulas.

4. First-degree modal formulas

As we have shown, the map σ assigns to each Σ_0-Boolean formula φ a modal transform $\sigma\varphi$ in a natural way, hence it may be asked what can be done in other direction. For that, we define a map $\sigma' : Sent_{S5} \to For_{L_{BA}} \cap \Sigma_0$ so that $\vdash_{S5} \psi$ iff $BA \vdash \sigma'\psi$. We use a well-known result that each modal propositional formula φ has so-called modal conjunctive normal form (MCNF) ψ in $S5$.

Theorem 4.1. *For each $\psi \in Sent_{S5}$ there is $\sigma'\psi \in For_{L_{BA}} \cap \Sigma_0$ so that $\vdash_{S5} \psi$ iff $BA \vdash \sigma'\psi$.*

Proof. Let φ be MCNF of ψ. Then $\vdash_{S5} \psi \leftrightarrow \varphi$, and φ is of the form $\varphi_1 \wedge \ldots \wedge \varphi_n$. Each φ_i is a disjunction of formulas of the following types : (1) θ, (2) $L\alpha_1 \wedge \ldots \wedge L\alpha_n$, (3) $M\beta$, where $\theta, \alpha_1, \ldots, \alpha_n, \beta$ are propositional formulas.

If φ_i is $\theta \vee L\alpha_1 \vee \ldots \vee L\alpha_n \vee M\beta$[14], let φ'_i be $L\theta \vee L\alpha_1 \vee \ldots \vee L\alpha_n \vee M\beta$. Then $\vdash_{S5} \varphi_i$ iff $\vdash_{S5} \varphi'_i$, what follows from the necessity rule and reduction theorem in $S5$ of modal formulas to first-degree modal formulas. Define $\sigma'\varphi_i$ to be $\sigma'\theta = 1 \vee \sigma'\alpha_1 = 1 \vee \ldots \vee \sigma'\alpha_n = 1 \vee \sigma'\beta > 0$, where for propositional formula γ $\sigma'\gamma$ is a Boolean term obtained from γ by replacing all occurrences of $\wedge, \vee, \neg, \Rightarrow$ by $\cdot$, $+$, $'$, $\to$ respectively. Finally, define $\sigma'\psi$ to be $\sigma'\varphi_1 \wedge \ldots \wedge \sigma'\varphi_n$. Then, by Theorem 3.6. we have

$BA \vdash \sigma'\psi$ iff $\vdash_{S5} \sigma\sigma'\psi$
iff $\vdash_{S5} \sigma\varphi \wedge \ldots \wedge \sigma'\varphi_n$
iff for all φ_i of the form $\theta \vee L\alpha_1 \vee \ldots \vee L\alpha_n \vee M\beta$
$\vdash_{S5} \theta \equiv \text{true} \vee \alpha_1 \equiv \text{true} \vee \ldots \vee \alpha_n \equiv \text{true} \vee \neg(\beta \equiv \text{false})$
i.e., $\vdash_{S5} L\theta \vee L\alpha_1 \vee \ldots \vee L\alpha_n \vee M\beta$
iff $\vdash_{S5} \psi$.

We illustrate this theorem by examples given in [2].

Example 4.2. (c.f. [2] p. 117) ψ is $(Mp \vee L\neg p \vee Lp) \wedge (M\neg p \vee L\neg p \vee Lp)$
$\sigma'\psi$ is $(p > 0 \vee p' = 1 \vee p = 1) \wedge (p' > 0 \vee p' = 1 \vee p = 1)$.

[14] The similar consideration is for other combinations of formulas (1), (2), (3).

Obviously $BA \models \sigma'\psi$, hence $\vdash_{S5} \psi$.

Example 4.3. (c.f. [2] p. 118) ψ is

$$(Lq \vee M \neg p \vee L \neg (p \wedge r) \vee (p \wedge q \Rightarrow r)) \wedge (Mp \vee L \neg p). \text{ Then } \sigma'\psi \text{ is}$$

$$(q=1 \vee p'>0 \vee \mathrm{G}pr)'=1 \vee (pq \to r)=1) \wedge (p>0 \vee p'=1).$$

$\sigma'\psi$ is not true in four-element Boolean algebra, thus not $\vdash_{S5} \psi$.

