Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

---

Mathematica Balkanica

New Series Vol. 2, 1988, Fasc. 2-3

Exact Error Bounds for the Periodic Cubic and Parabolic Spline Interpolation on the Uniform Mesh

V. L. Miroshnichenko

Presented by Bl. Sendov

1. Introduction

Let $\Delta : x_i = a + ih$, $i = 0, \dots, N$ be the uniform partition of the interval $[a, b]$ with spacing $h = (b-a)/N$. Suppose that in the points x_i the values $f_i = f(x_i)$, $i = 0, \dots, N$ of a $(b-a)$-periodic function $f(x) \in W^r_\infty$ are given. We write $W^r_\infty = W^r_\infty [a, b]$ for the class of $(b-a)$-periodic functions $f(x)$ such that $f^{(r-1)}$ is absolutely continuous and $f^{(r)} \in L_\infty [a, b]$. We denote $\|f\|_\infty = \|f\|_{L_\infty [a,b]}$.

Let $S(x)$ be the cubic [1-3] or parabolic [2, 4] $(b-a)$-periodic spline of deficiency 1 interpolating to $f(x)$ at the meshpoints x_i, i.e. $S(x_i) = f_i$, $i = 0, \dots, N$. We suppose that the knots (breakpoints) of cubic spline coincide with x_i, but the knots of parabolic spline are located at the points $x_i + h/2$, $i = 0, \dots, N-1$. We note that both types of parabolic splines [2, 4] are identical for interpolation on the uniform mesh.

The aim of this paper is to obtain the exact pointwise estimates

$$(1) \qquad |S^{(m)}(x) - f^{(m)}(x)| \leqq K_{rm}(N; t) h^{r-m} \|f^{(r)}\|_\infty, \quad m < r$$

and asymptotically exact pointwise estimates

$$(2) \qquad |S^{(m)}(x) - f^{(m)}(x)| \leqq K_{rm}(t) h^{r-m} \|f^{(r)}\|_\infty, \quad m < r,$$

where $f \in W^r_\infty$, $x = x_i + th$, $t \in [0, 1]$, $i = 0, \dots, N-1$; $r = 1, 2, 3$ for the parabolic splines and $r = 1, 2, 3, 4$ for the cubic splines.

We want to obtain best possible error functions $K_{rm}(N; t)$, $K_{rm}(t)$. It means that for given r, m the function $K_{rm}(N; t)$ cannot be diminished for any N, t and the function $K_{rm}(t)$ cannot be diminished for any t simultaneously for all N, i.e. $K_{rm}(t) = \sup_N K_{rm}(N; t)$.

In section 2 we describe briefly the method that have been used in order to obtain the estimates (1), (2) for the cubic splines. We can give the explicit expression of $K_{rm}(N\,;t)$ only for even N and some r, m. However, the functions $K_{rm}(t)$ are given for $r=1, 2, 3, 4$; $m<r$. Similar results for the parabolic splines we describe in section 3.

It is easy to derive from (2) the exact error bounds

$$\| S^{(m)}-f^{(m)} \|_{\infty} \leqq K_{rm} h^{r-m} \|f^{(r)}\|_{\infty}, \quad m<r, \tag{3}$$

where $K_{rm}=\max_t K_{rm}(t)$ are the best possible constants.

Using the explicit formulas for functions $K_{rm}(t)$ we can establish the following two theorems.

Theorem 1. *Let $S(x)\in C^2$ be the $(b-a)$-periodic cubic spline interpolating to $f(x)\in W^r_{\infty}$, $r=1, 2, 3, 4$ at the points of the mesh Δ. Then the estimates* (3) *holds, where*

$$K_{10}=(1+3\sqrt{3})/8, \quad K_{20}=3\sqrt{3}/32, \quad K_{21}=\sqrt{3}/3,$$
$$K_{30}=(3-\sqrt{3})(7/4+\sqrt{3}+\sqrt{2})/144, \quad K_{31}=(1+\sqrt{2})/18,$$
$$K_{32}=(21+5\sqrt{3}+4\sqrt{2}(3-\sqrt{3}))/72, \quad K_{40}=5/384,$$
$$K_{41}=1/24, \quad K_{42}=\sqrt{3}/12, \quad K_{43}=(1+\sqrt{3})/4$$

are the best possible constants.

Theorem 2. *Let $S(x)\in C^1$ be the $(b-a)$-periodic parabolic spline with knots $x_i+h/2$, $i=0,\dots,N-1$ interpolating to $f(x)\in W^r_{\infty}$, $r=1, 2, 3$ at the points of the mesh Δ. Then the estimates* (3) *holds, where*

$$K_{10}=\sqrt{2}/2, \quad K_{20}=(1+\sqrt{2})/16, \quad K_{21}=3\sqrt{2}/8,$$
$$K_{30}=1/24, \quad K_{31}=1/8, \quad K_{32}=(1+2\sqrt{2})/6$$

are the best possible constants.

