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Onesided Trigonometrical Approximation of Periodic Multivariate Functions

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Many papers have been consecrated to the problem of one-sided approximation of functions of one variable, see [1]-[7] and the references there. In this paper we shall consider the one-sided trigonometrical approximation of 2π-periodic multivariate functions. We shall prove Jackson's type and Bernstein type theorems for the best L_p-one-sided trigonometrical approximation using the averaged moduli of smoothness in L_p. These theorems are externally similar to the corresponding theorems in the one-dimensional case (see [3], [6], [7]), but really the dimension plays an essential role in onesided approximations (see theorem 1 and the remarks to the theorem).

The results were announced in [9] and [10] without proofs.

1. Notations and definitions

We shall denote by M the set of all 2π-periodic with respect to the each variable functions of m variables (i. e. $f(x_1, \dots, x_i+2\pi, \dots, x_m)=f(x_1, \dots, x_i, \dots, x_m)$), which are bounded and integrable on

$$\Pi^m=\{x=(x_1, \dots, x_m) : 0 \leqq x_i \leqq 2\pi, \quad i=1, \dots, m\}.$$

We shall denote the points (the vectors) in m-dimensional space R^m by x, y, t, h, $x=(x_1, \dots, x_m)$. For the multi-index $\alpha=(\alpha_1, \dots, \alpha_m)$ we use the usual notations :

$$D^\alpha f=\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_m^{\alpha_m}}, \quad |\alpha|=\sum_{i=1}^m \alpha_i.$$

We set also $|x|=\max\{|x_i| : i=1, \dots, m\}$.

For $f \in M$ and $x \in \mathrm{R}^m$ we define the local modulus of order k of f in the point x as follows :

$$(1) \qquad \omega_k(f, x, \delta)=\sup\{|\Delta_h^k f(t)| : t, t+kh \in \Omega_{k\delta/2}(x)\},$$

where

$$\Delta_h^k f(t) = \sum_{j=0}^{k} (-1)^{k+j} \binom{k}{j} f(t+jh)$$

is the k-difference of the function f with step h (in direction h) and

$$\Omega_{k\delta/2}(x) = \{y : |x-y| \leqq k\delta/2\}$$

is $k\delta/2$-neighbourhood of the point x with respect to $|\cdot|$.

The k-th averaged modulus of smoothness of $f \in M$ in L_p, $1 \leqq p \leqq \infty$, is given by

$$\tau_k(f, \delta)_p = \| \omega_k(f, \cdot, \delta) \|_{L_p}, \tag{2}$$

where

$$\| g \|_{L_p} : = \| g \|_{L_p(\pi^m)} = \{ \int_{\pi^m} |g(x)|^p dx \}^{1/p}, \quad 1 \leqq p < \infty$$

$$\| g \|_\infty : = \sup \{|g(x)| : x \in \Pi^m\}.$$

Both definitions (1) and (2) in the one-dimensional case coincide with definitions of k-th averaged moduli of smoothness, given in [6], [7]. For the history of the averaged moduli of smoothness see [3], [7], [8].

The averaged moduli of smoothness have the following properties :

1. $\tau_k(f, \delta')_p \leqq \tau_k(f, \delta'')_p$ if $\delta' \leqq \delta''$,
2. $\tau_k(f+g, \delta)_p \leqq \tau_k(f, \delta)_p + \tau_k(g, \delta)_p$,
3. $\tau_k(f, \delta)_p \leqq 2\tau_{k-1}(f, k\delta/(k-1))_p$.
4. If $D^\alpha f \in M$ for $\alpha : |\alpha| = 0,1$, then

$$\tau_k(f, \delta)_p \leqq \sqrt{m}\, \delta \sum_{|\alpha|=1} \tau_{k-1}(D^\alpha f, k\delta/(k-1))_p, \quad k > 1,$$

5. $\tau_k(f, \lambda\delta)_p \leqq (2(\lambda+1))^{k+m} \tau_k(f, \delta)_p$, $\lambda > 0$.
6. If $f \in M$, $D^\alpha f \in L_p$ for all $\alpha : \alpha_i = 0,1$, $i = 1, \dots, m$, then

$$\tau_1(f, \delta)_p \leqq 2 \sum_{\substack{|\alpha| \geqq 1 \\ \alpha_i = 0,1}} \delta^\alpha \| D^\alpha f \|_{L_p}.$$

7. Let $f \in L_p$ and $D^\alpha f \in L_p$ for all α such that $|\alpha| \geqq k$, $0 \leqslant \alpha_i \leqslant k$, $1 \leqq p \leqq \infty$.

$$\tau_k(f, \delta)_p \leqq c(k, m, p)\, \delta^k \sum_{\substack{|\alpha|=k \\ 0 \leqslant \alpha_i = k}} \| D^\alpha f \|_{L_p},$$

where the constant $c(k, m, p)$ depends only on k, m and p.

Properties 1)-3) follow immediately from the definition of τ_k. The proofs of properties 4)-6) are given in [14]. Property 7) can be found in [14]. Let us mention that the constant $(2(\lambda+1))^{k+m}$ in property 5) probably is not exact, but let us mention the dependence of the dimension m. Also we want to remark that in property 6) in many dimensional cases in the right-hand side mixed derivatives like $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_e}$ and s. l. are necessary.

