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Classification of Dragilev Spaces of Types-1 and 0*

Mefharet Kocatepe

Presented by T. Terzioğlu

1. Introduction

It was claimed in [2] by M. M. Dragilev that Dragilev spaces of different types determined by rapidly increasing functions cannot be isomorphic. It was shown in [4] and [5] by the author that Dragilev spaces of types 1 and ∞ and of types -1 and 0 can be isomorphic. In [6] Dragilev spaces of type 1 which are isomorphic to some Dragilev space of type ∞ were characterized.

In this note, we characterize those Dragilev spaces of type -1 which are isomorphic to some Dragilev space of type 0 and those spaces of type 0 which are isomorphic to some space of type -1, that is we find those Dragilev spaces which are in the intersection of the class of the spaces of type -1 and of type 0.

We note that Dragilev spaces of type -1, that is $L_f(a, -1)$ have the property that $Ext(L_f(a, -1), L_f(a, -1))=0$ (see [3]). On the other hand subspaces and quotients of nuclear and stable Dragilev spaces of type 0 have been characterized (see [1]).

2. Preliminaries

An $L_f(a,r)$ space, also called a Dragilev space, is the Köthe space $\lambda(A)$ generated by the matrix $A=(a_i^k)$, $a_i^k=\exp f(r_k a_i)$, where f is an increasing, odd, logarithmically convex function (i. e. the function $\varphi(x)=\log f(e^x)$ is convex on R), $a=(a_i)$ is a strictly increasing sequence of positive numbers which approaches infinity rapidly enough to make $L_f(a,r)$ nuclear and (r_k) a strictly increasing sequence with $r=\lim r_k$ and $-\infty<r\leqq+\infty$ (see [2]). An $L_f(a,r)$ space is isomorphic to $L_f(b,1)$ (resp. $L_f(b,-1)$) if $0<r<\infty$ (resp. $r<0$). Hence basically there are four types of L_f spaces : $-1, 0, 1, +\infty$. Since f is logarithmically convex, for all $a>1$ the function $f(ax)/f(x)$ is increasing and either has a finite limit for all $a>1$ or approaches infinity for all $a>1$ as $x\to\infty$. In the first case f is called slowly increasing and in the second case it is called rapidly increasing. It is well-known that $L_f(a,r)$ is isomorphic to a power series space

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if and only if f is slowly increasing. In this note we shall consider only rapidly increasing functions f, and nuclear $L_f(a,r)$ spaces.

For the definition and the properties of the functor $Ext(E,F)=Ext^1(E,F)$ on the category of Fréchet spaces we refer the reader to D. Vogt [9]. It was shown in [7] that for two Köthe spaces $\lambda(A)$ and $\lambda(B)$, $Ext(\lambda(A), \lambda(B))=0$ if and only if the following condition holds :

$$(S_2^*) : \forall\, q\; \exists\, p,k\;\; \forall\, K,m\;\; \exists\, n,S>0 : \forall_{i,j} \frac{a_i^m}{b_j^k} \leqq S \max\left(\frac{a_i^n}{b_j^K}, \frac{a_i^p}{b_j^q}\right).$$

It was shown in [3] that $Ext(L_f(a,-1), L_f(a,-1))=0$ for all f and a and $Ext(L_g(b,0), L_g(b,0))=0$ if and only if $(b,b)\in HP$, i.e. the set of all finite limit points of the set $\{b_i/b_j : i,j\in \mathbf{N}\}$ is bounded.

3. Results

We first note that the generating matrix of $L_f(a,-1)$ (resp. $L_g(b,0)$) can be given by $(\exp(-f(r_k a_i)))$ (resp. $(\exp(-g(s_k b_i))))$ where (r_k) strictly decreases to 1 (resp. (s_k) strictly decreases to 0).

The proof of the following proposition has been given in the proof of Proposition 2 in [5]. But for completeness we present it here.