We note that only constants K_{40} [5, 6], K_{41} [7], K_{42}, K_{43} [8] were known for the cubic splines. Also K_{30} [6] and K_{31} [10] were known for the parabolic splines. From [9] we have the expressions of the error functions $K_{40}(2k\,;t)=K_{40}(t)$ (cubic spline) and $K_{30}(2k\,;t)=K_{30}(t)$ (parabolic spline). Hence, the most of the results given in our paper are new. We should also mention that the papers [3, 11-14] contain the information about the constants in error estimates for the cubic and parabolic splines on nonuniform mesh for the various classes of functions. Moreover these constants are close to (some of them coincide with) the best possible values.

2. Error bounds for cubic splines

Let $S(x)\in C^2$ be the cubic spline interpolating to $f\in W^r_{\infty}$, $r=1, 2, 3, 4$ at the points of Δ. Then we have

$$(4)\qquad S_{i-1}+4S_i+S_{i+1}=f_{i-1}+4f_i+f_{i+1},$$

$$(5)\qquad S'_{i-1}+4S'_i+S'_{i+1}=3(f_{i+1}-f_{i-1})/h,$$

$$(6)\qquad S''_{i-1}+4S''_i+S''_{i+1}=6(f_{i+1}-2f_i+f_{i-1})/h^2,$$

$$(7)\qquad S'''_{i-1+o}+4S'''_{i+o}+S'''_{i+1+o}=\frac{6}{h}\left\{\frac{f_{i+2}-2f_{i+1}+f_i}{h^2}-\frac{f_{i+1}-2f_i+f_{i-1}}{h^2}\right\},$$

$$i=1,\dots,N,$$

where $S_i^{(m)}=S^{(m)}(x_i)$, $m=0,\ 1,\ 2$; $S'''_{i+o}=S'''(x_i+0)$ and in view of periodicity $S_{i+N}^{(m)}=S_i^{(m)}$, $m=0,\ 1,\ 2$; $S'''_{i+o}=S''_{i+N+o}$, $f_{i+N}=f_i$.

The equality (4) follows from the interpolation conditions. The next three equations (5)—(7) are well known relations between values of the interpolated function $f(x)$ and the derivatives of cubic spline [1—3].

By summing (4)—(7) multiplied by 1, th, $t^2h^2/2$, $t^3h^3/6$, $t\in[0,\ 1]$ correspondingly we have

$$(8)\qquad S(x_{i-1}+th)+4S(x_i+th)+S(x_{i+1}+th)=d_i,\quad t\in[0,\ 1],\quad i=1,\dots,N,$$

where $d_i=(1-t)^3f_{i-1}+(4-6t^2+3t^3)f_i+(1+3t+3t^2-3t^3)f_{i+1}+t^3f_{i+2}$.

Denote $\varepsilon_i(t)=S(x_i+th)-f(x_i+th)$. Then from (8) we obtain

$$(9)\qquad \varepsilon_{i-1}(t)+4\varepsilon_i(t)+\varepsilon_{i+1}(t)=\delta_i(t),\quad i=1,\dots,N,$$

where $\delta_i(t)=d_i-f(x_{i-1}+th)-4f(x_i+th)-f(x_{i+1}+th)$.

Let a_{ij}, $i,j=1,\dots,N$ be the elements of the inverse matrix to the coefficient matrix in (9). Then [1]

$$a_{ij}=\frac{\sigma^{|i-j|}+\sigma^{N-|i-j|}}{2\sqrt{3}(1-\sigma^N)},$$

where $\sigma=\sqrt{3}-2$ is the root of the equation $\sigma^2+4\sigma+1=0$.

Now from (9) we have

$$(10)\qquad \varepsilon_k(t)=\sum_{j=1}^{N}a_{kj}\delta_j(t),\quad t\in[0,\ 1],$$

where k is equals to the integral part of $N/2$.