In the case $p=\infty$ the k-th averaged modulus of smoothness coincides with the usual (uniform) k-th modulus of continuity (see [11], p. 117). The connection between the integral moduli given by

$\omega_k(f,\delta)_p=\sup\{\|\Delta_h^k f(\cdot)\|_{L_p}:|h|\leqq\delta\}$ (see [8]), is the following:

$$\omega_k(f,\delta)_p\leqq\tau_k(f,\delta)_p\leqq\omega_k(f,\delta)_\infty.$$

We shall approximate the functions in M by trigonometrical polynomials of order n. The set of all trigonometrical polynomials of order n with respect to each of m-variables we define by

$$\pi_n=\{p:p(x)=\sum_{\substack{0\leqq n_i'+n_i''\leqq n\\ i=1,\dots,m}} a_{n_1',n_1'',\dots,n_m',n_m''}\cos n_1' x_1 \sin n_1'' x_1 \dots \cos n_m' x_m \sin n_m'' x_m\}.$$

The best trigonometrical approximation of order n of the function $f\in L_p(\pi^m)$ in L_p is given by

$$E_n(f)_p=\inf\{\|f-P\|_p:P\in\pi_n\}.$$

The best upper (lower) trigonometrical approximation of order n of the function $f\in M$ in L_p is respectively given by

$$E_n^+(f)_p=\inf\{\|P-f\|_p:P\in\pi_n,\ P(x)\geqq f(x),\ x\in\mathrm{R}^m\}$$

$$E_n^-(f)_p=\inf\{\|f-Q\|_p:Q\in\pi_n,\ f(x)\geqq Q(x),\ x\in\mathrm{R}^m\}.$$

The best one-sided trigonometrical approximation of order n of the function $f\in L_p$ in L_p is given by

$$\tilde{E}_n(f)_p=\inf\{\|PN-Q\|_p:P,Q\in\pi_n,\ P(x)\geqq f(x)\geqq Q(x),\ x\in\mathrm{R}^m\}.$$

Here we enlist some of the basic properties of the just defined functionals, which we need below. When some of the following propositions is valid for each of $E_n^+(\cdot)_p$, $E_n^-(\cdot)_p$ and $\tilde{E}_n(\cdot)_p$ we write $E_n^*(\cdot)_p$ meaning any of them.

i) $E_n(f)_p\leqq E_n^*(f)_p$, $E_n^*(f)_\infty\leqq 2E_n(f)_\infty$.

ii) $\tilde{E}_n(f)_p=E_n^+(f)_p+E_n^-(f)_p$.

iii) If $P\in\Pi_n$, then $E_n(f-P)_p=E_n(f)_p$, $E_n^*(f-P)_p=E_n^*(f)_p$.

iv) The functionals $E_n^*(\cdot)_p$ and $E_n(\cdot)_p$ are semi-additive.

v) If $\lambda>0$, then $E(\pm\lambda f)_p=\lambda E_n(f)_p$, $E_n^*(\lambda f)_p=\lambda E_n^*(f)_p$

$$\tilde{E}_n(-\lambda f)_p=\lambda E_n(f)_p,\ E_n^+(-\lambda f)_p=\lambda E_n^-(f)_p.$$

vi) If $g\in M$ and $g\leqq 0$, then

$$E_n(f+g)_p\leqq \|g\|_p+E_n^+(f)_p.$$

vii) There exist unique P and Q in π_n for which

$$E_n^+(f)_p=\|P-f\|_p,\ E_n^-(f)_p=\|f-Q\|_p,\ \tilde{E}_n(f)_p=\|P-Q\|_p$$

and $P(x)\geqq f(x)\geqq Q(x)$ for any $x\in \mathrm{R}^m$.

Propositions i) — vi) follow directly from the definitions. The proposition vii) is well known in more general situation, see [14].

2. Integral representation of 2π-periodic functions with mixed derivatives

In what follows, we shall work with multi-indexes which contain only zeros or unities. The set of all such multyindexes will be denoted by Γ_0:

$$\Gamma_0=\{\alpha:\alpha=(\alpha_1,\ldots,\alpha_m),\quad \alpha_i=0,\ 1,\ i=1,\ldots,m\}.$$

We shall use also the set : $\Gamma=\{\alpha:\alpha\in\Gamma_0,\ \alpha\neq(0,\ldots,0)\}$.

We say that β precedes α if $\beta_i\leqq\alpha_i$, $i=1,\ldots,m$ (notation $\beta\leqq\alpha$), and that β precedes α strictly if $\beta\leqq\alpha$ and $\beta\neq\alpha$.

If $x\in\mathrm{R}^m$, $\alpha\in\Gamma$, then x^α denotes the point $(x_{i_1},\ldots,x_{i_{|\alpha|}})$ where $i_1,\ldots,i_{|\alpha|}$ are the coordinates i for which $\alpha_i=1(\alpha_{i_1}=\ldots=\alpha_{i_{|\alpha|}}=1)$. When x varies in Π^m, then x^α varies in Π^α.