Proposition 1. *If $L_f(a,-1)$ and $L_g(b,0)$ are isomorphic then there are subsequences (j_k) and (m_k) of $\mathbf{N}$ with $j_1=1$ and a sequence (i_k) of indices such that*

$$g(s_{j_k} b_i) \geqq f(r_{m_k} a_i), \quad f(r_{m_k+1} a_i) \geqq g(s_{j_{k+1}} b_i), \quad i \geqq i_k.$$

P r o o f. Since the spaces are isomorphic, their diametral dimensions are equal. Since both of them are nuclear G_1-spaces, we have $\Delta(L_g(b,0))=L_g(b,0)$ and $\Delta(L_f(a,-1))=L_f(a,-1)$, and so $L_g(b,0)=L_f(a,-1)$ (see [8]). This means that the coordinate bases of $L_g(b,0)$ and $L_f(a,-1)$ are equivalent and so the identity operator $I : L_g(b,0)\to L_f(a,-1)$ is an isomorphism. Also using the fact that the functions f and g are rapidly increasing, we have that

$$\text{(i)}\;\; \forall\; k\; \exists\; l\; \exists\; A>0 : g(s_k b_i) \geqq Af(r_l a_i) \geqq Af(r_{l+1} a_i), \text{ large } i,$$

$$\text{(ii)}\;\; \forall\; j\; \exists\; m\; \exists\; B>0 : f(r_j a_i) \geqq Bg(s_m b_i) \geqq Bg(s_{m+1} b_i), \text{ large } i.$$

Now the claim follows.

Proposition 2. *If for each $c>1$, the set $\{a_i/a_j : i,j\in \mathbf{N}\}$ has a limit point in the interval $(1,c)$ then $L_f(a,-1)$ is not isomorphic to any $L_g(b,0)$.*

P r o o f. We assume $L_f(a,-1)$ is isomorphic to some $L_g(b,0)$, and take $s_k \searrow 0$ with $s_k/s_{k+1}\to\infty$. By Proposition 1, we find (n_k) and (m_k) such that

$$g(s_{n_k} b_i) \geqq f(r_{m_k} a_i), \quad f(r_{m_k+1} a_i) \geqq g(s_{n_{k+1}} b_i), \quad i \geqq i_k.$$

Given $k \geqq 1$, by assumption the set $\{a_i/a_j\}$ has a limit point α contained in the interval $(1, r_{m_k}/r_{m_k+2})$. Since $\alpha > 1$, there is $q = q(k) \geqq k$ such that $\alpha > r_{m_q+1}/r_{m_q+2}$.
So the set

$$I_k = \{(i,j) : \frac{r_{m_q+1}}{r_{m_q+2}} \leqq \frac{a_i}{a_j} \leqq \frac{r_{m_k}}{r_{m_k+2}}\}$$

is infinite. Moreover, the projections of I_k onto the first and second components are infinite sets. Now for $(i,j) \in I_k, i \geqq i_{q+2}, j \geqq i_q$ we have

$$g(s_{n_q+2} b_i) \geqq f(r_{m_q+2} a_i) \geqq f(r_{m_q+1} a_j) \geqq g(s_{n_q+1} b_j),$$

and so $b_i/b_j \geqq s_{n_q+1}/s_{n_q+2}$.

Also for $(i,j) \in I_k, i \geqq i_{k+2}, j \geqq i_k$ we have

$$g(s_{n_k+3} b_i) \leqq f(r_{m_k+2+1} a_i) \leqq f(r_{m_k+2} a_i) \leqq f(r_{m_k} a_j) \leqq g(s_{n_k} b_j),$$

and so $b_i/b_j \leqq s_{n_k}/s_{n_k+3}$.

Hence the interval $[s_{n_q+1}/s_{n_q+2}, s_{n_k}/s_{n_k+3}]$ contains infinitely many b_i/b_j, and so at least one limit point of the set $\{b_i/b_j\}$. Since $s_{n_q+1}/s_{n_q+2} \to \infty$ as $k \nearrow \infty$, we have that the set of finite limit points of the set $\{b_i/b_j\}$ is unbounded, and $Ext(L_g(b,0), L_g(b,0)) \neq 0$.

Now we give some positive results.

