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On the Stable Approximation of Derivatives by Splines in the Convex Set

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While using some algorithms, in particular of the interpolating type, for the aims of calculating the derivatives of functions given by tables of its values containing some error, there often appears an unstability, i. e. the diminution of the steps of the mesh leads to the worse approximation of derivatives. In the present paper it is shown, that the derivatives of polinomial splines of odd degree, defect one, in the convex sets give stable approximation of the corresponding derivatives of the given function. The estimates of such approximation of the derivatives are obtained.

The extremal problem, leading to the splines of odd degree, defect one, was primarily suggested by M. Attia [2]. Both he and P. -J. Laurent [5] investigated the questions of existence, uniqueness and characterization of the solution of such problem. The stability of the first two derivatives approximation of the table function by the derivatives of cubic spline in the convex set was shown by V. A. Morozov [7]. The approximative properties of splines in the convex sets of small degree and some other aspects of the problem were examined by the authors in [8-9], [12-15].

In the first section of the paper we remind the extremal problem leading to the splines of degree $2k-1$ $(k=1)$. The error bounds of the k-th derivative approximation of the function $f(x)$ by spline in the convex set in the norm of space $L_2[a, b]$ are obtained in the section 2. The third section contains the estimates of $f^{(k-1)}$ approximation in norm of $C[a,b]$.

1. Let $\Delta: a=x_1<\ldots<x_N=b$ be a mesh on the interval $[a,b]$ and z_i, $\varepsilon_i \geqq 0$ $(i=1,\ldots,N)$ are the real numbers. We consider the sets of functions: $G=\{\varphi(x)\in W_2^k[a,b]: z_i-\varepsilon_i \leqq \varphi(x_i) \leqq z_i+\varepsilon_i,\ i=1,\ldots,N\}$, $\bar{G}=G\cap \bar{W}_2^k[a,b]$, $\tilde{G}=G\cap \tilde{W}_2^k[a,b]$, where $\bar{W}_2^k[a,b]$ and $\tilde{W}_2^k[a,b]$ are the subsets of the Sobolev space $W_2^k[a,b]$ such that: $\varphi^{(q)}(a)=z_a^q$, $\varphi^{(q)}(b)=z_b^q$, $q=1,\ldots,k-1$, if $\varphi\in\bar{W}_2^k[a,b]$, where z_a^q and z_b^q are real numbers, and φ is a $b-a$-periodic function if $\varphi\in\tilde{W}_2^k[a,b]$. It is evident that the sets $\bar{G}$ and $\tilde{G}$ are convex and closed.

The problem of minimizing the functional

$$(1)\qquad J(\varphi)=\int_a^b |\varphi^{(k)}(x)|^2 dx = \|\varphi^{(k)}\|_2^2$$

(we denote $\|\cdot\|_{L_q[a,b]}$ by $\|\cdot\|_q$) on the sets $\bar{G}$ and $\tilde{G}$ has the solutions in both cases. These solutions are the polinomial splines of degree $2k-1$ of defect 1, i.e. the functions $\bar{S}$ and $\tilde{S}$ on the interval $[a, b]$, such that :

1) their restriction on each small interval $[x_i, x_{i+1}]$ is polinomial of degree $2k-1$;

2) $\bar{S}$, $\tilde{S}\in C^{2(k-1)}[a,b]$;

3) $\bar{S}^{(q)}(a)=z_a^q$, $\bar{S}^{(q)}(b)=z_b^q$, $q=1,\ldots,k-1$,
$\tilde{S}^{(p)}(a)=\tilde{S}^{(p)}(b)$, $p=0,\ldots,2(k-1)$.