Let f and g be 2π-periodic integrable functions and $\alpha\in\Gamma$. Then α-convolution of f and g is $g *_\alpha f(x)=(2\pi)^{-|\alpha|}\int_{\Pi^\alpha} g(x^{\bar\alpha},\ x^\alpha-u^\alpha)f(x^{\bar\alpha},\ u^\alpha)\,du^\alpha$ where $\bar\alpha=(1,\ldots,1)-\alpha$, $f(x^{\bar\alpha},\ x^\alpha)=f(x_1,\ldots,x_m)$.

When $\alpha=(1,\ldots,1)$ this is the usual convolution.

We shall use the following property of α-convolutions : if $g=g(x^\alpha)$, $g\in L_1(\Pi^\alpha)$, $f\in L_p(\Pi^m)$, then

$$\|g *_\alpha f\|_p\leqq(2\pi)^{-|\alpha|}\quad \|g\|_{L_1(\pi^\alpha)}\|f\|_{L_p}.$$

The first Bernoulli functions in the multivariate case are defined by

$$B^\alpha(x)=\prod_{i=1}^{m} B_{\alpha_i}(x_i),\quad \text{where } \alpha\in\Gamma_0 \text{ and } B_0(s)\equiv 1,$$

$$B_1(s)=2\sum_{\nu=1}^{\infty}\frac{\sin \nu s}{s}=\begin{cases}\pi-s, & 0<s<2\pi\\ 0, & s=0.\end{cases}$$

Lemma 1. *If $B^\alpha f\in L_1(\Pi^m)$ for all $\alpha\in\Gamma_0$, then*

$$f(x)=a_0+\sum_{\alpha\in\Gamma} B^\alpha *_\alpha D^\alpha f(x),$$

where

$$a_0=(2\pi)^{-m}\int_{\pi^m} f(x)\,dx.$$

Lemma 2. *Let $\alpha\in\Gamma$ and $D^\alpha f\in L_p(\pi^m)$. If $\alpha<\beta\in\Gamma$, then*

$$\| B^\alpha *_\alpha D^\alpha f\|_p\leqq 2^m(2\pi)^{m-1}\|f\|_p.$$

The proofs of these two lemmas are routine and can be made by induction with respect to the dimension in the first one, and with respect to the number of nonzero indexes in the second, using Lemma 1.

Lemma 3. (T. Ganelius [2]). *Let $m=1$ and let n be a natural number. Then $\tilde{E}_n(B_1)_1=4\pi^2/(n+1)$, i.e. there exist two trigonometrical polynomials T_n and t_n of order n such that:*

(i) $T_n(s)\geqq B_1(s)\geqq t_n(s),\quad s\in\mathbb{R}^1$,
(ii) $\| T_n-t_n\|_{L_1[0,2\pi]}=4\pi^2/(n+1)$.
For the multi-index $\alpha\in\Gamma_0$ we set

$$T^\alpha(x)=\prod_{i=1}^{m} T_n^{\alpha_i}(x_i),\quad t^\alpha(x)=\prod_{i=1}^{m} t_n^{\alpha_i}(x_i),$$

where $T_n^0(s)=t_n^0(s)\equiv 1,\ T_n^1(s)=T_n(s),\ t_n^1(s)=t_n(s)$;

$$(B-t)^\alpha(x)=\prod_{i:\alpha_i=1}(B_1(x_i)-t_n(x_i)),$$

$$(T-B)^\alpha(x)=\prod_{i:\alpha_1=1}(T_n(x_i)-B_1(x_i)),$$

$$(T-t)^\alpha(x)=\prod_{i:\alpha_i=1}(T_n(x_i)-t_n(x_i)).$$

Lemma 4. *For $\alpha\in\Gamma$ we have*

$$B^\alpha-t^\alpha=\sum_{\beta<\alpha}(-1)^{|\alpha-\beta|+1}(B-t)^{\alpha-\beta}B^\beta.$$

The proof of this combinatoric Lemma can be done by induction with respect to the number of nonzero indexes of α.

3. One-sided approximation by means of differentiable functions

The following Lemma is a well-known tool for intermediate approximation in the classical approximation theory (see for example Ju. A. Brudnii [15]):

Lemma 5. *Let $f\in L_p(\Pi^m)$ and k and n be natural numbers. There exists $f_{k,n}\in L_p(\Pi^m)$ with the properties:*

a) $|f_{k,n}(x)-f(x)|\leqq \omega_k(f, x, 2/n)$, $x\in \mathrm{R}^m$,

b) $\|f_{k,n}-f\|_p\leqq c(k,m)\omega_k(f, 1/n)_p$,

c) for every multi-index α such that $\alpha_i\leqq k$, $i=1,\dots,m$ we have

$$\| D^\alpha f_{k,n}\|_p\leqq c(k,m)\, n^{|\alpha|}\,\omega_{|\alpha|}(f, 1/n)_p.$$

We shall use an analogue of this Lemma with the restriction that the function $f_{k,n}$ is over the function f. As a consequence at the place of $\omega_r(f, 1/n)_p$ will appear $\tau_r(f, 1/n)_p$.