Proposition 3. *If $r_k \searrow 1$ and $r_{k+1}^2 \leqq r_k$, then*

$$\frac{f(r_{k+1} a_i)}{f(r_t a_i)} \leqq \frac{f(r_k a_i)}{f(r_{k+1} a_i)}, \quad i, k, t \in \mathbf{N}.$$

Proof. Since for $t \leqq k+1$, the inequality is obvious, we consider the case $t > k+1$. $\varphi(x) = \log f(e^x)$ is convex. Given i and $t > k+1$,

$$\log \frac{f(r_{k+1} a_i)}{f(r_t a_i)} = \frac{\varphi(\log(r_{k+1} a_i)) - \varphi(\log(r_t a_i))}{\log r_{k+1} - \log r_t} (\log r_{k+1} - \log r_t)$$

$$\leqq \frac{\varphi(\log(r_k a_i)) - \varphi(\log(r_{k+1} a_i))}{\log r_k - \log r_{k+1}} (\log r_{k+1} - \log r_t)$$

$$= (\log \frac{f(r_k a_i)}{f(r_{k+1} a_i)}) \frac{\log r_{k+1} - \log r_t}{\log r_k - \log r_{k+1}}.$$

Since $r_t > 1$, $\log r_{k+1} - \log r_t \leqq \log r_{k+1} \leqq \log r_k - \log r_{k+1}$. So we have the result.

Proposition 4. *If lim* $\inf(a_{i+1}/a_i)>1$ *and* $L_f(a,-1)$ *is isomorphic to some* $L_g(b,0)$, *then there is a sequence* $u_k\searrow 1$ *and a sequence of indices* (p_k) *such that*

$$\frac{f(u_1 a_i)}{f(u_2 a_i)}\leqq\frac{f(u_k a_{i+1})}{f(u_{k+1} a_{i+1})},\quad k\in\mathbb{N},\ i\geqq p_k.$$

Proof. We choose (r_k) as in Proposition 3 and (s_k) any sequence which strictly decreases to 0. Since (a_i) is strictly increasing and lim $\inf(a_{i+1}/a_i)>1$, we have that $\inf(a_{i+1}/a_i)>1$. By passing to a subsequence of (r_k) (if necessary) we may assume that $a_{i+1}/a_i\geqq r_1$, for all i. Then by Proposition 1, we choose strictly increasing sequences of indices (j_k) and (m_k) with $j_1=1$ such that

$$\frac{s_{j_2}}{s_{j_4}}\leqq\frac{s_{j_k}}{s_{j_{k+1}}},\quad k\geqq 4$$

and

$$g(s_{j_k} b_i)\geqq f(r_{m_k} a_i),\quad f(r_{m_{k+1}} a_i)\geqq g(s_{j_{k+1}} b_i),\quad i\geqq i_k.$$

Then for any k,i,

$$r_{m_k} a_{i+1}>a_{i+1}\geqq r_{m_1+1} a_i$$

and so for $i\geqq i_k$,

$$g(s_{j_k} b_{i+1})\geqq f(r_{m_k} a_{i+1})\geqq f(r_{m_1+1} a_i)\geqq g(s_{j_2} b_i)\geqq g(s_{j_4} b_i).$$

Hence for $i\geqq i_{k+1}$, $s_{j_2} b_i\leqq s_{j_k} b_{i+1}$ and $s_{j_4} b_i\leqq s_{j_{k+1}} b_{i+1}$.
Let $\psi(x)=\log g(e^x)$, which is convex. Then for $k\geqq 4$, $i\geqq i_{k+1}$,

$$\begin{aligned}\log\frac{g(s_{j_2} b_i)}{g(s_{j_4} b_i)}&=\frac{\psi(\log(s_{j_2} b_i))-\psi(\log(s_{j_4} b_i))}{\log s_{j_2}-\log s_{j_4}}(\log s_{j_2}-\log s_{j_4})\\&\leqq\frac{\psi(\log(s_{j_k} b_{i+1}))-\psi(\log(s_{j_{k+1}} b_{i+1}))}{\log s_{j_k}-\log s_{j_{k+1}}}(\log s_{j_2}-\log s_{j_4})\\&=(\log\frac{g(s_{j_k} b_{i+1})}{g(s_{j_{k+1}} b_{i+1})})\frac{\log(s_{j_2}/s_{j_4})}{\log(s_{j_k}/s_{j_{k+1}})}\leqq\log\frac{g(s_{j_k} b_{i+1})}{g(s_{j_{k+1}} b_{i+1})}.\end{aligned}$$