Let $D_i=S^{(2k-1)}(x_i+0)-S^{(2k-1)}(x_i-0)$ be the values of the gaps of spline $2k-1$-th derivative in the points x_i. We also put $D_1=S^{(2k-1)}(x_1)$, $D_N=-S^{(2k-1)}(x_N)$ for the splines from $\bar{G}$ and $D_1=D_N=S^{(2k-1)}(x_1+0)-S^{(2k-1)}(x_N-0)$ for the splines from $\tilde{G}$. The spline S delivers the minimum to the functional (1) if and only if the following conditions, called the characterization conditions, are fulfilled :

$$(-1)^k D_i\leqq 0, \text{ if } S(x_i)=z_i+\varepsilon_i,$$

$$D_i=0, \text{ if } z_i-\varepsilon_i<S(x_i)<z_i+\varepsilon_i,$$

$$(-1)^k D_i\geqq 0, \text{ if } S(x_i)=z_i-\varepsilon_i.$$

2. Let z_i be the values of some function $f\in W_1^{2k}[a, b]$ in the knots of Δ given with some error, such that $|z_i-f(x_i)|\leqq\varepsilon_i$. A question may arise concerning the approximative properties of splines in the convex set generated by z_i, ε_i. We give the error bounds for the approximation of the k-th derivative by the k-th derivative of such spline in the norm of space $L_2[a, b]$.

Let S be a spline of degree $2k-1$ in the convex set $\bar{G}$ and let $f(x)$ be such that $f^{(q)}(a)=z_a^q$, $f^{(q)}(b)=z_b^q$, $q=1,\ldots,k-1$. Due to the fact that $\|S^{(k)}\|_2^2\leqq\|f^{(k)}\|_2^2$, the square of the L_2 norm of $f^{(k)}-S^{(k)}$ can be estimated as follows :

$$\|f^{(k)}-S^{(k)}\|_2^2=\|S^{(k)}\|_2^2-\|f^{(k)}\|_2^2+2\int_a^b f^{(k)}(f^{(k)}-S^{(k)})dx \tag{2}$$

$$\leqq 2\int_a^b f^{(k)}(f^{(k)}-S^{(k)})dx.$$

Using the integration by parts with respect to the fact that $f^{(q)}(a)=S^{(q)}(a)$, $f^{(q)}(b)=S^{(q)}(b)$, $q=1,\ldots,k-1$, we obtain the equality

$$\int_a^b f^{(k)}(f^{(k)}-S^{(k)})dx=(-1)^k[-f^{(2k-1)}(f-S)|_a^b+\int_a^b f^{(2k)}(f-S)dx]. \tag{3}$$

Suppose that the estimate $\|f-S\|_\infty\leqq E$ is known and $\varepsilon^*=\max\{\varepsilon_1,\varepsilon_N\}$, $R=\max\{|f^{(2k-1)}(a)|, |f^{(2k-1)}(b)|\}$.

Using the Hölder inequality, we obtain from (2) and (3)

(4) $$\|f^{(k)}-S^{(k)}\|_2 \leqq \sqrt{2}(2\varepsilon^* R + E\|f^{(2k)}\|_1)^{1/2}.$$

Now let f be a periodic function. Consider the spline in the convex set $\tilde{G}$. Repeating the speculations of the case $\bar{G}$, we get the estimate analogous with (4)

(5) $$\|f^{(k)}-S^{(k)}\|_2 \leqq \sqrt{2}(E\|f^{(2k)}\|_1)^{1/2}.$$

If $f \in W_\infty^{2k}[a, b]$, one can write $\|f^{(2k)}\|_1 \leqq (b-a)\|f^{(2k)}\|_\infty$ and get the following estimates :

(6) $$\|f^{(k)}-S^{(k)}\|_2 \leqq \sqrt{2}\,[2\varepsilon^* R + E(b-a)\|f^{(2k)}\|_\infty]^{1/2},$$

if $f \in \bar{G}$, and

(7) $$\|f^{(k)}-S^{(k)}\|_2 \leqq \sqrt{2}\,[E(b-a)\|f^{(2k)}\|_\infty]^{1/2},$$

if $f \in \tilde{G}$.