Lemma 6. *Let $f\in M$ and $1\leqq p\leqq\infty$. For any natural numbers k and n there is a function $F_{k,n}$ with the properties:*

a) $0\leqq F_{k,n}(x)-f(x)\leqq 2^{km-k+1}\,\omega_k(f, x, (2km+8\pi)/kn)$,

b) $\| F_{k,n}-f\|_p\leqq c(k,m)\,\tau_k(f, 1/n)_p$.

c) For every multi-index $\alpha:\alpha_i\leqq k$, $i=1,\dots,m$, we have

$$\| D^\alpha F_{k,n}\|_p\leqq c(k,m)\, n^{|\alpha|}\,\tau_{|\alpha|}(f, 1/n)_p.$$

Proof. Let $A_n=\{a:a=(\frac{2\pi}{n}a_1,\dots,\frac{2\pi}{n}a_m):a_i-\text{integers},\ i=1,\dots,m\}$. Let ψ_n be an infinitely many times differentiable function with the properties:

1. $0\leqq\psi_n(x)\leqq 1$ for every $x\in\mathrm{R}^m$,
2. $\psi_n(x)=0$ for $|x|\geqq 2\pi/n$,
3. $\sum_{a\in A_n}\psi_n(x-a)=1$.
4. For every multi-index α we have

$$| D^\alpha\psi_n(x)|\leqq c(|\alpha|)\, n^{|\alpha|}.$$

Such functions exist, for example we can take

$$\psi_n(x)=\Phi_n(x)/\sum_{a\in A_n}\Phi_n(x-a),$$

where

$$\Phi_n(x)=\prod_{i=1}^{m}\varphi_n(x_i),\quad \varphi_n(t)=\begin{cases}\exp(-1/(1-4\pi^2t^2/n^2), & |t|<2\pi/n,\\ 0, & |t|\geqq 2\pi/n.\end{cases}$$

Let $v=km$ and let us consider the function

$$F_{k,n}(x)=f_{v,n}(x)+\sum_{a\in A_n}\omega_v(f,a,(2v+4\pi)/vn)\,\psi_n(x-a),$$

where $f_{v,n}$ is the corresponding function from Lemma 5.

The function $F_{k,n}$ is well-defined for every x, because for every x in the sum on the right-hand side only a finite number of terms are different from zero (see property 2 of the function ψ_n). Let us show that the function $F_{k,n}$ satisfies the conditions of the Lemma.

Let $x\in\mathrm{R}^m$. Denoting $A_n(x)=\{a: a\in A_n,\ |x-a|\leqq\frac{2\pi}{n}\}$ we have

$$F_{k,n}(x)-f(x)=f_{v,n}(x)-f(x)+\sum_{a\in A_n(x)}\omega_v(f,a,(2v+4\pi)/vn)\psi_n(x-a)\geqq$$

(since $\psi_n(x-a)\neq 0$ only for $a\in A_n(x)$)

$$\geqq -\omega_v(f,x,2/\pi)+\min_{a\in A_n(x)}\omega_v(f,a,(2v+4\pi)/vn)$$

$$\geqq -\omega_v(f,x,2/n)+\omega_v(f,x,2/n)=0$$

since for every $a\in A_n(x)$, then $\Omega_{v/n}(x)\subset\Omega_{(v+2\pi)/n}(a)$.

On the other hand,

$$F_{k,n}(x)-f(x)\leqq\omega_v(f,x,2/n)+\max_{a\in A_n(x)}\omega_v(f,a,(2v+4\pi)/vn)$$

$$\leqq\omega_v(f,x,2/n)+\omega_v(f,x,(2v+8\pi)/vn)\leqq 2^{km-k+1}\,\omega_k(f,x,(2mk+8\pi)/kn).$$

Then b) follows from a) taking the L_p-norm of both sides.

At the end, using property 4) of ψ_n we have

$$|D^\alpha F_{k,n}(x)|\leqq|D^\alpha f_{v,n}(x)|+\sum_{a\in A_n(x)}\omega_v(f,a,(2v+4\pi)/vn)\,|D^\alpha\psi_n(x-a)|$$

$$\leqq|D^\alpha f_{v,n}(x)|+c(|\alpha|)_n{}^{|\alpha|}\,\omega_v(f,x,(2v+8\pi)/vn)\sum_{a\in A_n(x)}1.$$

Taking L_p-norm from the both sides, using the property c) of $f_{v,n}$ and the fact that we have only a finite number of terms on the right hand side, we obtain c), with constant $c=c(k,m)$.

4. Direct theorem

We shall prove the following Jackson's type theorem for the best multivariate one-sided approximations :

Theorem 1. *Let $f\in M$ and $1\leqq p\leqq\infty$. For every natural number k there exists a constant $c(k,m)$ depending only on k and the dimension m, such that*

$$\tilde{E}_n(f)_p\leqq c(k,m)\,\tau_k(f,1/n)_p.$$

First we shall prove the case $k=1$.

Lemma 7. *Let $f\in M$, $1\leqq p\leqq\infty$. Then*

$$E_n^+(f)_p\leqq c(m)\,\tau_1(f,1/n)_p.$$

Proof. We shall prove the Lemma using induction with respect to the dimension m. For $m=1$ the Lemma follows directly from theorem in [3], see also [6].