As an illustration, we shall transfer already mentioned Carnap's decision procedure for $S5$[15] to a decision procedure for validity of Σ_0-Boolean formulas.

Let φ be a Σ_0-Boolean formula. By conjunctive normal form (CNF) theorem for propositional Calculus, can be found effectively formulas $\varphi_1, \varphi_2, \dots, \varphi_n$ so that $BA \models \varphi \leftrightarrow \varphi_1 \wedge \dots \wedge \varphi_n$ and each φ_i is a disjunction of atomic and negations of atomic formulas.

Using the following reduction rules :

$$BA \vdash (A=B) \leftrightarrow (AFB)=1, \quad BA \vdash \neg(A=1) \leftrightarrow A'>0,$$

$$BA \vdash (A>0 \vee B>0) \leftrightarrow (A+B>0), \quad A, B \in Term_{L_{BA}}.$$

it may be assumed that each φ_i is of the form

4.4. $$A_1=1 \vee \dots \vee A_n=1 \vee B>0, \quad A_1, \dots, A_n B \in Term_{L_{BA}}.$$

Then, $BA \vdash \varphi$ iff for each $i \leqq n$ $BA \vdash \varphi_i$. If φ_i is of the form 4.4., we have :

$$BA \vdash \varphi_i \quad \text{iff} \quad \vdash_{S5} \sigma\varphi_i$$

$$\text{iff} \quad \vdash_{S5} L\sigma A_1 \vee \dots \vee L\sigma A_n \vee M\sigma B$$

(by Carnap's decision procedure)

iff there is $i \leqq n$ so that $\vdash_{S5} \sigma A_i \vee \sigma B$.

Therefore,

4.5. $$BA \vdash A_1=1 \vee \dots \vee A_n=1 \vee B>0 \text{ iff}$$

for some $i \leqq n$ $BA \vdash A_i+B=1$.

However, 4.5. can be obtained directly, thus, by means of Theorem **3.6.** we have an another proof of Carnap's decision procedure for $S5$. So we prove 4.5. As the part $(\leftarrow)$ is trivial, we prove only $(\rightarrow)$ direction.

Proof of 4.5. Assume $BA \vdash A_1=1 \vee \dots \vee A_n=1 \vee B>0$. Let Ω be a countable free Boolean algebra with $\{p_0, p_1 \dots\}$ as a set of free generators. Then

$$\Omega \models A_1(p_1, \dots, p_n)=1 \vee \dots \vee A_n(p_1, \dots, p_n)=1 \vee B(p_1, \dots, p_n)>0,$$

so for some $i \leqq n \Omega \models A_i(p_1, \dots, p_n)=1 \vee B(p_1, \dots, p_n)>0$. First assume $\Omega \models A_i(p_1, \dots, p_n)=1$ i.e. by Theorem 2.3. $BA \models A_i=1$, hence $BA \vdash A_i+B=1$. Assume $\Omega \models B(p_1, \dots, p_n)>0$. Then by disjunctive normal form theorem there is a

[15] c. f. [2] p. 116

nonempty set $T \leqq 2^n$ so that $B(p_1, \ldots, p_n) = \Sigma_{\alpha \in I} p_1^{\alpha_1} \ldots p_n^{\alpha_n}$[16]. Let $\beta \in 2^n - I$. Then $2 \models B(\beta_1, \ldots, \beta_n) = 0$, so as $2 \models A_i = 1 \vee B > 0$, it follows $2 \models A_i(\beta_1, \ldots, \beta_n) = 1$, hence $\Omega \models A_i(p_1, \ldots, p_n) \geqq p_1^{\beta_1} \ldots p_n^{\beta_n}$. Thus $\Omega \models A_i(p_1, \ldots, p_n) \geqq \Sigma_{\beta \in 2^n - I} p_1^{\beta_1}, \ldots, p_n^{\beta_2}$, hence