The estimates for the value $\|f-S\|_\infty$ may be obtained, for example, as follows (further we consider the periodic case and uniform mesh). We have the inequality

(8) $$\|f-S\|_\infty \leqq \|f-\hat{S}\|_\infty + \|S\|_\infty,$$

where S is the spline of degree $2k-1$ in the convex set $\tilde{G}$ and $\hat{S}$ is the spline of the same degree interpolating function f. If $f \in W_\infty^{2k}[a, b]$ we have the following exact estimate for the first summand in (8) [4, p. 195]

(9) $$\|f-S\|_\infty \leqq (K_{2k}/\pi^{2k})h^{2k}\|f^{(2k)}\|_\infty,$$

where $K_m = (4/\pi)\Sigma_{j=0}^{\infty}(-1)^{j(m+1)}/(2j+1)^{m+1}$ are the Favard constants, $K_m \leqq \pi/2$, $K_0 = 1$, $K_2 = \pi^2/8$, $K_4 = 5\pi^4/384$.

Let us estimate the second summand in (8). By L^{2k-1} we denote the spline interpolating operator of degree $2k-1$ from the space of the continious periodic functions $C[a, b]$ onto the space of periodic splines $S^{2k-1}[a, b]$, which is the finite dimensional subspace of $C[a, b]$. We have the inequality

(10) $$\|\hat{S}-S\|_\infty \leqq 2\varepsilon\|L^{2k-1}\|,$$

where $\varepsilon = \max_i \varepsilon_i$. The value of the norm of the operator L^{2k-1} may be estimated by the formula [6]

(11) $$\|L^{2k-1}\| \leqq (2/N)\sum_{j=0}^{N-1} H_{2k-1}(1/2, \zeta j)/[(1+\zeta j)H_{2k-1}(1, \zeta j)],$$

where $\zeta = \exp(2\pi i/N)$, $H_r(t, z)$ are the modified Euler-Frobenius polinomials, defined for $r \geqq 0$ and $|z| < 1$ by the formula

$$H_r(t, z) = (1-z)^{r+1} \sum_{j=0}^{\infty} (t+j)^r z^j.$$

Note that for $N = 2n+1$, we have the equality in (11). In particular the following result for the cubic splines is given in [6]:

$$\| L^3 \| = \frac{1}{4}[1 + 3\sqrt{3} \frac{1-(2-3)^N}{1+(2-3)^N}].$$

Finally from (8-10) we get

$$\| f - S \|_\infty \leqq (K_{2k}/\pi^{2k}) h^{2k} \| f^{(2k)} \|_\infty + 2\varepsilon \| L^{2k-1} \| = E.$$

These are some examples for a small k:

1) $k=1$, we have the spline of the first degree

$$\| f - S \|_\infty \leqq 2\varepsilon + (h^2/8) \| f'' \|_\infty \,; \tag{12}$$

2) $k=2$, we obtain the cubic spline

$$\| f - S \|_\infty \leqq [(1+3\sqrt{3})/2]\varepsilon + (5/384) h^4 \| f^{(4)} \|_\infty . \tag{13}$$

Note 1. The estimate (12) is true for the splines from $\bar{G}$ as well from $\tilde{G}$ and also for irregular mesh with $h = \max_i h_i$.

Note 2. The estimate (13) is not true for the irregular mesh. Moreover, the value of the norm of the operator L^{2k-1}, $k \geqq 2$, essentially depends on the structure of the mesh Δ not only on its maximal step h. In particular, there exists the sequence of meshes Δ_N, such that h_N tends to zero, but L^{2k-1} tends to infinity if N tends to infinity [16] (for cubic splines see also [17]).

The following estimates:

$$\| f - S \|_\infty \leqq 2\varepsilon + h^2 \| f'' \|_\infty + (\sqrt{3}/8) h^2 \underline{h}^{-1/2} \| f'' \|_2$$

for $\tilde{G}$ and

$$\| f - S \|_\infty \leqq 2\varepsilon + h^2 \| f'' \|_\infty + (\sqrt{6}/8) h^2 \underline{h}^{-1/2} \| f'' \|_2$$

for $\bar{G}$ and irregular mesh with $\underline{h} = \min_i h_i$, are obtained in [15] and may be taken instead of (13) in case of cubic splines.

