Mathematica Balkanica

Mathematical Society of South-Eastern Europe
A quarterly published by
the Bulgarian Academy of Sciences – National Committee for Mathematics

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Mathematica Balkanica

New Series Vol. 2, 1988, Fasc. 4

Zeros of Quasi-Polynomials and Linear Time-Optimal Control Problem

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Presented by P. Kenderov

An algorithm is presented for finding all the sign changes of the real-valued quasi-polynomial $f(t)=\sum_{k=1}^{n} c_k \exp(\lambda_k t)$ in a given interval, where c_k and λ_k are complex numbers. Such a question arises in numerical solving a linear time-optimal control problem when determining switches of controls. Particular attention is paid to the localization of the desired points. The idea of the algorithm is based on the fact that by means of factorization applied to the appropriate differential operator the problem can be reduced to a problem of finding the zeros of quasi-polynomials with smaller number of summands.

1. We consider a linear time-optimal control problem in the following form: to synthesize a control $u=u(t)\in U$, which steers in a minimal time t_1 the solution of the equation

$$\dot{x}(t)=Ax(t)+Bu(t), \quad x(0)=x_0(\neq 0)$$

to the origin: $x(t_1)=0$. Here U is a parallelepiped in $\mathbf{R}^r$, defined by the inequalities

$$(0>)\,\alpha_i \leqq U^i \leqq \beta_i \ (>0), \ i=1, 2, \ldots, r;$$

A and B are constant $n\times n$ and $n\times r$ matrices, respectively; $x(t)$ is an n-dimensional vector-valued function. It is assumed that the normality condition holds [1] and x_0 belongs to the controllable set (which is automatically fulfilled in case of a stable A). It is well-known that under these conditions the desired optimal control exists, that it is unique (to within a set of measure zero) and is represented by piecewise constant function, whose values coincide with the vertices of U. The points t in which $u(t)$ has jumps are called switches. According to the Pontryagin Maximum Principle $u(t)$ is uniquely determined by the condition

$$\psi(t)Bu(t)=\max_{u\in U} \psi(t)Bu,$$

where $\psi(t)$ is the solution of the adjoint system $\dot{\psi}=-A^*\psi$ with some initial condition $\psi(0)=\psi_0$. The value of ψ_0 is a priori unknown and this is the main

source of computing difficulties. Therefore, one has to use rather complicated constructions in order to find $u(t)$. The majority of them are connected with repeatedly synthesizing optimal controls for initial points different from x_0. It is clear that the use of such methods requires to find control switches many times. Thus, it is important to have effective computer procedure available for searching these points.

Every switch is characterized by a sign-change of the components of the vector-valued function

$$\psi(t)B=(f^1(t),\ f^2(t),\ldots,\ f^r(t)).$$

Each of these components is obviously a polynomial of the following kind

$$f(t)=\sum_{k=1}^{n'} c_k(t)\exp(\lambda_k t), \tag{1}$$

where $\lambda_1,\ \lambda_2,\ldots,\ \lambda_{n'}$, $(n'\leqq n)$ are distinct eigenvalues of matrix $-A^*$ (or, which is the same, of $-A$) and $c_1(t),\ c_2(t),\ldots,\ c_{n'},\ (t)$ are polynomials of t.

For the sake of presentation simplicity in the sequel we restrict ourselves to the case of simple eigenvalues; by this the conceptual aspect does not change, while presentation becomes less cumbersome. In the case that interests us $f(t)$ obtains the form

$$f(t)=\sum_{k=1}^{n} c_k\exp(\lambda_k t). \tag{2}$$

There may be complex numbers among $c_k,\ \lambda_k$ $(k=1,\ 2,\ldots,\ n)$, but it is clear that the quasi-polynomial is real-valued (complex summands participate in (2) as conjugate pairs).

In accordance with the nature of the time-optimal control problem we are interested only in values $t>0$. Without loss of generality we assume each coefficient c_k in (2) being different from zero, since zero members can be omitted (changing appropriately the enumeration). Besides, we suppose that $n\geqslant 3$, as for $n=2$ the problem becomes trivial.