$$\Omega \models A_i(p_1, \ldots, p_n) + B(p_1, \ldots, p_n) \geqq \sum_{\beta \in 2^n - I} p^{\beta_1}, \ldots, p_n^{\beta_n} + \sum_{\alpha \in I} p_I^{\alpha_n}, \ldots, p_n^{\alpha_n},$$

i.e. $\Omega \models A_i(p_1, \ldots, p_n) + b(p_1, \ldots, p_n) = 1$. By Theorem 2.3. $BA \models A_i + B = 1$ and by

Theorem 2.1. $BA \vdash A_i + B = 1$.

As consequences of 4.5. we have

Corollary 4.6. *Let $A, B \in Term_{L_{BA}}$. Then*
1° $BA \vdash B > 0$ *iff* $BA \vdash B = 1$
2° $BA \vdash A = 1 \vee B > 0$ *iff* $BA \vdash A + B = 1$
3° $BA \vdash A = 1 \vee B = 1$ *iff* $BA \vdash A = 1$ or $BA_1^{\alpha_1} B = 1$
(compare to Theorem 2.3.2°).

All these statements have appropriate versions in $S5$.

Corollary 4.7. *Let φ, ψ be propositional formulas. Then*
1° $\vdash_{S5} M\varphi$ *iff* $\vdash_{S5} \varphi$
2° $\vdash_{S5} L\varphi \vee M\psi$ *iff* $\vdash_{S5} \varphi \vee \psi$
3° $\vdash_{S5} L\varphi \vee L\psi$ *iff* $\vdash_{S5} \varphi$ or $\vdash_{S5} \psi$[17].

It is easily obtained as a consequence of 4.5. also the following:

Corollary 4.8. *Let $A_1, \ldots, A_n, B \in Term_{L_{BA}}$. Then*

$$BA \vdash A_1 = 1 \wedge \ldots \wedge A_n = 1 \Rightarrow B = 1 \text{ iff}$$

$$BA \vdash \left(\left(\prod_{i=1}^{n} A_i\right) \rightarrow B\right) = 1 \text{ iff}$$

$$2 \models \left(\left(\prod_{i=1}^{n} A_i\right) \rightarrow B\right) = 1.$$

In fact, one can prove much more easily, using that Skolem functions in finite BA's are definable in the theory BA (i.e. they are Boolean functions):

Theorem 4.9.[18] If φ is a Horn formula of L_{BA} and $\tilde{\varphi}$ the Boolean term obtained from φ by replacing all occurrences of $=, \leqq, \wedge, \Rightarrow$ in φ by $\leftrightarrow, \rightarrow, \cdot, \rightarrow$ respectively, and all subformulas $\forall x\psi$, $\exists x\theta$ of φ by $\tilde{\psi}_x(0) \cdot \tilde{\psi}_x(1)$, $\tilde{\theta}_x(0) + \tilde{\theta}_x(1)$ respectively, then

[16] $\alpha = (\alpha_1, \ldots, \alpha_n)$, $p^1 = p$, $p^0 = p'$

[17] 3° holds in $S4$ for arbitrary modal formulas φ, ψ, as it was shown by *I. C.* Mc. Kinsy and A. Tarski [6]. But 3° does not hold in $S5$ for arbitrary modal φ, ψ, e.g. $\varphi = p$, $\psi = M \neg p$ c.f. [6], [4a].

[18] A version of Vaught's theorem, c.f. [1], p. 368.

$$
\begin{aligned}
BA \models \varphi \quad & \text{iff} \quad BA \models \tilde{\varphi} = 1 \\
& \text{iff} \quad \mathbf{2} \models \tilde{\varphi} = 1 \\
& \text{iff} \quad \mathbf{2} \models \varphi \quad .
\end{aligned}
$$

Applications of Vaught theorem can be found in [7b].

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