Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New Series Vol. 2, 1988, Fasc. 2-3

Optimal Filtering of Linear Wiener Process Observed at Moments of First Reaching of Lines

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1. Statement of the problem

The present work deals with the problem of optimal filtering of one-dimensional Wiener process which is observed at moments of first reaching of lines. The lines are arbitrary chosen, but so that the moments of first reaching form an increasing sequence. The observation are presented in the form of a counting process and the theorem of filtering (a representation theorem) and the theorem of the predictable expectation due to D. Hadzhiev ([1]) are used. An explicit formula for the optimal filter is obtained and the needed distributions are calculated. The special cases when the linear Wiener process is observed at moments of first reaching of : a set of horisontal lines ; a bundle of lines and a set of parallel inclined lines are considered. In these cases the received formulae have more simple form.

Let $(\Omega, \mathscr{F}_\infty, P)$ be a complete probability space with a filtration $\mathscr{F}=(\mathscr{F}_t)$, $\mathscr{F}_t \subseteq \mathscr{F}_\infty$, $t \in R_+=[0, \infty)$, i.e. $\mathscr{F}$ satisfies the conditions :

a) $\mathscr{F}_s \subseteq \mathscr{F}_t$, $s \leqq t$ (nondecreasing) ;

b) $\mathscr{F}_t = \cap \mathscr{F}_{u>t} u$ (right-continuity) ;

c) $\mathscr{F}_0$ is completed by P-null sets from $\mathscr{F}_\infty$.

Let $W=(W_t, \mathscr{F}_t)$, $t \in R_+$, be a standart linear Wiener process which is $\mathscr{D}$-adapted in the probability space $(\Omega, \mathscr{F}_\infty, P)$.

Let $K(0ty)$ be a Cartesian coordinate system and l_n, $n \geqq 1$, be a set of lines in $K: l_n: y=a_n-b_n t$. Denote by

$$T_n=\inf(t \in R_+: W_t=a_n-b_n t), \quad n>0, \quad T_0=0,$$

the moment of first reaching of line l_n by the process W. We choose the stopping times T_n in such a way, so they form a sequence which is strong increasing to $T_\infty=\infty$. This requirement implies some restrictions of the coefficients a_n and b_n of the lines l_n. We suppose, that $\{a_n\}$, $n \geqq 1$, is nondecreasing sequence with positive members $(a_n>0)$, and $\{b_n\}$, $n \geqq 1$, is nonincreasing sequence with non-negative members ($b_n \geqq 0$). Under these assumptions we construct a counting process $N=(N_t, \mathscr{F}_t)$, $t \geqq 0$, in the following way :

$N_t = \Sigma_{n \geqq 0} l_{\{T_n \leqq t\}}$, where $l_{\{\cdot\}}$ is an indicator of set $\{\cdot\}$.

Set $S_{n+1} = T_{n+1} - T_n$, $n \geqq 0$. Designate by $F_n(t) = P(S_{n+1} \leqq t)$ the distribution functions of the random variables S_{n+1}.

The nondecreasing family $\mathscr{F}^N = (\mathscr{F}_t^N)$, $t \in R_+$, of (completed by P-null sets from $\mathscr{F}_\infty$) σ-algebras $\mathscr{F}_t^N = \sigma\{N_s, s \leqq t\}$ is generated by the counting process N. This family is right-continuous, i. e. it is a filtration too ([4], ch. II, § 2), and will be interpreted as a flow of observations.

From observations $\mathscr{F}^N$ over the counting process N up to time $t \in R_+$ we shall estimate a nonobservable process W. An unique cadlag modifications $\pi(W) = (\pi_t(W), \mathscr{F}_t^N)$, $t \in R_+$, of the optimal estimate $E(W_t/\mathscr{F}_t^N)$, $t \in R_+$, exists. The process $\pi(W)$ is said to be an optimal filter of nonobservable process W (see [1]).

Our purpose in the present note is to derive an explit formula for the optimal filter on the set $[\![0, T_\infty]\!]$.