Let us suppose, that if $f\in M(\Pi^{m_1})^-$, $m_1<m$, then $E_n^+(f)_p\leqq c(m_1)\,\tau_1(f,1/n)_{p,m_1}$.

Let now $f\in M(\Pi^m)$ and let us denote by F the function $F_{1,n}$ from Lemma 6. Then, using the properties of $E_n^+(\cdot)_p$ and of $F_{1,n}$ we get :

$$E_n^+(f)_p\leqq \|f-F\|_p+E_n^+(F)_p\leqq c(m)\,\tau_1(f,1/n)_p+E_n^+(F)_p. \tag{3}$$

So we must estimate $E_n^+(F)_p$.

Using Lemma 1, we obtain

$$E_n^+(F)_p\leqq\sum_{\alpha\in\Gamma}E_n^+(B^\alpha * D^\alpha F)_p=\sum_{\alpha\in\Gamma}E_n^+((B^\alpha-t^\alpha) * D^\alpha F)_p$$

$$\leqq\sum_{\alpha\in\Gamma}\sum_{\beta<\alpha}E_n^+((B-t)^{\alpha-\beta}B^\beta * (-1)^{|\alpha-\beta|+1}D^\alpha F)_p.$$

We shall estimate all terms in the sum on the right-hand side in the same way. Let $\alpha\in\Gamma$ and $\beta<\alpha$. If we denote $g(x)=B^\beta *_\beta(-1)^{|\alpha-\beta|+1}D^\alpha F(x)$, then

$$E_n^+((B-t)^{\alpha-\beta}B^\beta * (-1)^{|\alpha-\beta|+1}D^\alpha F)_p=E_n^+((B-t)^{\alpha-\beta} *_\beta g)_p.$$

We shall prove that

$$E_n^+((B-t)^{\alpha-\beta} *_\beta g)_p\leqq c(m)\,n^{-|\alpha-\beta|}\sum_{\gamma_1<\bar\gamma}n^{-|\gamma_1|}\,\|D^{\gamma_1}g\|_p, \tag{4}$$

where $\bar\gamma=(1,\dots,1)-\alpha+\beta$.

We set as usual $g_+(x)=\max(0, g(x))$, $g_-(x)=\min(0, g(x))$.

Using the fact, that for every φ, $\varphi \leqq 0$, we have $E_n^+(\varphi)_p \leqq \|\varphi\|_p$, we obtain :

$$E_n^+((B-t)^{\alpha-\beta} *_\beta g)_p \leqq E_n^+((B-t)^{\alpha-\beta} *_\beta g_-)_p$$

$$+E_n^+((B-t)^{\alpha-\beta} *_\beta g_+)_p \leqq \|(B-t)^{\alpha-\beta} *_\beta g_-\|_p + E_n^+((B-t)^{\alpha-\beta} *_\beta g_+)_p$$

$$\leqq (2\pi)^{-|\beta|}\left(\frac{4\pi^2}{n+1}\right)^{|\alpha-\beta|} \|g_-\|_p + E_n^+((B-t)^{\alpha-\beta} *_\beta g_+)_p$$

$$\leqq c(m)\, n^{-|\alpha-\beta|} \|g\|_p + E_n^+((B-t)^{\alpha-\beta} *_\beta g_+)_p.$$

Let us denote $\gamma=\alpha-\beta$, $\bar{\gamma}=(1, \ldots, 1)-\alpha+\beta$. Since $\alpha<\beta \Rightarrow \gamma\in\Gamma$, $\bar{\gamma}\in\Gamma$ and $\bar{\gamma}\neq(1, \ldots, 1)$, therefore $|\bar{\gamma}|<m$.

Let us consider the function $H_n(x)=H_n(x^\gamma, x^{\bar{\gamma}})$, which for every fixed x^γ is the trigonometrical polynomial of order n of best upper approximation of the function $g_+(x^\gamma, x^{\bar{\gamma}})$, considered as a function of the variables $x^{\bar{\gamma}}$. $H_n(x^\gamma, x^{\bar{\gamma}})$ is a trigonometrical polynomial of order n of the variables $x^{\bar{\gamma}}$ with coefficients, which depend on x^γ. It is not difficult to see by induction that these coefficients are measurable functions of x^γ. In the case when $\bar{\gamma}=(0, \ldots, 0)$ we have $H_n=g_+$. For the so defined function H_n we have $0\leqq g_+(x) \leqq H_n(x)$ for every x.