Applying Taylor's expansion with the integral remainder at the point x_j+th to the values f_n, $n=j-1,\ j,\ j+1,\ j+2$; $f(x_n+th)$, $n=j-1,\ j+1$ in $\delta_j(t)$, after some transformations, which are omitted here, we obtain

$$\varepsilon_k(t)=\frac{h^r}{(r-1)!}\{\sum_{\substack{j=1\\j\neq k}}^{N}\int_0^1 \alpha_{kjr}(t,\tau)f^{(r)}(x_j+\tau h)\,d\tau$$

$$+\int_0^t \alpha_{kkr}(t,\tau)f^{(r)}(x_k+\tau h)\,d\tau+(-1)^r\int_t^1 \alpha_{kkr}(1-t,1-\tau)f^{(r)}(x_k+\tau h)\,d\tau\},$$

where

$$\alpha_{kjr}(t,\tau)=a_{kj-1}\,\varphi_{-1,r}(t,\tau)+a_{kj}\,\varphi_{o,r}(t,\tau)+a_{kj+1}\,\varphi_{1r}(t,\tau),$$

$$\varphi_{-1,r}(t,\tau)=t^3(1-\tau)^{r-1}-(t-\tau)^{r-1},$$

$$\varphi_{o,r}(t,\tau)=(-1)^r[(1-t)^3(1+\tau)^{r-1}+(4-6t^2+3t^3)\tau^{r-1}-(1+\tau-t)^{r-1}],$$

$$\varphi_{1,r}(t,\tau)=(-1)^r(1-t)^3\tau^{r-1}.$$

Performing the differentiation with respect to x and using Hölder's inequality we obtain the estimate

$$|\varepsilon_k^{(m)}(t)|\leqq h^{r-m}\,\|f^{(r)}\|_\infty\,K_{rm}(N;t),\quad t\in[0,\ 1], \tag{11}$$

where

$$K_{rm}(N;t)=\frac{1}{(r-1)!}\{\sum_{\substack{j=1\\j\neq k}}^{N}\int_0^1|\alpha_{kjr}^{(m)}(t,\tau)|\,d\tau$$

$$+\int_0^t|\alpha_{kkr}^{(m)}(t,\tau)|\,d\tau+\int_0^{1-t}|\alpha_{kkr}^{(m)}(1-t,\tau)|\,d\tau\}, \tag{12}$$

$$0\leqq m<r,\ r=1,\ 2,\ 3,\ 4;$$

$$\varepsilon_k^{(m)}(t)=S^{(m)}(x_k+th)-f^{(m)}(x_k+th),\ \alpha_{kjr}^{(m)}=\frac{\partial^m}{\partial t^m}\alpha_{kjr}(t,\tau).$$

As a result we have the estimates (1). It is evident that the formulae (12) give the best possible error functions $K_{rm}(N;t)$ for arbitrary N, r, m. However, we can derive the explicit expression of $K_{rm}(N;t)$ only for N even and some special values of r, m.

It can be easily shown that the functions $K_{rm}(t)$ in (2) can be obtained from (12) by taking the limit as $N\to\infty$. Then

$$K_{rm}(t)=\frac{\sqrt{3}}{6(r-1)!}\{\int_0^t|A_r^{(m)}(t,\tau)|\,d\tau+\int_0^{1-t}|A_r^{(m)}(1-t,\tau)|\,d\tau$$

$$+\frac{1}{1+\sigma}(\int_0^1|B_r^{(m)}(t,\tau)|\,d\tau+\int_0^1|C_r^{(m)}(t,\tau)|\,d\tau)\} \tag{13}$$

$$0\leqq m<r,\quad r=1,\ 2,\ 3,\ 4,$$

where

$$A_r(t,\tau)=\varphi_{o,r}(t,\tau)+\sigma[\varphi_{-1,r}(t,\tau)+\varphi_{1,r}(t,\tau)],$$

$$B_r(t,\tau)=\varphi_{-1,r}(t,\tau)+\sigma\varphi_{o,r}(t,\tau)+\sigma^2\varphi_{1,r}(t,\tau),$$

$$C_r(t,\tau)=\sigma^2\varphi_{-1,r}(t,\tau)+\sigma\varphi_{o,r}(t,\tau)+\varphi_{1,r}(t,\tau).$$

Hence in order to obtain any function $K_{rm}(t)$, it is sufficient in contrast to $K_{rm}(N;t)$ to find only four integrals. It can be done for example, by the numerical integrating. In fact all error functions $K_{rm}(t)$, $r=1, 2, 3, 4$; $m<r$ were obtained as the explicit analytical expressions.

We'll state without proof the results obtained from (12), (13). The following notations are used: $u=t(1-t)$, $w=1-2t$, $t_1=(3-\sqrt{3})/6\approx\cdot 21132$, $t_2=\dfrac{1+\sqrt{3}-\sqrt{2}}{2\sqrt{3}}\approx\cdot 38043$.