2. Results

We establish two theorems.

Theorem 1: Under considerations and notations in § 1 the following representation of the optimal filter $\pi_t(W)$:

$$(1) \qquad \pi_t(W) = \sum_{n \geqq 0} (1 - F_n(t - T_n))^{-1} \; (a_n - b_n T_n - a_{n+1} F_n(t - T_n) + b_{n+1} \int_{T_n}^{t} s dF_n(s - T_n)) \, . \, 1_{\{T_n \leqq t < T_{n+1}\}},$$

where $a_0 = 0$, $b_0 = 0$, holds on the set $[\![0, T_\infty[\![$.

Theorem 2. *The distribution functions of the random variables S_{n+1}, $n \geqq 0$, are determined by formulae:*

$$(2) \qquad F_n(t) = \begin{cases} a'_{n+1} \sqrt{2/\pi} \int_{1/\sqrt{t}}^{\infty} \exp(-(b_{n+1} s^{-2} - a'_{n+1})^2 s^2/2) ds, & t > 0 \, ; \\ 0, \; t \leqq 0, & \end{cases}$$

where $a'_{n+1} = a_{n+1} - a_n - (b_{n+1} - b_n) T_n$; $\quad a_0 = 0, \quad b_0 = 0$.

3. Proofs of the theorem

Proof of the theorem 1: The Wiener process is a continuous martingale which "drift" is zero and $W_0 = 0$. Applying the theorem of filtering ([1], § 3), we derive the form of the optimal filter $\pi(W)$ on the set $[\![0, T_\infty[\![$:

(3) $$\pi_t(W)=\int_{(0,t]}(\langle W/N\rangle_s-\pi_{s-}(W))d(N_s-\bar{A}_s).$$

Here $\langle W/N\rangle$ is the predictable expectation of W w. r. t. the counting process N, $\pi_{s-}(W)=\lim_{u\uparrow s}\pi_u(W)$ for $s>0$ and $\bar{A}=(\bar{A}_s)$, $s\geqq 0$, is the minimal compensator of N, i. e. $\bar{A}$ is the dual $\mathscr{F}^N$-predictable projection of the process N ([1], [2]).

The random variable $S_{n+1}=T_{n+1}-T_n$ does not depend on the σ-algebra $\mathscr{F}^N_{T_n}=\sigma\{N_s, s\leqq T_n\}$ because the Wiener process is a process with independent increments. Therefore :

$$P(S_{n+1}\leqq t/\mathscr{F}^N_{T_n})=P(S_{n+1}\leqq t).$$

According to [2], ch. V, T57 we know that

$$\bar{A}_t=\sum_{n\geqq 0}\ln(1-F_n((t-T_n)\wedge S_{n+1}))^{-1}.$$

From the theorem of the predictable expectation ([1], § 4) it holds on the set $[\![0, T_\infty[\![$:

$$\langle W/N\rangle_t=\sum_{n\geqq 0}g_{n+1}(T_1, T_2,\dots, T_n, t).1_{\{T_n<t\leqq T_{n+1}\}},$$

where $g_n=g_n(t_1, t_2,\dots,t_n)$, $n\geqq 1$, are Borel's functions with the property

$$g_n(T_1,\dots,T_n)=\mathsf{E}(W_{T_n}.1_{\{T_n<\infty\}}/\mathscr{F}^N_{T_n}).$$

On the set $]\!]T_n, T_{n+1}[\![$, $n\geqq 0$, we receive

$$g_{n+1}=a_{n+1}-b_{n+1}t\ ;$$

$$\langle W/N\rangle_t=a_{n+1}-b_{n+1}t\ ;$$

$\pi_{t-}=\pi_t$ since π_t is continuous on the set $]\!]T_n, T_{n+1}[\![$ ([1], § 1) ;