Using the induction assumption $|\bar{\gamma}|<m$ and the properties of τ_1, we get

$$\|H_n-g_+\|_p = \left(\int_{\pi^\gamma}\int_{\pi^{\bar{\gamma}}} |(H_n-g_+)(x^\gamma, x^{\bar{\gamma}})|^p dx^{\bar{\gamma}}\, dx^\gamma\right)^{1/p}$$

$$\leqq \left(\int_{\pi^\gamma} c^p(|\bar{\gamma}|)\, \tau_1(g_+(x^\gamma, \cdot), 1/n)^p_{L_p(\pi^{\bar{\gamma}})} dx^\gamma\right)^{1/p}$$

$$\leqq c(|\bar{\gamma}|)\left(\int_{\pi^\gamma} \tau_1(g(x^\gamma, \cdot), 1/n)^p_{L_p(\pi^{\bar{\gamma}})} dx^\gamma\right)^{1/p}$$

$$\leqq 2c(|\bar{\gamma}|)\left(\int_{\pi^\gamma}\Big(\sum_{\substack{\gamma_1\leqq\bar{\gamma}\\ |\gamma_1|>0}} n^{-|\gamma_1|} \|D^{\gamma_1} g(x^\gamma, \cdot)\|_{L_p(\pi^{\bar{\gamma}})}\Big)^p dx^\gamma\right)^{1/p}$$

$$\leqq 2c(|\bar{\gamma}|)(2^{|\bar{\gamma}|}-1)\left(\int_{\pi^\gamma} \sum_{\substack{\gamma_1\leqq\bar{\gamma}\\ |\gamma_1|>0}} n^{-|\gamma_1|p} \|D^{\gamma_1} g(x^\gamma, \cdot)\|^p_{L_p(\pi^{\bar{\gamma}})} dx^\gamma\right)^{1/p}$$

$$\leqq c'(|\bar{\gamma}|) \sum_{\substack{\gamma_1\leqq\bar{\gamma}\\ |\gamma_1|>0}} n^{-|\gamma_1|} \|D^{\gamma_1} g\|_p.$$

Here we used that obviously $\tau_1(g_+,\delta)_p \leqq \tau_1(g,\delta)_p$.

Let us consider the trigonometrical polinomial $(T-t)^{\alpha-\beta} *_\beta H_n \in \Pi_n$. We have

$$(B-t)^{\alpha-\beta} *_\beta g_+ \leqq (T-t)^{\alpha-\beta} *_\beta H_n$$

since $(T-t)^{\alpha-\beta} \geqq (B-t)^{\alpha-\beta} \geqq 0$ and $H_n \geqq g_+ \geqq 0$.

Therefore

$$E_n^+((B-t)^{\alpha-\beta} *_\beta g_+)_p \leqq \|(T-t)^{\alpha-\beta} *_\beta H_n - (B-t)^{\alpha-\beta} *_\beta g_+\|_p$$

$$\leqq \|(T-t)^{\alpha-\beta} *_\beta (H_n - g_+)\|_p + \|(T-t)^{\alpha-\beta} *_\beta g_+\|_p + \|(B-t)^{\alpha-\beta} *_\beta g_+\|_p$$

$$\leqq c(m) n^{-|\alpha-\beta|} \sum_{\substack{\gamma_1<\gamma \\ |\gamma_1|>0}} n^{-|\gamma_1|} \|D^{\gamma_1} g\|_p + c(m) n^{-|\alpha-\beta|} \|g\|_p$$

which gives us (4).

It remains to obtain estimates for the derivatives $D^{\gamma_1} g$ of the function g. For every $\gamma_1 < (1,\dots,1) - \alpha + \beta$, using L2, L6, we have

$$\|D^{\gamma_1}(B^\beta *_\beta D^\alpha F)\|_p \leqq c(m) \|D^{\alpha-\beta+\gamma_1} F\|_p \leqq c(m) n^{|\alpha-\beta+\gamma_1|} \tau_1(f, 1/n)_p.$$

Since $\alpha<\beta$, $\gamma_1 < (1,\dots,1) - \alpha + \beta$, then $\alpha - \beta + \gamma_1 \in \Gamma$. From (3) and (4) the Lemma follows.

To prove Theorem 1 we need one more Lemma.

Lemma 8. *Let r be a natural number, $r \geqq m$, and $D^\alpha f \in L_p$ for $|\alpha| \leqq r$, $1 \leqq p \leqq \infty$. Then there exists a polynomial $S \in \pi_n$ such that*

$$\|D^\beta(f-S)\|_p \leqq c(m,r) n^{|\beta|-r} \sum_{|\alpha|=r} \|D^\alpha f\|_p$$

for every multi-index β such that $|\beta| \leqq r$.

The statement follows immediately from the construction given in [11], p. 231, if we take σ in the formula (6) on p. 231 sufficiently large (in dependence of r and m).

Now we are ready to prove Theorem 1.

P r o o f o f T h e o r e m 1.