3. Finally here we obtain the error bounds for the $k-1$-th derivative approximation in the norm of space $L_\infty[a, b]$. For this purpose we use the

inequality for the norms of intermediate derivatives of functions from $W_r^k[a, b]$. With the technique of [3] it is easy to prove the inequality

$$\| \varphi^{(k-1)} \|_\infty \leqq A \gamma^{-k+1+1/p} \| \varphi \|_p + B \gamma^{1-1/r} \| \varphi^{(k)} \|_r \tag{14}$$

true for $1 \leqq k < +\infty$, $1 \leqq p$, $r \leqq +\infty$, where parameter γ is such that $0 < \gamma \leqq b-a$. The exact constant A has the form

$$A = (k-1)! / \| Q_{k-1} \|_{L_p[0,1]},$$

here $Q_{n-1}(x)$ is the polinomial of the minimal norm in the space $L_p[0, 1]$. The constant B is such that $0 < B \leqq 1$. Minimizing (14) by γ, we obtain the estimate that has two cases :

1) the inequality (14) takes place with $\gamma = b-a$, if

$$\| \varphi^{(k)} \|_r / \| \varphi \|_p \leqq Ar\mu (b-a)^{-\nu/pr} / [Bp(r-1)],$$

where $\mu = kp - p - 1$, $\nu = krp - p - r$;

2) in the opposite case we get the inequality

$$\| \varphi^{(k-1)} \|_\infty \leqq \{ [r\mu/p(r-1)]^{-r\mu/\nu} + [r\mu/p(r-1)]^{p(r-1)/\nu} \}$$
$$\times (A \| \varphi \|_p)^{p(r-1)/\nu} (B \| \varphi^{(k)} \|_r)^{r\mu/\nu}. \tag{15}$$

We are interested in the case, when $p = \infty$, $r = 2$. Suppose, that $\varphi = f - S$ and $\| f - S \|_\infty \leqq E$, $\| f^{(k)} - S^{(k)} \|_2 \leqq E_2$, where E_2 is given by right hand parts of (4-7), then we get the estimate from (14)

$$\| f^{(k-1)} - S^{(k-1)} \|_\infty \leqq A \gamma^{-k+1} E + B \gamma^{1/2} E_2. \tag{16}$$

Minimizing the right hand part of (16) by γ and computing the constant A, we obtain

$$\| f^{(k-1)} - S^{(k-1)} \|_\infty \leqq 2^{2k-3} (k-1)! \, E/(b-a)^{k-1} + B(b-a)^{1/2} E_2, \tag{17}$$

if $E_2/E \leqq 2^{2k-2} (k-1)! (k-1) (b-a)^{-(2k-1)/2} / B$, in the opposite case we have

$$\| f^{(k-1)} - S^{(k-1)} \|_\infty \leqq (2k-1) \{ (k-1)! \, E [BE_2/(k-1)]^{2(2k-1)/2} \}^{1/(2k-1)}. \tag{18}$$

The inequality (17, 18) is true for any $k \geqq 1$ for both types of convex sets : $\tilde{G}$ and $\bar{G}$.

We shall consider more carefully the case $k = 2$, corresponding to the cubic splines. In [14] the way is shown how the inequalities between the norms of derivatives are obtained. One begins with an equality

$$\varphi'(x) = -\int_a^b \varphi(\xi)\tilde{\omega}(\xi)d\xi + \int_a^b \varphi''(\xi)\tilde{\Lambda}(x, \xi)d\xi, \tag{19}$$

where $\tilde{\omega}(\xi)$ is a function on $[a, b]$ with absolutely continuous derivative $\tilde{\omega}'(\xi)$ and such that

$$\tilde{\omega}(a) = \tilde{\omega}(b) = 0, \int_a^b \tilde{\omega}(\xi)d\xi = 1,$$

and $\tilde{\Lambda}(x, \xi)$ is defined by the formula

$$\tilde{\Lambda}(x, \xi) = \begin{cases} \int_a^\xi \tilde{\omega}(\eta)\, d\eta, & a \leqq \xi \leqq x, \\ -\int_\xi^b \tilde{\omega}(\eta)\, d\eta, & x \leqq \xi \leqq b. \end{cases}$$