Taking into account the existence of well-known and fairly quick procedures for determination of localized zeros of smooth function (for example, bisection method, chord method, etc.), it is easy to understand that the problem actually reduces to the problem of localizing the zeros of $f(t)$. To localize the zero t_0 of the function $f(t)$ means to find an interval $(t_0',\ t_0'')$, at the ends of which $f(t)$ has different signs and inside it vanishes at the point t_0 only. It is clear that the problem of localization makes sense only for odd-multiple zeros.

Note that there are no effective methods for localizing the zeros of an arbitrary continuous function f. Clearly, if it is required to find the zeros of a function f in some interval $[a,\ b]$ (supposing that their number is finite), it is possible to consecutively calculate the values of f at the points $a,\ a+h,\ a+2h,\ldots,$ b, where $h=(b-a)/m,\ m\gg 1$ and watch for sign changes of f. For small h this method demands too much computing time; as for not too small h, there appear an actual possibility to omit (an even number) sign changes of f. Thus, additional

assumptions for the function are needed. One of the possible ways here is to use estimates for the derivatives of f. We cite a simplest example for such kind of "fight with zeros omission". Let $f(t_1)f(t_2)>0$ and

$$|\dot{f}(t)|<M=\frac{|f(t_1)+f(t_2)|}{t_2-t_1} \qquad (t_1<t<t_2). \tag{3}$$

Then f has no zeros between t_1 and t_2. In fact, it may be assumed that $f(t_i)>0$, $i=1, 2$. Let

$$t^*=\frac{f(t_1)t_2+f(t_2)t_1}{f(t_1)+f(t_2)}.$$

If f has a zero t_0 in $(t_1, t^*]$, then

$$\frac{|f(t_0)-f(t_1)|}{t_0-t_1}\geqq\frac{f(t_1)}{t^*-t_1}=\frac{f(t_1)+f(t_2)}{t_2-t_1}=M,$$

which contradicts (3). Analogously one can eliminate the possibility for a zero $t_0\in[t^*, t_2)$. If (3) is not fulfilled, we bisect (t_1, t_2) and so on until (3) is fulfilled in all obtained intervals (or we come to an interval with negligibly small length). The given trivial way is general, but it requires estimation for f (naturally most precise possible). We also note without going into details that in order to estimate the computational complexity of the given method it will be necessary in particular to estimate $|\dot{f}|$ too.

The method, proposed below, takes into account the characteristics of quasi-polynomials and thus is less general. Here a problem of estimating derivatives does not arise (such estimates are not required) or of choosing step length. However, it is not excluded that in some cases (especially for great n) the trivial method which uses some kind of rough estimates for $|\dot{f}|$ turns out to be practically more effective.

Naturally, if f belongs to a specific class of functions some new possibilities may be induced. The classical case when f is a polynomial has been investigated in an exhaustive way. But already for quasi-polynomials the problem becomes more difficult and it has been examined apparently far less.

2. We begin with a simpler case, when $\lambda_1, \lambda_2, \ldots, \lambda_n$ (hence $c_1, c_2, \ldots, c_n$) are real and

$$\lambda_1<\lambda_2<\ldots<\lambda_n. \tag{4}$$

It is known that the quasi-polynomial (2) has at most $n-1$ zeros. A more precise estimate can be obtained in a following way. According to Descartes rule [3] applied to the system of functions

$$\exp(\lambda_1 t), \exp(\lambda_2 t), \ldots, \exp(\lambda_n t) \tag{5}$$

the number of zeros (and hence the number of sign changes) of the quasi-polynomial (2) does not exceed the number of sign changes in the sequence of coefficients $c_1, c_2, \ldots, c_n$.

We assume that all coefficients c_k have not the same signs simultaneously (otherwise $f(t)$ would have no zeros at all) and that $c_n>0$ (otherwise we can go over from f to $-f$).