Let first $k \geqq 1$. Using the function $F_{k,n}$ from Lemma 6, we get

$$E_n^+(f - F_{k,n} + F_{k,n})_p \leqq \|f - F_{k,n}\|_p + E_n^+(F_{k,n})_p \leqq c(k,m) \tau_k(f, 1/n)_p + E_n^+(F_{k,n})_p.$$

To estimate $E_n^+(F_{k,n})_p$ we use the polynomial from Lemma 8, Lemma 7, the estimate of the first averaged modulus by means of mixed derivatives, Lemma 8 and Lemma 6. We get

$$E_n^+(F_{k,n})_p = E_n^+(F_{k,n}-S)_p \leqq c(m)\,\tau_1(F_{k,n}-S, 1/n)_p$$

$$\leqq c(m) \sum_{\beta\in\Gamma} n^{-|\beta|} \| D^\beta (F_{k,n}-S)\|_p$$

$$\leqq c(k,m) \sum_{\beta\in\Gamma} n^{-|\beta|} n^{|\beta|-k} \sum_{|\alpha|=k} \| D^\alpha F_{k,n}\|_p \leqq c(k,m)\,\tau_k(f, 1/n)_p.$$

We have obtained that for $k \geqq m$ we have

$$E_n^+(f)_p \leqq c(k,m)\,\tau_k(f, 1/n)_p.$$

In the case when $k<m$ we use the fact that $\tau_m(f, 1/n)_p \leqq c(k,m)\,\tau_k(f, 1/n)_p$ and therefore we have the needed inequality again.

We end the proof with the following :

$$\tilde{E}_n(f)_p = E_n^+(f)_p + E_n^-(f)_p = E_n^+(f)_p + E_n^+(-f)_n$$

$$\leqq c(k,m)\,\tau_k(f, 1/n)_p + c(k,m)\,\tau_k(-f, 1/n)_p = 2c(k,m)\,\tau_k(f, 1/n)_p.$$

5. Converse theorem

Theorem 2. *Let $f\in M$. For every natural number k there exists a constant $c(k,m)$ depending only on k and m such that*

$$\tau_k(f, 1/n)_p \leqq c(k,m)\, n^{-k} \sum_{\nu=0}^{n} (\nu+1)^{k-1} \tilde{E}_\nu(f)_p.$$

Proof. We shall use the scheme of [8]. Let for every natural number n the trigonometrical polynomials $P_n\in\pi_n$, $Q_n\in\pi_n$ be such that

$$\tilde{E}_n(f)_p = \| P_n - Q_n \|_p$$

$$Q_n(x) \leqq f(x) \leqq P_n(x), \quad x\in \mathrm{R}^m.$$

Let $x\in\pi^m$ be fixed and $t, t+kh \in \Omega_{k\delta/2}(x)$.

If $0 \leqq \Delta_h^k f(t)$, then

$$0 \leqq \Delta_h^k f(t) = \sum_{j=0}^{k} (-1)^{j+k} \binom{k}{j} f(t+jh)$$

$$\leqq \sum_{\substack{j=0 \\ j\equiv k \pmod 2}}^{k} \binom{k}{j} P_n(t+jh) - \sum_{\substack{j=0 \\ j\equiv k-1 \pmod 2}}^{k} \binom{k}{j} Q_n(t+jh)$$

$$=\Delta_h^k P_n(t)-\sum_{\substack{j=0\\ j\equiv k-1\,(\mathrm{mod}\,2)}}^{k}\binom{k}{j}(Q_n(t+jh)-P_n(t+jh))$$

$$=\Delta_h^k P_n(t)+\sum_{\substack{j=0\\ j\equiv k-1\,(\mathrm{mod}\,2)}}^{k}\binom{k}{j}[P_n(t+jh)-Q_n(t+jh)-(P_n(x)-Q_n(x))]$$

$$+\sum_{\substack{j=0\\ j\equiv k-1\,(\mathrm{mod}\,2)}}^{k}\binom{k}{j}(P_n(x)-Q_n(x))$$

$$\leqq \Delta_h^k P_n(t)+2^{k-1}\omega_1(P_n-Q_n,x,k\delta)+2^{k-1}(P_n(x)-Q_n(x)),$$

(5) i.e. $0\leqq \Delta_h^k f(t)\leqq \Delta_h^k P_n(t)+2^{k-1}\omega_1(P_n-Q_n,x,k\delta)+2^{k-1}(P_n(x)-Q_n(x))$.

Analogically in the case $\Delta_h^k f(t)\leqq 0$ we obtain :

(6) $0\leqq -\Delta_h^k f(t)\leqq |\Delta_h^k Q_n(t)|+2^{k-1}\omega_1(P_n-Q_n,x,k\delta)+2^{k-1}(P_n(x)-Q_n(x))$.

From (5) and (6) it follows :

$$\omega_k(f,x,\delta)\leqq \omega_k(P_n,x,\delta)+\omega_k(Q_n,x,\delta)$$

$$+2^{k-1}\omega_1(P_n-Q_n,x,k\delta)+2^{k-1}(P_n(x)-Q_n(x)),$$

e. g.

(7) $\tau_k(f,\delta)_p\leqq \tau_k(P_n,\delta)_p+\tau_k(Q_n,\delta)_p+2^{k-1}\tau_1(P_n-Q_n,k\delta)_p+2^{k-1}\tilde{E}_n(f)_p$.