So for $k\geqq 4$ and $i\geqq i_{k+1}$ we have

$$\frac{g(s_{j_2} b_i)}{g(s_{j_4} b_i)}\leqq\frac{g(s_{j_k} b_{i+1})}{g(s_{j_{k+1}} b_{i+1})},$$

hence by Propositions 1 and 3, for $k\geqq 1$, $i\geqq i_{k+4}$,

$$\frac{f(r_{m_2} a_i)}{f(r_{m_3} a_i)}\leqq\frac{g(s_{j_2} b_i)}{f(r_{m_3+1} a_i)}\leqq\frac{g(s_{j_2} b_i)}{g(s_{j_4} b_i)}\leqq\frac{g(s_{j_{k+3}} b_{i+1})}{g(s_{j_{k+4}} b_{i+1})}$$

$$\leq \frac{f(r_{m_{k+2}+1}a_{i+1})}{f(r_{m_{k+4}}a_{i+1})} \leq \frac{f(r_{m_{k+1}+1}a_{i+1})}{f(r_{m_{k+4}}a_{i+1})} \leq \frac{f(r_{m_{k+1}}a_{i+1})}{f(r_{m_{k+1}+1}a_{i+1})} \leq \frac{f(r_{m_{k+1}}a_{i+1})}{f(r_{m_{k+2}}a_{i+1})}.$$

Let $u_k = r_{m_{k+1}}$, $p_k = i_{k+4}$.

Proposition 5. *Suppose that* $\liminf(a_{i+1}/a_i) > 1$ *and there is a sequence* (u_k) *which strictly decreases to* 1 *such that*

$$\frac{f(u_1 a_i)}{f(u_2 a_i)} \leq \frac{f(u_k a_{i+1})}{f(u_{k+1} a_{i+1})}, \quad k \geq 1, \ i \geq i_k.$$

Then $L_f(a, -1)$ *is isomorphic to some* $L_g(b, 0)$.

Proof. Since (a_i) is strictly increasing, the condition $\liminf(a_{i+1}/a_i) > 1$ implies that $\inf(a_{i+1}/a_i) > 1$. By passing to a subsequence of (u_k) if necessary, we may assume that $u_{k+1}^2 \leq u_k$ and $a_{i+1}/a_i > u_1$. Then Proposition 3 also holds. Then we have

$$(1) \qquad \frac{f(u_{k+1} a_i)}{f(u_{k+2} a_i)} \leq \frac{f(u_k a_i)}{f(u_{k+1} a_i)}, \quad i, k \in \mathbf{N}$$

$$(2) \qquad \frac{f(u_1 a_i)}{f(u_2 a_i)} \leq \frac{f(u_k a_{i+1})}{f(u_{k+1} a_{i+1})}, \quad k \geq 1, \ i \geq p(k),$$

where $p: \mathbf{N} \to \mathbf{N}$ is a strictly increasing function.

We let $i_0 - 1 = \min\{p(k-1) : k \geq 2\} = p(1)$, and for $i \geq i_0$ we define $k(i) = \max\{k : p(k-1) \leq i-1\}$. Then

$$2 = k(i_0) \leq k(i) \leq k(i+1), \quad i \geq i_0,$$

$\lim k(i) = \infty$ and $p(k(i)-1) \leq i-1$. If we write (2) above with $k = k(i+1)-1$ (we have $p(k(i+1)-1) \leq i$), then

$$(3) \qquad \frac{f(u_1 a_i)}{f(u_2 a_i)} \leq \frac{f(u_{k(i+1)-1} a_i)}{f(u_{k(i+1)} a_i)}, \quad i \geq i_0.$$

Next we choose $s > 1$ and fix it, and define $s_k = s^{-k}$. Then we define a sequence $b = (b_i)$, $i \geq i_0$ as follows: $b_{i_0} = 1$ and b_{i+1} is inductively defined by

$$\log\left(\frac{s_{k(i+1)} b_{i+1}}{s_1 b_i}\right) = \log s \frac{\log f(u_{k(i+1)} a_{i+1}) - \log f(u_1 a_i)}{\log f(u_{k(i+1)-1} a_{i+1}) - \log f(u_{k(i+1)} a_{i+1})}.$$

Since $a_{i+1}/a_i > u_1 > u_1/u_k$ for all i, k, the right hand side is positive, so the left hand side is positive, that is $s_1 b_i < s_{k(i+1)} b_{i+1}$.