$f(x)\in W^1_\infty$.

$$K_{10}(N;t)=u\{1-2u+\sqrt{3}(1+2u)\frac{1-|\sigma|^k}{1+|\sigma|^k}+\sqrt{3}\frac{|\sigma|^k}{1-\sigma^N}|w|(1-|w|)\},\quad N=2k,$$

$$K_{10}(t)=u\{1+\sqrt{3}+2u(\sqrt{3}-1\},\quad t\in[0,\ 1];$$

$$\max K_{10}(t)=K_{10}(1/2)=(1+3\sqrt{3})/8.$$

$f(x)\in W^2_\infty$.

$$K_{20}(t)=\frac{u}{4}\{1+\sqrt{3}-\frac{2-2u-8u^2}{3+\sqrt{3}-u+4u(1+\sqrt{3})+4u^2}\},\quad t\in[0,\ 1],$$

$$\max K_{20}(t)=K_{20}(1/2)=3\sqrt{3}/32.$$

$$K_{21}(t)=\frac{3-\sqrt{3}}{6}\{\frac{\sqrt{3}}{2}+3u-6\sqrt{3}u^2+\sqrt{3}|w|+\frac{(1-\sqrt{3}|w|-6u)^2}{4(1+2\sqrt{3}u)}\},$$

$$t\in[0,t_2]\cup[1-t_2,1],$$

$$K_{21}(t)=\frac{3-\sqrt{3}}{6}\{\sqrt{3}/2-1+9u-6\sqrt{3}u^2+\frac{2(1-3u)^2}{1+2\sqrt{3}u}\},\quad t\in[t_2,1-t_2],$$

$$\max K_{21}(t)=K_{21}(0)=K_{21}(1)=\sqrt{3}/3.$$

$\underline{f(x)\in W^3_\infty .}$

$$K_{30}(t)=\frac{3-\sqrt{3}}{36}u\{1+\sqrt{2}+\tau_0(\sqrt{3}-|w|)(2+\sqrt{3}\tau_0)\},\quad t\in[0,1],$$

where $\tau_0(t)$ is the nonnegative root of the equation

$$2\sqrt{3}(\frac{4}{\sqrt{3}-1}-1+4u-|w|)\tau^2-(1+2\tau\sqrt{3})[(\sqrt{3}-1)(1+|w|)+4u]=0\ ;$$

$$\max K_{30}(t)=K_{30}(1/2)=(3-\sqrt{3})(7/4+\sqrt{2}+\sqrt{3})/144.$$

$$K_{31}(t)=\frac{3-\sqrt{3}}{36}\{(1+\sqrt{2})|w|-\tau_1(2+\sqrt{3}\tau_1)(1-\sqrt{3}|w|-6u)\},$$

$$t\in[0,t_2]\cup[1-t_2,1],$$

$$K_{31}(t)=\frac{3-\sqrt{3}}{36}\{(6u-1)\frac{1+\sqrt{2}}{\sqrt{3}}-\tau_1(2+\sqrt{3}\tau_1)(1-\sqrt{3}|w|-6u)$$

$$-\bar{\tau}_1(2+\sqrt{3}\bar{\tau}_1)(1+\sqrt{3}|w|-6u)\},\quad t\in[t_2,1-t_2],$$

where $\tau_1(t)$ and $\bar{\tau}_1(t)$ are the nonnegative roots of the equations respectively

$$6(1+2\sqrt{3}u)\tau^2+(1-\sqrt{3}|w|-6u)(1+2\sqrt{3}\tau)=0,$$

$$6(1+2\sqrt{3}u)\tau^2+(1+\sqrt{3}|w|-6u)(1+2\sqrt{3}\tau)=0\ ;$$

$$\max K_{31}(t)=K_{31}(0)=K_{31}(1)=(1+\sqrt{2})/18.$$

$$K_{32}(t)=\frac{\sqrt{3}-1}{6}[(1+\sqrt{2})|w|-\tau_2(2+\sqrt{3}\tau_2)(1-\sqrt{3}|w|)]$$

$$+u[1+2(\sqrt{3}-1)u],\quad t\in[0,t_1]\cup[1-t_1,1],$$

where $\tau_2(t)$ is the nonnegative root of the equation

$$6|w|\tau^2+(1+2\sqrt{3}\tau)(1-\sqrt{3}|w|)=0\ ;$$

$$K_{32}(t)=u[1+2u(\sqrt{3}-1)]+(3-\sqrt{3})(1+\sqrt{2})/18, \quad t\in[t_1, 1-t_1],$$

$$\max K_{32}(t)=K_{32}(1/2)=(21+5\sqrt{3}+4\sqrt{2}(3-\sqrt{3}))/72.$$

$\underline{f(x)\in W^4_\infty}$.