Using Hölder inequality and Minkowski's generalized integral inequality we obtain from (19)

$$\| \varphi' \|_\infty \leqq A(\tilde{\omega}) \| \varphi \|_p + B(\tilde{\omega}) \| \varphi'' \|_r,$$

where $A(\tilde{\omega}) = \| \tilde{\omega}' \|_{L_{p'}[a,b]}$, $p' = p/(p-1)$, $r' = r/(r-1)$, $B(\tilde{\omega}) = \{\| \ \|\tilde{\Lambda}(x, \xi)\|_{r',\xi} \|_{\infty,x}$, $\| \ \|\tilde{\Lambda}(x, \xi)\|_{\infty,x}\|_{r',\xi}\}$. We have $A(\tilde{\omega}) = (b-a)^{-1/p} \| \omega \|_{L_{p'}[0,1]}$, where $\omega(\xi) = \tilde{\omega}[a + (b-a)\xi]$ and is defined on [0,1]. It is proved in [3] that $A(\tilde{\omega})$ will be minimal if $\| \omega \|_{L_{p'}[0,1]} = 1/\| Q_1 \|_{L_p[a,b]}$, where $Q_1(x)$ is the polinomial of minimal norm in the space $L_p[0, 1]$, $Q_1(x) = x - 1/2$. We compute

$$\omega(\xi) = \begin{cases} (p+1)[1-(1-2\xi)^p]/p, & 0 \leqq \xi \leqq 1/2, \\ (p+1)[1-(2\xi-1)^p]/p, & 1/2 \leqq \xi \leqq 1. \end{cases}$$

and then the function $\tilde{\Lambda}(x, \xi)$. After that we determine the limit of $B(\tilde{\omega}, p)$ when p tends to infinity and obtain inequality $\| \varphi' \|_\infty \leqq 2 \| \varphi \|_\infty + \gamma^{1/2} (\sqrt{3}/3) \| \varphi'' \|_2$.

Substituting the value $B = \sqrt{3}/3$ into (15), we get an exact inequality [1]. Hence B is an exact constant.

The main characteristic feature of the obtained estimates (4)-(7) and (17, 18) is the following : their right hand parts tend to zero, no matter how h and ε are tending to zero. This shows the stability of the approximation of the derivatives of any order with the help of splines in the convex sets.

Let $k = 2$, then we can see from (18) that the order of ε is equal to 2/3, that coincides with the order of ε in the optimal estimate in [10, p. 198]. Note, however, that there no table functions were considered but functions defined on $(-\infty, +\infty)$ such that $f \in KW^3$, i.e. belonging to the class of functions with the second derivative satisfying to Lipschitz condition with constant K. In our case we suppose that $f \in W_1^4[a, b]$, or $f \in W_\infty^4[a, b]$.

It is worthwhile to underline that the described approach to the approximation of derivatives is a constructive one. To obtain the spline in the convex set, we need only the approximate values of function $f(x)-z_i$ and values of error ε_i. This distinguishes this method among the ones connected, for example, with the choice of optimal steps of differentiation that needs additional information about the function (norm of elter derivatives, etc.).

The numerical solution of the problem, leading to the problem of quadratic programming, was discussed in [8, 9], [11]. There the method of penalty functions was used.

Table 1

ε	0.5	0.05	0.005
$\|f'-S'\|_\infty$	1.253	0.174	0.025
E_1	3.427	0.753	0.184

Table 1 represents the results of numerical example. We consider the approximate data of the periodic function $f(x)=\sin(x)+\cos(x)$, $x\in[0, 2\pi]$, given in the knots of the uniform mesh with the step $h=\pi/60$ and with the different values of error ε. This table also contains the values of the norm $\|f'-S'\|_\infty$ and right hand parts of the estimate (17), (18) is E_1, in case $k=2$. One can see that the derivative of cubic spline approximates the derivative of function f well enough for different ε and that the estimate (17), (18) are effective for $\varepsilon/h \gg 1$ and for ε comparable with h. Its efficiency decreases if $\varepsilon/h \ll 1$. In this case the interpolating spline estimates are appropriate.

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