By (3) definition of b_{i+1} and $\log s_k - \log s_{k+1} = \log s$, we have that

$$\frac{\log f(u_1 a_i) - \log f(u_2 a_i)}{\log(s_1 b_i) - \log(s_2 b_i)} \leq \frac{\log f(u_{k(i+1)} a_{i+1}) - \log f(u_1 a_i)}{\log(s_{k(i+1)} b_{i+1}) - \log(s_1 b_i)}$$

$$(4)\qquad \leq \frac{\log f(u_{k(i+1)-1}\, a_{i+1}) - \log f(u_{k(i+1)}\, a_{i+1})}{\log (s_{k(i+1)-1}\, b_{i+1}) - \log (s_{k(i+1)}\, b_{i+1})}.$$

By (3) we have that

$$\frac{f(u_{k(i)-1}\, a_i)}{f(u_{k(i)}\, a_i)} \geq \frac{f(u_1\, a_{i-1})}{f(u_2\, a_{i-1})},\quad i \geq i_0 + 1.$$

Since right hand side approaches infinity as $i \to \infty$ we choose $i_1 \geq i_0 + 1$ such that $f(u_1\, a_{i_1-1})/f(u_2\, a_{i_1-1}) > s$. Then

$$1 < \frac{\log f(u_{k(i_1)-1}\, a_{i_1}) - \log f(u_{k(i_1)}\, a_{i_1})}{\log s}.$$

Now for $i \geq i_1$ and for $1 \leq k \leq k(i)$ we define $\psi(\log(s_k\, b_i)) = \log f(r_k\, a_i)$ and $Q_{k,i} = (\log(s_k\, b_i),\ \psi(\log(s_k\, b_i)))$ and join the points

$$Q_{k(i_1),i_1} \to \cdots \to Q_{1,i_1} \to \cdots \to Q_{k(i),i} \to Q_{k(i)-1,i} \to \cdots$$
$$\to Q_{2,i} \to Q_{1,i} \to Q_{k(i+1),i+1} \to Q_{k(i+1)-1,i+1} \to \cdots$$

by line segments. This way ψ is defined for all $x \geq \log(s_{k(i_1)}\, b_{i_1})$. For $x \leq \log(s_{k(i_1)}\, b_{i_1})$, we define

$$\psi(x) = \log f(u_{k(i_1)}\, a_{i_1}) - \log(s_{k(i_1)}\, b_{i_1}) + x.$$

It is clear that ψ is increasing. By (1) we have that ψ is convex within the i-th block, and from (4) it follows that the slope of ψ increases when we pass from the i-th block to the $(i+1)$-st block. Finally i_1 was chosen in such a way that the slope of ψ from $(0, \psi(0))$ to $Q_{k(i_1),i_1}$ is smaller than the slope from $Q_{k(i_1),i_1}$ to $Q_{k(i_1)-1,i_1}$.

Now we define

$$g(x) = \begin{cases} \dfrac{f(u_{k(i_1)}\, a_{i_1})}{s_{k(i_1)}\, b_{i_1}}\, x, & 0 \leq x \leq s_{k(i_1)}\, b_{i_1} \\ e^{\psi(\log x)} & s_{k(i_1)}\, b_{i_1} \leq x \\ -g(-x), & x \leq 0 \end{cases}.$$

Then g is an increasing, odd function with $\log g(e^x) = \psi(x)$. Finally for $i \geq i_1$ and $2 \leq k \leq k(i)$ and so for $k \geq 2$ and $i \geq \max\{p(k-1)+1,\ i_1\}$ we have $f(u_k\, a_i) = g(s_k\, b_i)$, which shows that $L_f(a, -1)$ is isomorphic to some $L_g(b, 0)$.

We summarize the last two propositions in the following theorem.

Theorem 1. *Suppose $L_f(a, -1)$ is nuclear and $\liminf(a_{i+1}/a_i) > 1$. Then $L_f(a, -1)$ is isomorphic to some $L_g(b, 0)$ if and only if there is a sequence (u_k) which strictly decreases to 1 such that*

$$\frac{f(u_1\, a_i)}{f(u_2\, a_i)} \leq \frac{f(u_k\, a_{i+1})}{f(u_{k+1}\, a_{i+1})},\quad k \geq 1,\ i \geq i_k.$$

For the general case we have the following theorem.

Theorem 2. *Suppose $L_f(a,-1)$ is nuclear. Then $L_f(a,-1)$ is isomorphic to some $L_g(b,0)$ if and only if the following conditions hold: there is a subsequence (a_{i_n}) of (a_i) and a sequence (u_k) which strictly decreases to 1 such that*

$$\text{(i)} \qquad \lim\inf \frac{a_{i_{n+1}}}{a_{i_n}} > 1,$$

$$\text{(ii)} \qquad \lim_{n\to\infty} \frac{a_{i_{n+1}-1}}{a_{i_n}} = 1,$$

$$\text{(iii)} \qquad \frac{f(u_1 a_{i_n})}{f(u_2 a_{i_n})} \leqq \frac{f(u_k a_{i_{n+1}})}{f(u_{k+1} a_{i_{n+1}})}, \quad n \geqq p_k$$

for some sequence (p_k) of indices.