We are interested in the zeros, which are in the interval $(0, \infty)$. We can, however, deal with the finite interval $(0, t_{\max})$, where $t_{\max}$ is some upper bound for zeros of f. For example, assume

$$t_{\max}=(\lambda_n-\max_{k\in K}\lambda_k)^{-1}\ln(|K|\max_{k\in K}|c_k|/c_n),$$

where $K=\{k: c_k<0\}$, $|K|$ is the number of elements of K. In order to justify this choice we show that $f(t)>0$ for $t\geqq t_{\max}$ Indeed, we have

$$f(t)\geqq c_n\exp(\lambda_n t)+\sum_{k\in K}c_k\exp(\lambda_k t)$$

$$\geqq c_n\exp(\lambda_n t)-|K|\max_{k\in K}|c_k\exp(\lambda_k t)|$$

$$\geqq c_n\exp(\lambda_n t)-|K|\max_{k\in K}|c_k|\exp(\max_{k\in K}\lambda_k t).$$

A direct calculation shows that the right-hand side has an unique zero $t=t_{\max}$ Since for $t>t_{\max}$ it is evidently positive, the same holds for $f(t)$ as well. Further, $f(t_{\max})>0$ as far as $n\geqq 3$. If $t_{\max}\leqq 0$, it is obvious that $f(t)$ has no positive zeros. That is why later on we consider that $t_{\max}>0$.

Thus, all zeros of $f(t)$ we are interested in lie in the interval $(0, t_{\max})$. Our aim is to find the localization intervals for each of these zeros (more precisely, for each sign-change point of f) separately.

Let p $(1\leqslant p<n)$ be the minimal index such that there is exactly one sign change in the sequence $c_p, c_{p+1}, \ldots, c_n$ (since not each c_k has one and the same sign, such a p obviously exists). Let us consider the following function system

$$f(t)=f_1(t),\quad f_2(t),\ldots,f_p(t), \tag{6}$$

where the functions $f_k(k=2, 3,\ldots, p)$ are defined as follows:

$$f_k=\frac{df_{k-1}}{dt}-\lambda_{k-1}f_{k-1}=\exp(\lambda_{k-1}t)\frac{d}{dt}(\exp(-\lambda_{k-1}t)f_{k-1}). \tag{7}$$

If we introduce

$$\nu_{kj}=(\lambda_j-\lambda_1)(\lambda_j-\lambda_2)\ldots(\lambda_j-\lambda_{k-1}),\quad k=2,\ldots p;\quad j=k,\ldots, n$$

the system (6) can be written in the form

$$f_1(t)=c_1\exp(\lambda_1 t)+c_2\exp(\lambda_2 t)+\ldots+c_n\exp(\lambda_n t)$$

$$f_2(t)=c_2\nu_{22}\exp(\lambda_2 t)+c_3\nu_{23}\exp(\lambda_3 t)+\ldots+c_n\nu_{2n}\exp(\lambda_n t)$$

$$\vdots$$

$$f_p(t)=c_p\nu_{pp}\exp(\lambda_p t)+c_{p+1}\nu_{p,p+1}\exp(\lambda_{p+1}t)+\ldots+c_n\nu_{pn}\exp(\lambda_n t).$$

Since, by virtue of (4), each $v_{kj}>0$, in accordance to Descartes rule the number of zeros of each function f_k is not greater than the number of sign changes in the sequence $c_k, c_{k+1}, \ldots, c_n$. Therefore, f_p has no more than one zero. As we consider only the zeros which lie in the interval $(0, t_{\max})$, we have to compute the values of $f_p(t)$ at the ends of this interval. If $f_p(0)f_p(t_{\max})\geqq 0$, then f_p has no zeros in $(0, t_{\max})$. If $f_p(0)f_p(t_{\max})<0$, then $(0, t_{\max})$ is a localization interval for the unique zero t_1^p of the function f_p. It can be found by one of the above mentioned classical procedures for determination of a localized zero.