$$K_{40}(N\,;t)=K_{40}(t)=u(1+u)/24, \quad N=2k, \quad t\in[0,1],$$

$$\max K_{40}(t)=K_{40}(1/2)=5/384.$$

$$K_{41}(N\,;t)=K_{41}(t)=|w|(1+2u)/24, \quad N=2k, \quad t\in[0,t_2]\cup[1-t_2,1],$$

$$K_{41}(t)=\frac{1}{24}\{\frac{1-\sqrt{3}}{2}v_1+\frac{v_1^2(\sqrt{3}-1)(\sqrt{3}\tau_1^2+4\tau_1+\sqrt{3})}{4(1+2\sqrt{3}u)}+|w|(1+2u)\},$$

$$t\in[t_2, 1-t_2],$$

where $v_1=1+\sqrt{3}|w|-6u$ and $\tau_1(t)$ is the nonnegative root of the equation $2(1+2\sqrt{3}u)\tau^2+v_1(1+\sqrt{3}\tau)=0$;

$$\max K_{41}(t)=K_{41}(0)=K_{41}(1)=1/24.$$

$$K_{42}(t)=\frac{(3-\sqrt{3})v}{24}\{\frac{v(\sqrt{3}\tau_2^2+4\tau_2+\sqrt{3})}{2|w|}-1\}+\frac{u}{2}, \quad t\in[0,t_1]\cup[1-t_1,1],$$

where $v=1-\sqrt{3}|w|$ and $\tau_2(t)$ is the nonnegative root of the equation $2|w|\tau^2+v(1+\sqrt{3}\tau)=0$;

$$K_{42}(N\,;t)=K_{42}(t)=u/2, \quad t\in[t_1,1-t_1], \quad N=2k,$$

$$\max K_{42}(t)=K_{42}(0)=K_{42}(1)=\sqrt{3}/12.$$

$$K_{43}(t)=\frac{1+\sqrt{3}}{4}-u-u^2(\sqrt{3}-1)/2, \quad t\in[0,1],$$

$$K_{43}(N\,;t)=K_{43}(t)-\frac{\sqrt{3}|\sigma|^k}{16(1-\sigma^N)}\{7-8|\sigma|^k(1-2u^2)\}, \quad t\in[0,1], \quad N=2k.$$

$$\max K_{43}(t)=K_{43}(0)=K_{43}(1)=(1+\sqrt{3})/4.$$

We expose the graphs of all functions $K_{rm}(t)$ for the cubic spline interpolation in figure 1.