$$(t-T_n)\wedge S_{n+1}=t-T_n.$$

Therefore, on the set $]\!]T_n, T_{n+1}[\![$ the equation (3) takes the form

(4) $$d\pi_t(W)=((a_{n+1}-b_{n+1}t)-\pi_t(W))d(-\ln(1-F_n(t-T_n))^{-1}).$$

Considering t as a function with an argument $\ln(1-F_n(t-T_n))^{-1}$ from (4) we derive a linear differential equation which together with the initial condition $\pi_T=a_n-b_nT_n$ leads to a Cauchy problem :

$$\left| \begin{array}{l} d\pi_t = (\pi_t - a_{n+1} + b_{n+1} t) d(\ln(1 - F_n(t - T_n))^{-1}), \\ \pi_T = a_n - b_n T_n, \quad (\pi_T = W_T = 0) \end{array} \right.$$

on the interval $[T_n, T_{n+1})$, $n \geqq 1$.

The decision of this Cauchy problem is

$$\pi_t = (1 - F_n(t - T_n))^{-1} (a_n - b_n T_n - a_{n+1} F_n(t - T_n) + b_{n+1} \int_{T_n}^{t} s dF_n(s - T_n))$$

for $t \in [T_n, T_{n+1})$ and all $\omega \in \{\omega \in \Omega : \exists\ t \in R_+$ with $(\omega, t) \in [\![T_n, T_{n+1}[\![\}$.

Finally, on the set $[\![0, T_\infty[\![$, i.e. for $t \in R_+$, we derive

$$\pi_t(W) = \sum_{n \geqq 0} (1 - F_n(t - T_n))^{-1} (a_n - b_n T_n - a_{n+1} F_n(t - T_{n+1})$$

$$+ b_{n+1} \int_{T_n}^{t} s dF_n(s - T_n)) . 1_{\{T_n \leqq t < T_{n+1}\}},$$

where $a_0 = b_0$ since $\pi_0(W) = W_0 = 0$.

Proof of the theorem 2: It is well known that the distribution functions of the stopping times T_n have the form (c. f. [3]):

$$P(T_n \leqq t) = \begin{cases} a_n \sqrt{1/2\pi} \int_0^t s^{-3/2} \exp(-(b_n s - a_n)^2/(2s)) ds, & t > 0,\ n \geqq 1\ ;\ T_0 = 0. \\ 0, \ t \leqq 0. \end{cases}$$

(5) $$F_0(t) = P(S_1 \leqq t) = P(T_1 \leqq t) \quad \text{for } n = 0.$$

To find the form of $F_1(t) = P(S_2 \leqq t)$, we make a change of the coordinate system $K(Oty)$ in $K_1(O^1 t^1 y^1)$ with the formulas:

$$K_1 \quad \left| \begin{array}{l} t^1 = t - T_1 \\ y^1 = y - (a_1 - b_1 T_1). \end{array} \right.$$

Using the strong Markov property of the Wiener process we note that in K_1 the stopping time $T_2' = T_2 - T_1$ will be a moment of first reaching of line l_2 by a one-dimensional Wiener process with an initial point 0^1. Hence:

$$F_1(t) = P(T_2' \leqq t) = a_2' \sqrt{1/2\pi} \int_0^t s^{-3/2} \exp(-(b_2' s - a_2')^2/(2s)) ds,$$

where a_2' и b_2' are the coefficients in equention of line l_2 в K_1, i.e.

$$l_2 : y^1 = a'_2 - b'_2 t^1 \text{ in } K_1,$$

$$a'_2 = (a_2 - a_1) - (b_2 - b_1) T_1$$

$b'_2 = b_2$. We have $a'_2 \geqq 0$ because $a_2 \geqq a_1$, $b_2 \leqq b_1$ and $b'_2 \geqq 0$.