Proof. Necessity. Suppose $L_f(a,-1)$ is isomorphic to some $L_g(b,0)$. Then by Proposition 2, there is $c>1$ such that the set of limit points of $\{a_i/a_j\}$ is contained in the set $[0, 1]\cup[c, +\infty]$. We let $\alpha=(1+c)/2>1$, and $i_1=1$. We choose i_2 as the smallest integer n such that $a_n/a_{i_1}\geqq\alpha$, and then we choose i_3 as the smallest integer n such that $a_n/a_{i_2}\geqq\alpha$. We continue this way and get a strictly increasing sequence (i_n) of indices such that

$$\frac{a_{i_{n+1}}}{a_{i_n}} \geqq \alpha, \quad \frac{a_{i_{n+1}-1}}{a_{i_n}} < \alpha.$$

It follows from the property of c that $\lim(a_{i_{n+1}-1}/a_{i_n})=1$. So we have (i) and (ii). From Proposition 1, it follows that if $L_f(a,-1)$ is isomorphic to $L_g(b,0)$, then $L_f((a_{i_n}),-1)$ is isomorphic to $L_g((b_{i_n}),0)$. This together with (i) gives (iii).

Sufficiency. By Proposition 5, from (i) and (iii) we have that $L_f((a_{i_n}),-1)$ is isomorphic to some $L_g((b_{i_n}),0)$ with $f(u_k a_{i_n})=g(s_k b_{i_n})$, $n\geqq n_k$. If for some n, there is an i such that $i_n<i<i_{n+1}$, we define b_i in such a way that the sequence (b_i) is strictly increasing and $\lim(b_{i_{n+1}-1}/b_{i_n})=1$.

Now given k we find n_0 such that for $n\geqq n_0$,

$$\frac{a_{i_{n+1}-1}}{a_{i_n}} < \frac{u_k}{u_{k+1}}, \quad \frac{b_{i_{n+1}-1}}{b_{i_n}} < \frac{s_k}{s_{k+1}}.$$

If $n\geqq n_0$ and $i_n<i<i_{n+1}$, then

$$\frac{a_i}{a_{i_n}} < \frac{u_k}{u_{k+1}}, \quad \frac{b_i}{b_{i_n}} < \frac{s_k}{s_{k+1}}.$$

and so for $n\geqq\max\{n_0, n_{k+1}\}$ and $i_n<i_{n+1}$, we have

$$f(u_{k+1} a_i) \leqq f(u_k a_{i_n}) = g(s_k b_{i_n}) \leqq g(s_k b_i),$$

$$g(s_{k+1} b_i) \leqq g(s_k b_{i_n}) = f(u_k a_{i_n}) \leqq f(u_k a_i).$$

So $L_f(a,-1)$ is isomorphic to $L_g(b,0)$.

Now we consider the other case, namely we try to characterize those $L_g(b,0)$ which are isomorphic to some $L_f(a,-1)$.

Proposition 6. *If $L_g(b,0)$ is isomorphic to some $L_f(a,-1)$ then there is a sequence (s_k) which strictly decreases to 0 and a collection of indices $(p_{k,u})$ such that*

$$\frac{g(s_{k+1}b_i)}{g(s_u b_i)} \leqq \frac{g(s_k b_i)}{g(s_{k+1}b_i)}, \quad k \geqq 1, \quad u > k+1, \quad i \geqq p_{k,u}.$$

Proof. Let $(r_k) \searrow 1$ be such that $r_{k+1}^2 \leqq r_k$. By Proposition 3 we have that

$$\frac{f(r_{k+1}a_i)}{f(r_u a_i)} \leqq \frac{f(r_k a_i)}{f(r_{k+1}a_i)}, \quad i, k, u \in \mathbb{N}, \quad u > k+1.$$