The subsequent part of the algorithm consists of consequently finding the sign-change points of functions f_{p-1}, f_{p-2}, etc. up to f_1. Suppose we have already obtained all the sign changes $t_1^k, t_2^k, \ldots, t_{mk}^k$ of the function f_k $(2\leqq k\leqq p)$. We show that it allows to localize and, therefore to calculate, all sign changes of the function f_{k-1}. Consider the intervals

$$(0,\ t_1^k),\ (t_1^k,\ t_2^k), \ldots,\ (t_{mk}^k,\ t_{\max}). \tag{8}$$

(If $mk=0$ this system reduces to the unique interval $(0, t_{\max})$.) In each of these intervals f_k, and hence $\frac{d}{dt}(\exp(-\lambda_{k-1}t)f_{k-1}(t))$, does not change its sign. Thus, $\exp(-\lambda_{k-1}t)f_{k-1}(t)$ is strongly monotonic. Therefore, this function, just as the function f_{k-1} has no more than one zero in each interval from (8). We compute the values

$$f_{k-1}(0),\quad f_{k-1}(t_1^k), \ldots,\ f_{k-1}(t_{mk}^k),\ f_{k-1}(t_{\max}). \tag{9}$$

The number of odd-multiple zeros of f_{k-1} is equal to the number of sign changes in the sequence (9). In addition, each interval (t_j^k, t_{j+1}^k), $0\leqq j\leqq mk$, is a localization interval for one of these zeros if and only if

$$f_{k-1}(t_j^k)f_{k-1}(t_{j+1}^k)<0 \quad (t_0^k=0,\ t_{mk+1}^k=t_{\max}).$$

It is trivial to prove this assertion if there are no zeros in the sequence (9). However, it holds also in case there are such zeros. In fact, let $f_{k-1}(t_j^k)=0$, $0\leqq j\leqq mk+1$. In the first place, t_j^k cannot be a sign change-point of function f_{k-1}, which we are interested in. Indeed, for $1\leqq j\leqq mk$ the point t_j^k being odd-multiple zero of f_k, could be only even-multiple zero of f_{k-1}; as to t_0^k and t_{mk+1}^k, they do not belong to the considered interval $(0, t_{\max})$ at all. Secondly, the adjacent intervals $(t_{j-1}^k,\ t_j^k)$, $(t_j^k,\ t_{j+1}^k)$ cannot contain odd-multiple zeros of f_{k-1}, since $\exp(-\lambda_{k-1}t)f_{k-1}(t)$ is monotonic in both intervals and vanishes at t_j^k.

After we have obtained localization intervals for all the sign-change points of function f_{k-1} in $(0, t_{\max})$, we apply again one of the known methods for determination of these points. Next we analogously obtain sign-change points of f_{k-2} and so on until coming to $f_1=f$. Thus, the algorithm can be characterized as an alternation of localization processes with classical procedures for computing localized zeros.

3. Before we go on to consider the case when eigenvalues may be not real, we shall make some remarks.

Above it was implicitly assumed that all computations (determination of localized zeros, calculation of $f_k(t)$ values, etc.) are exact. In such an idealized situation the presented algorithm finds, in fact, all zeros with regard to their multiplicity. But it is clear that, as a rule, there are errors in real computations. In this connection an even-multiple zero can, depending on the case, either "disappear" or "provide" two very close sign changes. Inversely, it is possible as well to loose a pair of actual close-to-each-other sign changes. We note, however, that appearance or disappearance of such pairs is, in some sense, an unusual event. Moreover, as it is easy to see, such effects do not have noticeable influence on the quality of the control.