Analogously, after sequential changes of coordinate systems we obtain in $K_n(0^n t^n y^n)$

$$K^n \left| \begin{array}{l} t^n = t^{n-1} - T^n \\ y^n = y^{n-1} - (a_n - b_n T_n). \end{array} \right.$$

T'_{n+1} is a moment of first reaching of line l_{n+1} by a one-dimensional Wiener process with an initial point 0^n. Therefore, for all $n \geqq 0$ we get

$$(6) \quad F_n(t) = P(T'_{n+1} \leqq t) = a'_{n+1} \sqrt{1/2\pi} \int_0^t s^{-3/2} \exp(-(b_{n+1} s - a'_{n+1})^2/(2s)) ds,$$

where $a'_{n+1} = (a_{n+1} - a_n) - (b_{n+1} - b_n) T_n$. Obviously $a'_{n+1} \geqq 0$.

Remark that $F_n(t) = 0$ for $t \leqq 0$ because $P(T_{n+1} - T_n \leqq 0) = 0$ for $(a_{n+1}, b_{n+1}) \neq (a_n, b_n)$.

Finally, making the substitution $\xi = s^{-1/2}$ in (5) и (6) we derive the desired form (2).

4. Special cases

Case 1: Let us have a set of horizontal lines, i.e. $a_n = n$, $b_n = 0$, $n \geqq 1$. Then $T_n = \inf\{t \in R_+ : W_t = n\}$ is the moment of first reaching of the integer level n. Applying Theorem 1 and Theorem 2, we get

$$\pi_t(W) = \sum_{n \geqq 0} (1 - F_n(t - T_n))^{-1} (n - (n+1) F_n(t - T_n)) \cdot 1_{\{T_n \leqq t < T_{n+1}\}};$$

$$F_n(t) = \begin{cases} \sqrt{\frac{2}{\pi}} \int_{1/\sqrt{t}}^{\infty} \exp(-y^2/2) dy, & t > 0 \\ 0, & t \leqq 0, \end{cases}$$

for $n \geqq 0$, because $a'_{n+1} = 1$. This result coincides with the result in [1], § 5.

Case 2: Let $\{l_n\}$, $n \geqq 1$, be a bundle of lines which pass through a point $(0, a)$, i.e.

$$l_n : y = a - b_n t, \quad a > 0, \quad b_n \geqq 0$$

and let $\{b_n\}$, $n \geqq 1$, be a decreasing sequence.

It follows from Theorem 1 and Theorem 2 that

$$\pi_t(W)=\sum_{n\geqq 0}(a+(1-F_n(t-T_n))^{-1}(b_{n+1}\int_{T_n}^{t} s dF_n(s-T_n)-b_nT_n))\,1_{\{T_n\leqq t<T_{n+1}\}},$$

$$a'_{n+1}=(b_n-b_{n+1})T_n \text{ and for } n\geqq 0:$$

$$F_n(t)=\begin{cases}(b_n-b_{n+1})T_n\sqrt{2/\pi}\int_{1/\sqrt{t}}^{\infty}\exp(-(b_{n+1}s^{-2}-(b_n-b_{n+1})T_n)^2 s^2/2)\,ds, & t>0;\\ 0, & t\leqq 0.\end{cases}$$

Case 3: Let $\{l_n\}$, $n\geqq 1$ be a set of parallel inclined lines, which satisfy the conditions
$l_n: y=a_n-bt$, $b>0$, $a_n\geqq 0$, and $\{a_n\}$, $n\geqq 1$, be an increasing sequence.

It follows from Theorem 1 and Theorem 2 that

$$\pi_t(W)=\sum_{n\geqq 0}(1-F_n(t-T_n))^{-1}(a_n-a_{n+1}F_n(t-T_n)$$

$$+b\int_{T_n}^{t} s dF_n(s-T_n)-bT_n)\,.\,1_{\{T_n\leqq t<T_{n+1}\}}.$$

Since $a'_{n+1}=a_{n+1}$, we get for $n\geqq 0$

$$F_n(t)=\begin{cases}(a_{n+1}-a_n)\sqrt{2/\pi}\int_{1/\sqrt{t}}^{\infty}\exp(-(bs^{-2}-(a_{n+1}-a_n))^2 s^2/2)\,ds, & t>0\\ 0, & t\leqq 0.\end{cases}$$

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Received 30. 09. 1987