Let (t_k) be any sequence such that $(t_k) \searrow 0$. Then by Proposition 1 we have subsequences (j_k) and (m_k) of $\mathbb{N}$ with $j_1 = 1$ and a sequence (i_k) of indices such that

$$g(t_{j_k}b_i) \geqq f(r_{m_k}a_i), \quad f(r_{m_k+1}a_i) \geqq g(t_{j_{k+1}}b_i), \quad i \geqq i_k.$$

We let $s_k = t_{j_k}$ and $p_{k,u} = \max\{i_k, i_u\}$. Then for $u > k+1$, $m_u > m_{k+1} \geqq m_k + 1$ and so for $i \geqq p_{k,u}$,

$$\frac{g(s_k b_i)}{g(s_{k+1}b_i)} = \frac{g(t_{j_k}b_i)}{g(t_{j_{k+1}}b_i)} \geqq \frac{f(r_{m_k}a_i)}{f(r_{m_k+1}a_i)} \geqq \frac{f(r_{m_k+1}a_i)}{f(r_{m_u}a_i)}$$

$$\geqq \frac{g(t_{j_{k+1}}b_i)}{g(t_{j_u}b_i)} = \frac{g(s_{k+1}b_i)}{g(s_u b_i)}.$$

Proposition 7. *Suppose that $\lim(b_{i+1}/b_i) = \infty$ and there is a sequence (s_k) which strictly decreases to 0 such that*

$$\frac{g(s_{k+1}b_i)}{g(s_u b_i)} \leqq \frac{g(s_k b_i)}{g(s_{k+1}b_i)}, \quad k \geqq 1, \quad u > k+1, \quad i \geqq p_{k,u}.$$

Then $L_g(b,0)$ is isomorphic to some $L_f(a,-1)$.

Proof. The hypothesis is satisfied by subsequences of (s_k) as well. By passing to a subsequence of (s_k) we may assume that $s_1/s_2 \leqq s_k/s_{k+1}$.

Given k, we find i_k such that $s_1/s_{k+1} \leqq b_{i+1}/b_i$, for $i \geqq i_k$. Then for $i \geqq i_k$,

$$s_2 b_i \leqq s_1 b_i \leqq s_{k+1} b_{i+1} \leqq s_k b_{i+1}.$$

If $\psi(x) = \log g(e^x)$, then for $i \geqq i_k$ we have

$$\log\frac{g(s_1 b_i)}{g(s_2 b_i)} = \frac{\psi(\log(s_1 b_i)) - \psi(\log(s_2 b_i))}{\log s_1 - \log s_2}(\log s_1 - \log s_2)$$

$$\leqq \frac{\psi(\log(s_k b_{i+1})) - \psi(\log(s_{k+1}b_{i+1}))}{\log s_k - \log s_{k+1}}(\log s_1 - \log s_2)$$

$$=\left(\log\frac{g(s_k b_{i+1})}{g(s_{k+1} b_{i+1})}\right)\frac{\log(s_1/s_2)}{\log(s_k/s_{k+1})}\leqq\log\frac{g(s_k b_{i+1})}{g(s_{k+1} b_{i+1})}.$$

We take $u=k+2$ in the hypothesis and find a strictly increasing function $p:\mathrm{N}\to\mathrm{N}$ such that

$$\frac{g(s_{k+1} b_i)}{g(s_{k+2} b_i)}\leqq\frac{g(s_k b_i)}{g(s_{k+1} b_i)},\quad k\geqq 1,\ i\geqq p(k),$$

$$\frac{g(s_1 b_i)}{g(s_2 b_i)}\leqq\frac{g(s_k b_{i+1})}{g(s_{k+1} b_{i+1})},\quad k\geqq 1,\ i\geqq p(k).$$

We define i_0 and $k(i)$ as in Proposition 5. Then we take any sequence $(r_k)\searrow 1$ such that $r_k/r_{k+1}\leqq r_1/r_2$ (This is possible since $\lim(r_k/r_{k+1})=1$ and $r_1/r_2>1$.) Then

$$\frac{\log g(s_1 b_i)-\log g(s_2 b_i)}{\log r_1-\log r_2}\leqq\frac{\log g(s_{k(i+1)-1} b_{i+1})-\log g(s_{k(i+1)} b_{i+1})}{\log r_{k(i+1)-1}-\log r_{k(i+1)}}.$$

Next we define a sequence $a=(a_i)$, $i\geqq i_0$ as follows : $a_{i_0}=1$ and a_{i+1} is inductively defined by

$$\frac{\log g(s_1 b_i)-\log g(s_2 b_i)}{\log r_1-\log r_2}\leqq\frac{\log g(s_{k(i+1)} b_{i+1})-\log g(s_1 b_i)}{\log(r_{k(i+1)} a_{i+1})-\log(r_1 a_i)}$$

$$\leqq\frac{\log g(s_{k(i+1)-1} b_{i+1})-\log g(s_{k(i+1)} b_{i+1})}{\log r_{k(i+1)-1}-\log r_{k(i+1)}}.$$

The rest of the proof is exactly the same as the proof of Proposition 5, so we do not repeat it.