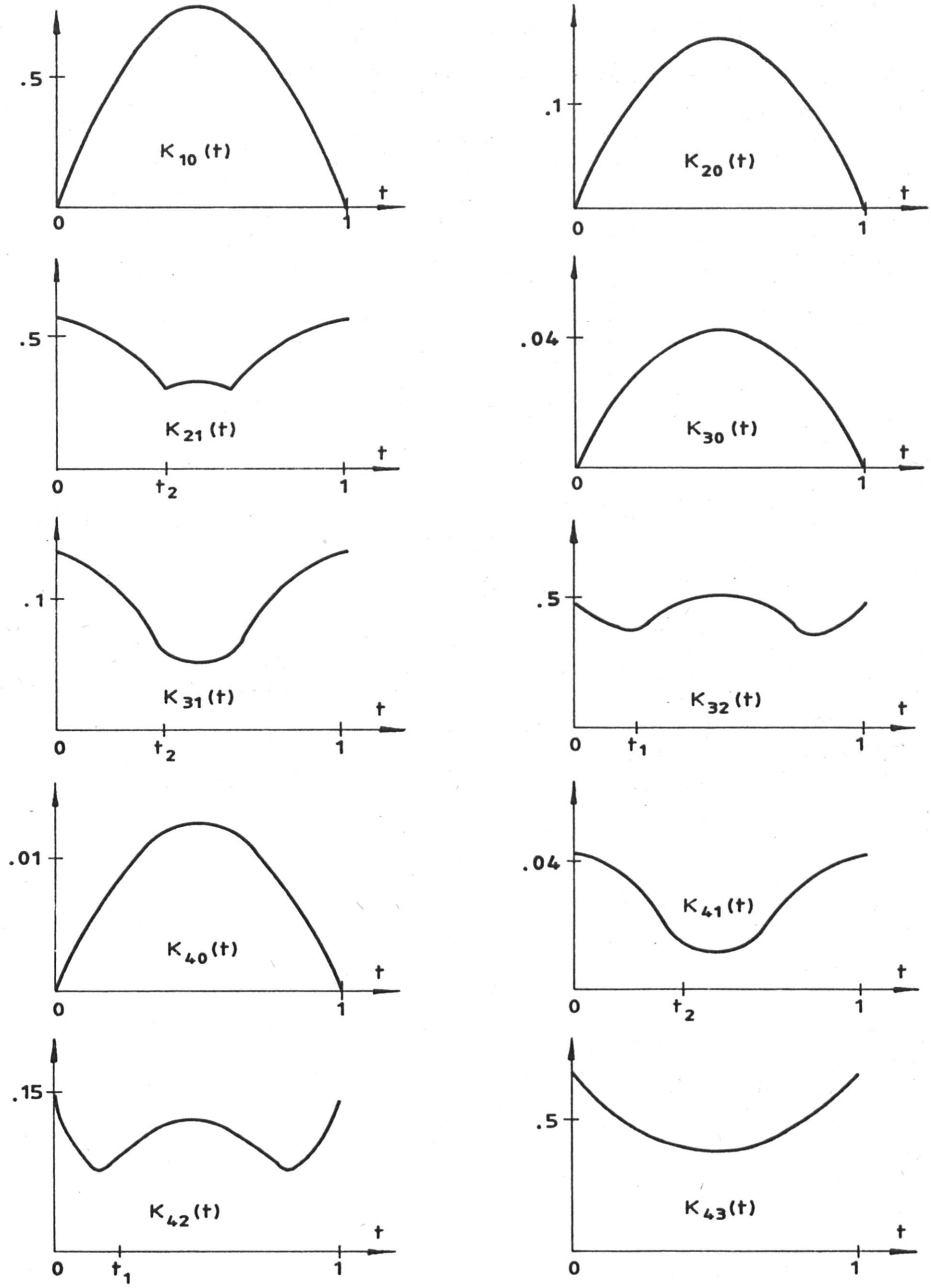

Fig. 1. Functions $K_{rm}(t)$ for cubic splines

3. Error bounds for parabolic splines

Let $S(x) \in C^1$ be the parabolic spline with the knots $x_i + h/2$, $i = 0, \ldots, N-1$ interpolating to $f \in W^r_\infty$, $r = 1, 2, 3$ at the points of Δ. The error functions $K_{rm}(N;t)$, $K_{rm}(t)$ in the case of the parabolic splines can be obtained by the same method as in previous section of this paper. We'll mention only the difference in speculations.

First we must replace the relations (4)—(7) by the relations

$$S_{i-1} + 6S_i + S_{i+1} = f_{i-1} + 6f_i + f_{i+1},$$

$$S'_{i-1} + 6S'_i + S'_{i+1} = 4(f_{i+1} - f_{i-1})/h,$$

$$S''_{i-1} + 6S''_i + S''_{i+1} = 8(f_{i+1} - 2f_i + f_{i-1})/h^2,$$

$$i = 1, \ldots, N.$$

Further we define

$$a_{ij} = \frac{\sigma^{|i-j|} + \sigma^{N-|i-j|}}{4\sqrt{2}(1-\sigma^N)}, \quad i, j = 1, \ldots, N,$$

where $\sigma = 2\sqrt{2} - 3$ is the root of the equation $\sigma^2 + 6\sigma + 1 = 0$.

Then using Taylor's expantion and Hölder's inequality we obtain

$$K_{rm}(N;t) = \frac{1}{(r-1)!}\{\sum_{\substack{j=1 \\ j \neq k}}^{N} \int_0^1 |\beta^{(m)}_{kjr}(t,\tau)|\, d\tau + \int_0^t |\beta^{(m)}_{kkr}(t,\tau)$$

$$+ \psi^{(m)}_{-1r}(t,\tau)|\, d\tau + \int_t^1 |\beta^{(m)}_{kkr}(t,\tau)|\, d\tau\}, \quad t \in [0, 1/2], \tag{14}$$

$$0 \leqq m < r; \quad r = 1, 2, 3,$$

where

$$\beta_{kjr}(t,\tau) = a_{kj}\psi_{or}(t,\tau) + a_{kj+1}\psi_{1r}(t,\tau),$$

$$\psi_{or}(t,\tau) = (1+2t)^2(1-\tau)^{r-1} - (1+t-\tau)^{r-1}, \quad \psi_{-1r}(t,\tau) = -(t-\tau)^{r-1},$$

$$\psi_{1r}(t,\tau) = (-1)^r[(1-2t)^2\tau^{r-1} - (\tau - t)^{r-1}].$$

The functions $K_{rm}(t)$ in (2) have the following expressions

$$K_{rm}(t)=\frac{\sqrt{2}}{8(r-1)!}\{\int_0^t |A_r^{(m)}(t,\tau)|\,d\tau-\int_0^t |C_r^{(m)}(t,\tau)|\,d\tau$$