The computer realization of the algorithm just presented does not cause peculiar difficulties, although it is appreciably more complicated than trivially calculating values of f at the points $h, 2h, \ldots$, where $h \ll 1$. The computing complexity of the latter method is equal to $Cnt_{\max}h^{-1}$, where C is a constant. The computing complexity of the algorithm presented is a sum of two components. The first component does not depend on the desired precision h and, as it is easy to calculate, is not greater than $C_1 n^3$ (here the following actions are contained: constructing f_k functions, computing f_{k-1}-values at the sign-change points of f_k, defining the number of sign changes in corresponding sequences). The second component is connected with the computing complexity D of the classical procedure for determination (to within h) a localized zero and the number of turns to this procedure, which as it is easy to check, does not surpass $C_2 n^2$ (C_1 and C_2 are constants). In particular, if the bisection method is used, then $D \leqq \log_2 Ct_{\max}h^{-1}$). Hence, the total computing complexity of the algorithm is not greater than $C_1 n^3 + C_2 n^2 D$. Thus, for small h and not large n (i.e. in a situation typical for optimal control problems) our algorithm requires considerably less computing time. Clearly, at the expense of increasing h, it is possible to shorten the time used for calculating values $f(h), f(2h), \ldots$, but in this way one may omit a real sign change of f (which, as it has been noted, cannot occur with our algorithm).

Principally new difficulties do not arise in case the matrix A has multiple (real) eigenvalues. It is true that then the presentation becomes more cumbersome — constants c_k in (2) are exchanged by polynomials of t, Descartes system (5) is more complicated, expression for $t_{\max}$ may need modification (if λ_n is a multiple eigenvalue). But the basic idea — the one of constructing the function sequence (6) by formula (7), applying Descartes rule and arguments implied by monotonicity for localizing zeros of f_k — remains unchanged.

Unlike presence of multiple eigenvalues, presence of complex eigenvalues considerably affects the algorithm structure. We go on to consider this case.

4. One of the basic new aspects is that the number of zeros of f in $(0, \infty)$ can be infinite and therefore a question about their upper bound, generally speaking, does not make sense.

Note, however, that in case one of the real eigenvalues — say λ_n — is greater than the real parts of the others, the number of zeros of f in $(0, \infty)$ is finite. As an upper bound for these zeros we can take the value

$$t_{\max} = (\lambda_n - \max_{k \in K} \operatorname{Re} \lambda_k)^{-1} \ln(|K| \max_{k \in K} |c_k|/c_n),$$

where $K=\{k: c_k c_n<0$ or $Im\lambda_k\neq 0\}$. The proof of this assertion is similar to the one given for real λ_k.

Returning to the time-optimal control problem we note that we need control switches only for some interval $(0, T)$. Hence, independently of λ_k nature, the question is always about determining finite number of f zeros.

For the sake of presentation clarity we begin with the case when there is only one pair $\lambda_{1,2}=\rho\pm i\eta$ $(\eta\neq 0)$ of complex conjugate eigenvalues, while $\lambda_3, \lambda_4, \ldots, \lambda_n$ are real and, for notational simplicity, distinct. We go over from $f(t)$ to the quasi-polynomial $g(t)=\exp(-\rho t)f(t)$, which has the same sign as $f(t)$ and we choose ηt as a new independent variable. In this way the problem reduces to a problem of determining sign changes of the quasi-polynomial

$$g(t)=c_1\sin t+c_2\cos t+\sum_{k=3}^{n} c_k\exp(\delta_k t),$$

where $\delta_k=(\lambda_k-\rho)/\eta$, $k=3, 4, \ldots, n$.

Consider the function

$$g_1(t)=\left(\frac{d^2}{dt^2}+1\right)g=\sum_{k=3}^{n} c_k(\delta_k^2+1)\exp(\delta_k t). \tag{10}$$

Since in the quasi-polynomial (10) every δ_k (as well as c_k) is real, we can use the algorithm presented above for determining all its sign changes (their number is obviously not greater than $n-3$) in $(0, \infty)$.

