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A Differential Geometric Proof of the Local Geometric Characterization of Spheres and Cylinders by Boas

Bernd Wegner

Presented by Đ. Kurepa

In connection with the study of the Bochner-Martinelli integral formula H. P. Boas characterised spheres and cylinders as hypersurfaces of Euclidean space satisfying the following condition globally [1], resp. locally [2]:

(*) For each two points of the hypersurface, the chord joining them meets the normal to the surface in equal angles at the two points.

The aim of this comment is to give a short proof of the (stronger) local version using methods from differential geometry.
Since the focal set of the surface is involved in the proof, it has to be assumed twice differentiable.

Theorem [2]: *Let M be a connected open piece of a C^2-immersed hypersurface in Euclidean n-space E^n, $n>2$. Then M satisfies (*) if and only if M is an open piece of the Riemannian product $S^{n-j-1}\times E^j$ of a round $(n-j-1)$-sphere and a Euclidean j-space, $0\leqslant j\leqslant n-1$.*

Proof: Let M satisfy (*). Without loss of generality we assume M to be oriented. Let N_q denote the unit normal to M in q. Then (*) implies for all $p, q\in M$

$$\langle p-q,\ N_p\rangle=\langle q-p,\ N_q\rangle. \tag{1}$$

The shape operator A_p on the tangent plane of M in p is defined by $\nabla_X N|_p=-A_p(X)$. A_p is self-adjoint. Let v_p, v_q denote principal directions (eigen directions) of A_p, A_q respectively with corresponding principal curvatures (eigenvalues) k_p, k_q. Keeping q fixed we get by differentiation of (1)

$$0=\langle v_p,\ N_p+N_q\rangle+\langle -A_p(v_p),\ p-q\rangle$$

for all $q\in M$ and therefore by differentiation in direction of v_q

$$0=\langle v_p,\ A_q(v_q)\rangle+\langle -A_p(v_p),\ -v_q\rangle.$$

Thus principal directions and principal curvatures in p and q are related by

$$0=\langle v_p, v_q\rangle(k_q-k_p), \tag{2}$$

implying that principal directions of different principal curvatures in p and q are perpendicular to each other.
Restricting our considerations to a small neighbourhood of a deliberately chosen point in M, we can assume that there are no orthogonal pairs of tangent planes in this neighbourhood. Now take an orthonormal system $e_1, \ldots, e_{n-1}$ of principal directions of M in p and add $e_n := N_p$. Then (2) implies

$$v_q = \sum_{i\in I} c_i e_i + c_n e_n, \tag{3}$$

where the sum is taken over all i, such that e_i has k_q as eigenvalue. Since by the assumptions on the tangent planes made above v_q cannot be proportional to e_n, we see that the multiplicities of k_q as principal curvature of M in p and q are equal. Furthermore taking a second principal curvature $\tilde{k}_q$ of M in q with principal direction $\tilde{v}_q$ we get from $k_q \neq \tilde{k}_q$ and (3)

$$0=\langle v_q, \tilde{v}_q\rangle = c_n \tilde{c}_n.$$

If $N_p \neq N_q$, this implies that the orthogonal projection of $N_q - N_p$ onto the tangent plane of M in p is proportional to a principal direction of M at p, i.e., the orthogonal projection on this hyperplane of the local image of the Gauss map around p spans a starlike subset of the set of eigenvectors of A_p. Then the projected Gauss image must contain a submanifold of dimension μ and 0 is principal curvature of M in p of multiplicity $n-\mu-1$. Therefore the starlike set defined above must be the eigenspace of the uniquely defined second principal curvature of M in p of multiplicity μ in the case $\mu \neq 0$. Considering the other properties of the principal curvatures and directions given above we have that M is umbilical or has exactly two constant principal curvatures of constant multiplicity, one of them being 0. Standard arguments ([3], [4]) lead to the product decomposition proposed in the theorem. The other direction of the proof is obvious.

References

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Fachbereich Mathematik
TU Berlin
Straße des 17 Juni 135
D-1000 Berlin 12
GERMANY

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