$$+\frac{1}{1+\sigma}(\int_0^1 |B_r^{(m)}(t,\tau)|\,d\tau+\int_0^1 |C_r^{(m)}(t,\tau)|\,d\tau)\}, \tag{15}$$

$$t\in[0,1/2],\quad 0\leqq m<r,\quad r=1,\ 2,\ 3,$$

where

$$A_r(t,\tau)=4\sqrt{2}\psi_{-1r}(t,\tau)+C_r(t,\tau),$$

$$B_r(t,\tau)=\sigma\psi_{or}(t,\tau)+\psi_{1r}(t,\tau),\quad C_r(t,\tau)=\psi_{or}(t,\tau)+\sigma\psi_{1r}(t,\tau).$$

Below we'll give the explicit expressions of $K_{rm}(N;t)$, $K_{rm}(t)$ for $t\in[0,1/2]$. To obtain the formulae for $t\in[1/2,1]$, we must replace t by $1-t$ at right-hand sides of this expressions.

$f(x)\in W_\infty^1$.

$$K_{10}(t)=t[1-4t(1-t)+\sqrt{2}(1+2t(1-2t))],\quad t\in[0,1/2],$$

$$K_{10}(N;t)=K_{10}(t)-\frac{2\sqrt{2}t|\sigma|^k}{1-\sigma^N}[1+|\sigma|^k(1+2t(1-2t))],\quad t\in[0,1/2],$$

$$\max K_{10}(t)=K_{10}(1/2)=\sqrt{2}/2.$$

$f(x)\in W_\infty^2$.

$$K_{20}(t)=\frac{t}{8}\{3\sqrt{2}-4t+\frac{2t(1-\sqrt{2}t)^2}{1-(2-\sqrt{2})(1+\sqrt{2}t)t}\},\quad t\in[0,1/2],$$

$$\max K_{20}(t)=K_{20}(1/2)=(1+\sqrt{2})/16.$$

$$K_{21}(t)=3\sqrt{2}/8-\sqrt{2}t^2(3-\sqrt{2})+4t^3(\sqrt{2}-1),\quad t\in[0,t_3],\quad t_3=\sqrt{2}/4,$$

$$K_{21}(t)=\frac{3}{2}t-(3\sqrt{2}-2)t^2+4t^3(\sqrt{2}-1)+\frac{(2+\sqrt{2})(2\sqrt{2}t-1)^2}{8(2\sqrt{2}t+1)},\quad t\in[t_3,1/2],$$

$$\max K_{21}(t)=K_{21}(0)=3\sqrt{2}/8.$$

$f(x)\in W_\infty^3$.