We show now, how knowing all the sign changes of g_1, one can localize and obtain the sign changes of g. The specifity of complex eigenvalues implies, in particular, that we have to consider separately the intervals $(0, \pi)$, $(\pi, 2\pi), \ldots$ The choice of these intervals is in connection with the fact, that in each of them a real factorization of the operator $\frac{d^2}{dt^2}+1$ can be performed:

$$\left(\frac{d^2}{dt^2}+1\right)g(t)=\frac{1}{\sin t}\frac{d}{dt}\left(\sin^2 t\,\frac{d}{dt}\left(\frac{1}{\sin t}\,g(t)\right)\right), \tag{11}$$

$$j\pi<t<(j+1)\pi, \quad j=0, 1, \ldots$$

(The concrete kind of a smooth function g is of no importance here.) This is a particular case of a well-known Polya theorem, which has been considered by Poincare [3]. Since all the construction for the intervals $(0, \pi)$, $(\pi, 2\pi), \ldots$ are similar, it is sufficient to examine one of them, for example, $(0, \pi)$.

Let $t_1^1, t_2^1, \ldots, t_m^1$ $(m\geqq 0)$ be sign-change points of the function g_1 in $(0, \pi)$. By virtue of the factorization (11) the number of sign changes of the function g in the interval $(0, \pi)$ is not greater than $m+2$. We go to localize these points. Assume

$$g_2(t)=\frac{dg}{dt}\sin t+g(t)\cos t=\sin^2 t\,\frac{d}{dt}\left(\frac{1}{\sin t}\,g(t)\right). \tag{12}$$

By virtue of (10) and (11) $\frac{dg_2}{dt} = \sin t\, g_1(t)$. From here it can be deduced that the function g_2 is strongly monotonic in each of the intervals

$$(t_0^1,\ t_1^1),\ (t_1^1,\ t_2^1),\ldots,\ (t_{m-1}^1,\ t_m^1),\ (t_m^1,\ t_{m+1}^1) \quad (t_0^1=0,\ t_{m+1}^1=\pi)$$

since in each of them g_1 does not change its sign (to within even-multiple zeros). By computing the values

$$g_2(t_0^1),\ g_2(t_1^1),\ldots,\ g_2(t_m^1),\ g_2(t_{m+1}^1)$$

and fixing the sign changes in this sequence we find localization intervals for all the sign-change points of g_2 in $(0, \pi)$, as it has been done earlier. By applying again some classical procedures for determining localized zeros we obtain the sign-change points of g_2; let us denote them by $t_1^2,\ t_2^2,\ldots,\ t_j^2$ $(0 \leqq j \leqq m+1)$. By virtue of (12) the intervals

$$(t_0^2,\ t_1^2),\ (t_1^2,\ t_2^2),\ldots,\ (t_j^2,\ t_{j+1}^2) \quad (t_0^2=0,\ t_{j+1}^2=\pi) \tag{13}$$

are intervals of strong monotonicity for the function $\frac{g(t)}{\sin t}$ and, therefore, in each of them g has not more than one sign change. In order to select the intervals, which contain a sign-change point we compute the values

$$g(t_0^2),\ g(t_1^2),\ldots,\ g(t_{j+1}^2). \tag{14}$$

Let us consider at first the case $g(t_0^2)g(t_{j+1}^2)\neq 0$. We can use the previous rule: $(t_k^2,\ t_{k+1}^2)$, $0 \leqq k \leqq j$, is a localization interval for a sign-change point of the function g if and only if $g(t_k^2)g(t_{k+1}^2)<0$. The proof for all intervals (13) except the extreme intervals $(t_0^2,\ t_1^2)$ and $(t_j^2,\ t_{j+1}^2)$ is similar to the one given above for such a situation. As to the extreme intervals, the monotonic function $\frac{g(t)}{\sin t}$ has a singularity in the points t_0^2 and t_{j+1}^2. Note, however, that the sign of $\frac{g(t)}{\sin t}$ in the right half-neighbourhood of t_0^2 (left half-neighbourhood of t_{j+1}^2) coincides with the sign of $g(t_0^2)$ $(g(t_{j+1}^2))$.