Remark. Similarly for every two functions $f^+, f^-, f^+\geqq f\geqq f^-$ we obtain

Lemma 9. Let $f\in M$, $f^+\in M$, $f^-\in M$ and $f^+\geqq f\geqq f^-$.
Then

$$\tau_k(f,\delta)_p\leqq \tau_k(f^+,\delta)_p+\tau_k(f^-,\delta)_p+2^{k-1}\tau_1(f^+-f^-,k\delta)_p+2^{k-1}\|f^+-f^-\|_p.$$

Till now everything is the same as in one-dimensional case. The essential difference in the many-dimensional case is the estimation of $\tau_1(P_n-Q_n,k\delta)_p$. From property 6 of τ_1 we have :

(8) $$\tau_1(P_n-Q_n,k\delta)_p\leqq 2\sum_{\substack{\alpha:\alpha_i=0,1\\ |\alpha|\geqq 1}}(k\delta)^{|\alpha|}\|D^\alpha(P_n-Q_n)\|_p.$$

Since $\|D^\alpha(P_n-Q_n)\|_p\leqq n^{|\alpha|}\|P_n-Q_n\|_p$ (Bernstein inequality, $P_n, Q_n\in\pi_n$, see, p. 111) we obtain from (8) :

(9) $$\tau_1(P_n-Q_n,k\delta)_p\leqq 2\sum_{\substack{\alpha:\alpha_i=0,1\\ |\alpha|>0}}(k\delta n)^{|\alpha|}\|P_n-Q_n\|_p=2\tilde{E}_n(f)_p\sum_{\substack{\alpha:\alpha_i=0,1\\ |\alpha|>0}}(k\delta n)^{|\alpha|}.$$

From (7) and (9) we obtain :

$$(10)\qquad \tau_k(f,\delta)_p \leqq \tau_k(P_n,\delta)_p + \tau_k(Q_n,\delta)_p + 2^{k-1}(1+2 \sum_{\substack{\alpha:\alpha_i=0,1\\ |\alpha|>0}} (k\delta n)^{|\alpha|}\tilde{E}_n(f))_p.$$

Now we shall use Bernstein-Salem-Stečkin scheme. Let us estimate $\tau_k(P_n,\delta)_p$ (the estimation of $\tau_k(Q_n,\delta)_p$ follows the same way). Setting $n=2^{s_0}$ and using property 2 of τ_k, we have :

$$(11)\qquad \tau_k(P_n,\delta)_p \leqq \sum_{i=0}^{s_0} \tau_k(P_{2^i}-P_{2^{i-1}},\delta)_p + \tau_k(P_1-P_0,\delta)_p.$$

Since $g=P_{2^i}-P_{2^{i-1}}\in\pi_n$, using property 7 of τ_k and Bernstein inequality, we obtain :

$$(12)\quad \tau_k(g,\delta)_p \leqq c(k,m) \sum_{\substack{|\alpha|\geqq k\\ 0\leqq\alpha_i\leqq k}} \delta^{|\alpha|}\,\|D^\alpha g\|_p \leqq c(k,m) \sum_{\substack{|\alpha|\geqq k\\ \alpha_i=0,\dots,k}} \delta^{|\alpha|}2^{i|\alpha|}\,\|g\|_p.$$

Let $\delta=n^{-1}=2^{-s_0}$. Then $\delta\leqq 2^{-i}$ for $i\leqq s_0$ and $\delta^{|\beta|}2^{i|\beta|}\leqq 1$. Therefore from (12) we obtain

$$(13)\qquad \tau_k(P_{2^i}-P_{2^{i-1}},\delta)_p \leqq c_1(k,m)\delta^k 2^{ik}\,\|P_{2^i}-P_{2^{i-1}}\|_p$$

$$\leqq 2c_1(k,m)\delta^k 2^{ik}\tilde{E}_{2^{i-1}}(f)_p,$$

where the constant $c_1(k,m)$ depends only on k and m.

From (11) and (13) we obtain $(\tilde{E}_{-1}\equiv\tilde{E}_0, \delta=n^{-1})$

$$\tau_k(P_n,n^{-1})_p \leqq \sum_{i=0}^{s_0} 2c_1(k,m)n^{-k}2^{ik}\tilde{E}_{2^{i-1}}(f)_p$$

$$(14)\qquad \leqq c_2(k,m)n^{-k}\sum_{s=0}^{n}(s+1)^{k-1}\tilde{E}_s(f)_p,$$

where the constant $c_2(k,m)$ depends only on k and m.

Similarly

$$(15)\qquad \tau_k(Q_n,n^{-1})_p \leqq c_3(k,m)n^{-k}\sum_{s=0}^{n}(s+1)^{k-1}\tilde{E}_s(f)_p.$$

Inequalities (10), (14) and (15) (for $\delta=n^{-1}$) give Theorem 2 in the case when $n=2^{s_0}$. Transition to arbitrary n is standard.

Theorems 1 and 2 give us

Theorem 3. *Let $f \in M$. For $0<\alpha<k$, $1 \leqq p \leqq \infty$, the following two conditions are equivalent:*

i) $\tau_k(f, \delta)_p = O(\delta^\alpha)$

ii) $\tilde{E}_n(f)_p = O(n^{-\alpha})$.

This theorem gives characterization of the best one-sided trigonometrical approximations in L_p, $1 \leqq p \leqq \infty$, by means of the averaged moduli of smoothness in multivariate case, what is the same as in the one-dimensional case. This characterization is similar to the classical characterization of the best trigonometrical approximations in L_p by means of the classical integral moduli of smoothness $\omega_k(f, \delta)_p$.

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