Now we consider the most general case.

Proposition 8. *Suppose $Ext(L_g(b,0), L_g(b,0))=0$ and there is a sequence (s_k) which strictly decreases to 0 such that*

$$\frac{g(s_{k+1} b_i)}{g(s_u b_i)}\leqq\frac{g(s_k b_i)}{g(s_{k+1} b_i)},\quad k\geqq 1,\ u>k+1,\ i\geqq p_{k,u}.$$

Then $L_g(b,0)$ is isomorphic to some $L_f(a,-1)$.

Proof. Since $Ext(L_g(b,0), L_g(b,0))=0$, there is $M>0$ such that the set of finite limit points of the set $\{b_i/b_j : i,j\in\mathrm{N}\}$ is contained in $[0, M]$. As in the proof of Theorem 2, we choose a strictly increasing sequence (i_n) of indices such that o

$$\frac{b_{i_{n+1}}}{b_{i_n}}\geqq M+1,\quad\frac{b_{i_{n+1}-1}}{b_{i_n}}<M+1.$$

Then we have that $\lim(b_{i_{n+1}}/b_{i_n})=\infty$ (if not, then it would have a bounded subsequence and so $\{b_i/b_j\}$ would have a finite limit point greater than $M+1$). Then by Proposition 7 we have that $L_g((b_{i_n}),0)$ is isomorphic to some

$L_f((a_{i_n}), -1)$ with $g(s_k b_{i_n}) = f(r_k a_{i_n})$, $n \geqq n_k$. For $i_n < i < i_{n+1}$ we define a_i such that (a_i) is strictly increasing and $\lim(a_{i_{n+1}-1}/a_{i_n}) = 1$.

Now given k, we find $m > k$ such that $M+1 < s_k/s_m$. Then there is n_0 such that $a_{i_{n+1}-1}/a_{i_n} \leqq r_k/r_m$ for $n \geqq n_0$. If $n \geqq n_0$ and $i_n < i < i_{n+1}$,

$$\frac{b_i}{b_{i_n}} \leqq \frac{b_{i_{n+1}-1}}{b_{i_n}} \leqq M+1 \leqq \frac{s_k}{s_m}, \quad \frac{a_i}{a_{i_n}} \leqq \frac{a_{i_{n+1}-1}}{a_{i_n}} \leqq \frac{r_k}{r_m},$$

and so for $n \geqq \max\{n_0, n_k\}$ and $i_n < i < i_{n+1}$,

$$g(s_m b_i) \leqq g(s_k b_{i_n}) = f(r_k a_{i_n}) \leqq f(r_k a_i),$$

$$f(r_m a_i) \leqq f(r_k a_{i_n}) = g(s_k b_{i_n}) \leqq g(s_k b_i).$$

Now we combine Proposition 6, Proposition 8 and the fact that $Ext(L_g(b,0), L_g(b,0)) = 0$ is a necessary condition for $L_g(b,0)$ to be isomorphic to some $L_f(a,-1)$ in the following theorem.

Theorem 3. *Suppose $L_g(b,0)$ is nuclear. Then $L_g(b,0)$ is isomorphic to some $L_f(a,-1)$ if and only if the following conditions are satisfied:*

(i) $Ext(L\ (b,0), L_g(b,0)) = 0$.

(ii) *There is a sequence (s_k) which strictly decreases to 0 and a collection $(p_{k,u})$ of indices such that*

$$\frac{g(s_{k+1} b_i)}{g(s_u b_i)} \leqq \frac{g(s_k b_i)}{g(s_{k+1} b_i)}, \quad k \geqq 1, \quad u > k+1, \quad i \geqq p_{k,u}.$$

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Bilkent University
Faculty of Engineering and Science
Department of Mathematics
Ankara,
TURKEY

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