In the case $g(t_0^2)g(t_{j+1}^2)=0$ we can act in the following way. If $g(t_0^2)=0$, one is to replace $g(t_0^2)$ in (14) by $\dot{g}(t_0^2)$; if $g(t_{j+1}^2)\doteq 0$, one is to change this value by $-\dot{g}(t_{j+1}^2)$. We clarify this, for example, for $t_0^2(=0)$. It is easy to see, that because of $g(0)=0$ the value $\dot{g}(0)$ is equal, in fact, to the value of $\frac{g(t)}{\sin t}$ (predetermined up to continuity) at $t=0$. Hence, if $\dot{g}(0)g(t_1^2)<0$ then $(0,\ t_1^2)$ is a localization interval for some sign-change point of g and if $\dot{g}(0)g(t_1^2)\geqq 0$ then $(0,\ t_1^2)$ does not contain a sign-change point of g.

After we have found all localization intervals for sign-change points of g, we again use some classical procedure to determine them. As a result we have all

sign-change points of g in $(0, \pi)$ and we find out the sign of g between these points. Next, the same is repeated for the interval $(\pi, 2\pi)$ and so on. There is no problem with the extreme points $t = k\pi$, $k = 1, 2, \ldots$, since on passing we clarify the sign of g at the both sides of these points. The process is going on until we obtain all sign change points of g, i. e. all switches of the control $u(t)$, in the interval of interest $(0, T)$ (naturally with regard to their upper bound t_{max}, if such exists).

Up to now we spoke about the case of one pair of complex conjugate eigenvalues. In order to cover the general case we can use the same technique for "decreasing" the number of complex λ_k. Namely, suppose there already exists an algorithm for determining the sign-change points of a quasi-polynomial with $m-1$ pairs of complex λ_k in an arbitrary interval $(0, T)$. We are to obtain all sign-change points of a quasi-polynomial (2), lying in $(0, T)$, with m pairs of complex λ_k. Let $\lambda_{1,2} = \rho \pm i\eta$ $(\eta \neq 0)$. By coming, as above, to the quasi-polynomial $g(t) = \exp(-\rho t) f(t)$ and choosing ηt as a new independent variable we obtain an equation

$$\left(\frac{d^2}{dt^2} + 1\right) g = g_1(t),$$

where g_1 is a quasi-polynomial with $m-1$ pairs of complex λ_k. By assumption we can obtain all sign-change points of g_1 in $(0, T)$. After that the search for sign-change points of g can be performed exactly in the same way as described above.

As a pair λ_1, λ_2 any complex conjugate eigenvalue pair could be chosen. For practical convenience it is advisable to chose the pair with the greatest real part. The purpose of this recommendation originates from the desire to come, as quick as possible, to a quasi-polynomial whose greatest $Re\lambda_k$ is realized for some real λ_k. As it was already noted, such a quasi-polynomial has a finite number of zeros in $(0, \infty)$ and they are not greater than t_{max} mentioned above. It is clear, however, that such a choice is of no use if all real eigenvalues lie "on the left" of all complex ones.

The computational complexity depends not only on desired precision, length of the considered interval and n, but as well on the number of complex λ_k and values of $Im\lambda_k$. Usually, great $|Im\lambda_k|$ correspond to "quickly oscillating" quasi-polynomials for which the problem is evidently more complicated. Without going into details of determining the complexity estimate, we just note that the algorithm presented gains an advantage to a considerable extent when high precision is required.

As far as the linear time-optimal control problem is concerned there is no significant difference between stationary and non-stationary (when $A = A(t)$, $B = B(t)$) cases. A part of the constructions described above can be extended to a non-stationary case too. However, the practical value of such extensions is obviously insignificant. The efficiency of the algorithm proposed is determined first of all by the fact that we have an explicit analytic expressions for the arising functions (quasi-polynomials). For the non-stationary case, as it is well-known, such a situation is not characteristic.

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Received 12.07.1987