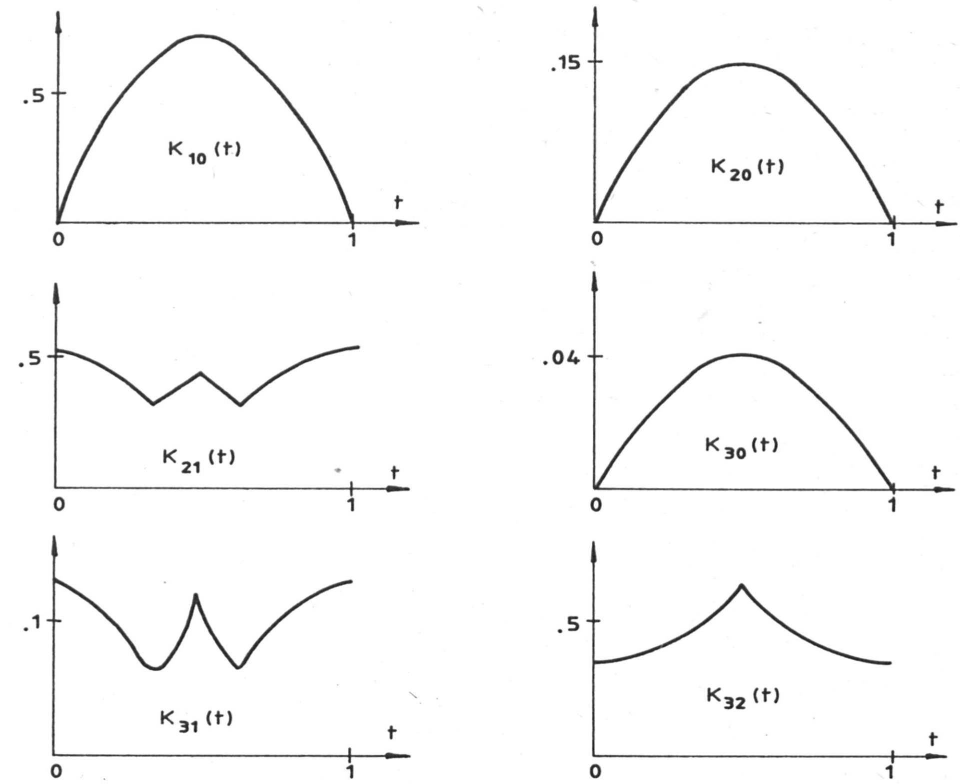

Fig. 2. Functions $K_{rm}(t)$ for parabolic spline

$$K_{30}(N\,;t)=K_{30}(t)=(3t-4t^3)/24, \quad t\in[0,1/2], \quad N=2k,$$
$$\max K_{30}(t)=K_{30}(1/2)=1/24.$$

$$K_{31}(N\,;t)=K_{31}(t)=(1-4t^2)/8, \quad t\in[0,t_3], \quad N=2k,$$

$$K_{31}(t)=\sqrt{2}\,[3t-3\sqrt{2}t^2-(1-2\sqrt{2}t)(2+\sqrt{2}\tau)\tau]/12, \quad t\in[t_3,1/2],$$

where $\tau(t)$ is the nonnegative root of the equation

$$2\tau^2(2\sqrt{2}-2)-(2\sqrt{2}\tau+1)(2\sqrt{2}t-1)/(2\sqrt{2}t+1)=0\,;$$

$$\max K_{31}(t)=K_{31}(0)=1/8.$$

$$K_{32}(t)=[3+12t^2-8t^3(2-\sqrt{2})]/(6\sqrt{2}), \quad t\in[0,1/2],$$

$$K_{32}(N\,;t)=K_{32}(t)-\frac{\sqrt{2}\,|\sigma|^k}{2(1+|\sigma|^k)}\{1-\frac{4t^2(3-4t)\,|\sigma|^k}{3(1-|\sigma|^k)}\},\quad t\in[0,1/2],$$

$$\max K_{32}(t)=K_{32}(1/2)=(1+2\sqrt{2})/6.$$

The graphs of the functions $K_{rm}(t)$ are shown in Fig. 2.

References

1. J. H. Ahlberg, E. N. Nilson, J. L. Walsh. The Theory of Splines and their Applications. New York, 1967.
2. S. B. Stechkin, Yu. N. Subbotin. Splines in Numerical Mathematics. Moscow, 1976 (Russian).
3. Yu. S. Zav'yalov, B. I. Kvasov, V. L. Miroshnichenko. The Methods of Spline Functions. Moscow, 1980 (Russian).
4. M. J. Marsden. Quadratic spline interpolation. *Bull. Amer. Math. Soc.*, **80**, 1974, 903-906.
5. C. A. Hall. On error bounds for spline interpolation. *J. Approxim. Theory*, **1**, 1968, 209-219.
6. V. M. Tihomirov. Best methods of approximation and interpolation of differentiable functions in the space C(−1, 1). *Mat. Sbornic*, **80**, 1969, 290-304 (Russian).
7. C. A. Hall, W. W. Meyer. Optimal error bounds for cubic spline interpolation. *J. Approxim. Theory*, 16, 1976, 105-121.
8. V. L. Miroshnichenko. Approximative properties of the cubic splines. — In : International conference on the theory of approximation of functions. Abstr. of papers. Kiev, 1983, 127 (Russian).
9. A. A. Zensykbaev. Approximation of differentiable periodic functions by splines on a uniform subdivision. *Mat. Zametki*, **13**, 1973, 807-816 (Russian).
10. N. P. Korneichuk. Splines in the Approximation Theory. Moscow, 1984 (Russian).
11. V. L. Miroshnichenko. Error of approximation by cubic interpolation splines. — II. In: Methods of spline functions (*Vychisl. Systemy*. **93**), Novosibirsk, 1982, 3-29 (Russian).
12. V. L. Miroshnichenko. Error of approximation by cubic interpolation splines. — II. In : Methods of spline functions in the numerical analysis (*Vychisl. Systemy*, **98**), Novosibirsk, 1983, 51-66 (Russian).
13. B. I. Kvasov. Quadric spline interpolation. Preprint 3, Inst. Theoret. and Appl. Mech., Novosibirsk, 1981 (Russian).
14. B. I. Kvasov. Error bounds for parabolic interpolating splines. Preprint 2-84. Inst. Theoret. and Appl. Mech., Novosibirsk, 1984 (Russian).

USSR, 630090, Novosibirsk, 90
Institute of Mathematics,
Siberian Branch of the USSR Academy of Sciences

Received 10. 10. 1987