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A Topological Dehn's Lemma

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Presented by P. Kenderov

This paper is devoted to an extension of the classical Dehn's lemma [6] to continuous (hence *not* necessarily PL) Dehn maps of disks into 3-manifolds. Precisely speaking, we (almost) prove the topological version of D. W. Henderson's (PL) extension of the classical Dehn's lemma [4].

Unless stated otherwise all maps are only assumed to be continuous. A map $f: D \to X$ of a disk (resp. disk-with-holes) D into a space X is a Dehn disk (resp. Dehn disk-with-holes) if $S_f \cap \partial D = \emptyset$, where $S_f = \mathrm{cl}\{x \in D \mid f^{-1}f(x) \neq x\}$ is the singular set of f. Let $\Sigma_f = f(S_f)$.

Theorem. (TOPOLOGICAL DEHN'S LEMMA). *Suppose $f: D^2 \to M^3$ is a Dehn disk in a 3-manifold with boundary M^3. Then for every neighborhood $U \subset M^3$ of Σ_f there is an embedding $F: D^2 \to M^3$ such that $F|\partial D^2 = f|\partial D^2$ and $F(D^2) - U = f(D^2) - U$.*

Corollary. (BING'S EXTENSION OF DEHN'S LEMMA [3]). *Suppose $f: D^2 \to M^3$ is a Dehn disk in a 3-manifold with boundary M^3. Then for every neighborhood $U \subset M^3$ of $f(\mathrm{int}\ D^2)$ there is an embedding $F: D^2 \to f(D) \cup U$ such that $F|\mathrm{int}\ D^2$ is locally PL.*

Proof. Follows by Theorem and [1].

Proof of the Theorem. We first consider the case when $f(D) \subset \mathrm{int}\, M$. Here is an outline of the proof: Put S_f inside pairwise disjoint PL disks with holes $C_1, \ldots, C_m \subset f^{-1}(U)$. Let $C = \bigcup_{i=1}^m C_i$. Assume that on some neighborhood of ∂C, f is a locally PL embedding.

Step 1. Consider the surface $H = f(D\text{-int } C)$. Use [1] to make H PL.

Step 2. Consider the singular surface $L = f(C)$. Use [7] to make L polyhedral.

Step 3. Now $H \cup L$ is a desired PL Dehn disk. Apply [4] to get an embedded disk $T \subset M$.

Step 4. Replace the portions of T which lie outside U by corresponding pieces of H. (See Figure 1.)

In general, the curves from $f(\partial C)$ are going to be "wildly" embedded in M so additional care must be taken to improve f near ∂C. This is achieved by using four

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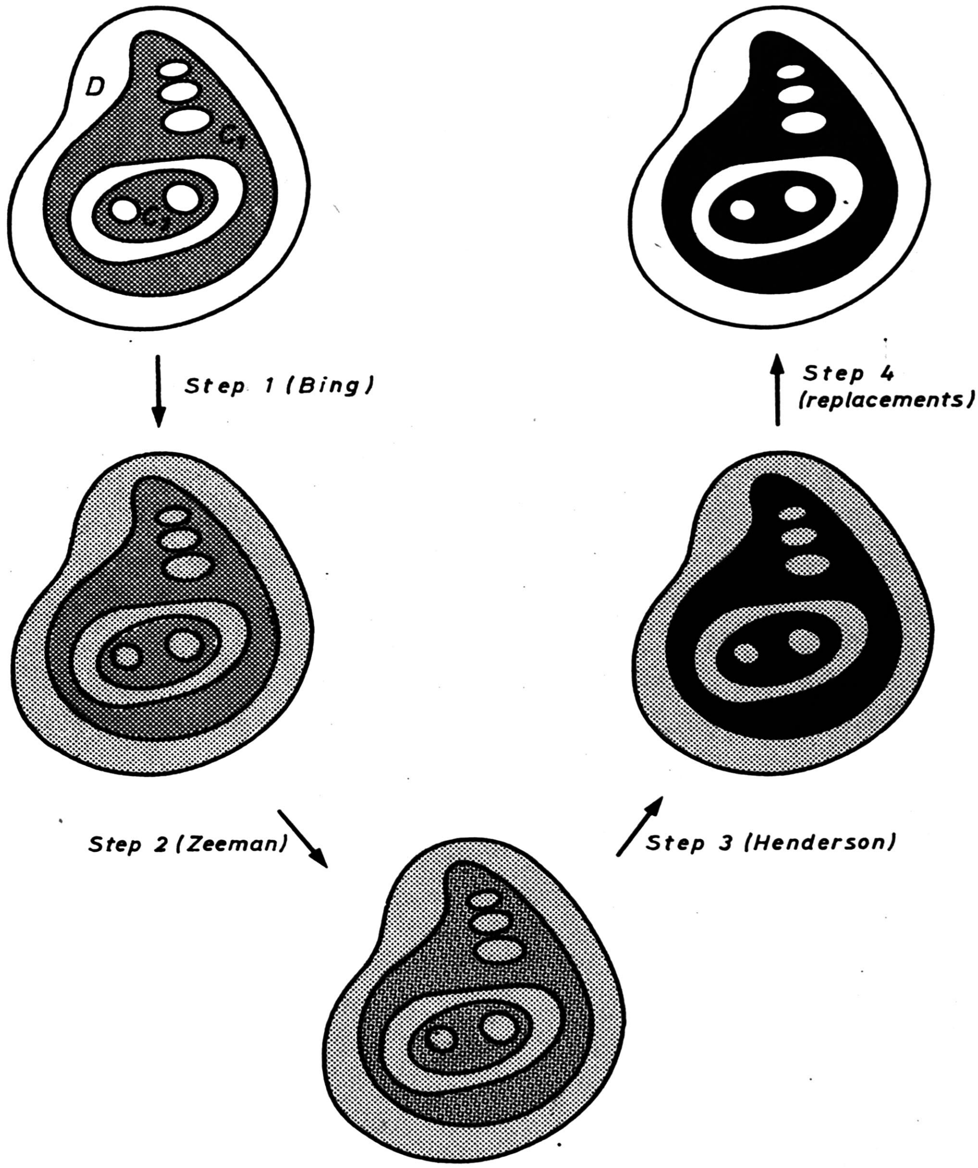

Fig. 1

concentric families of pairwise disjoint PL disks with holes rather than just one such family (our C).

Now, the details. Let $U'=f^{-1}(U)$. By [3; Theorem (4.8.3)], there exist families $\{A_i^{(j)} \mid 1 \leq i < t\}$, $1 \leq j < 4$, of pairwise disjoint PL disks with holes in U' such that:

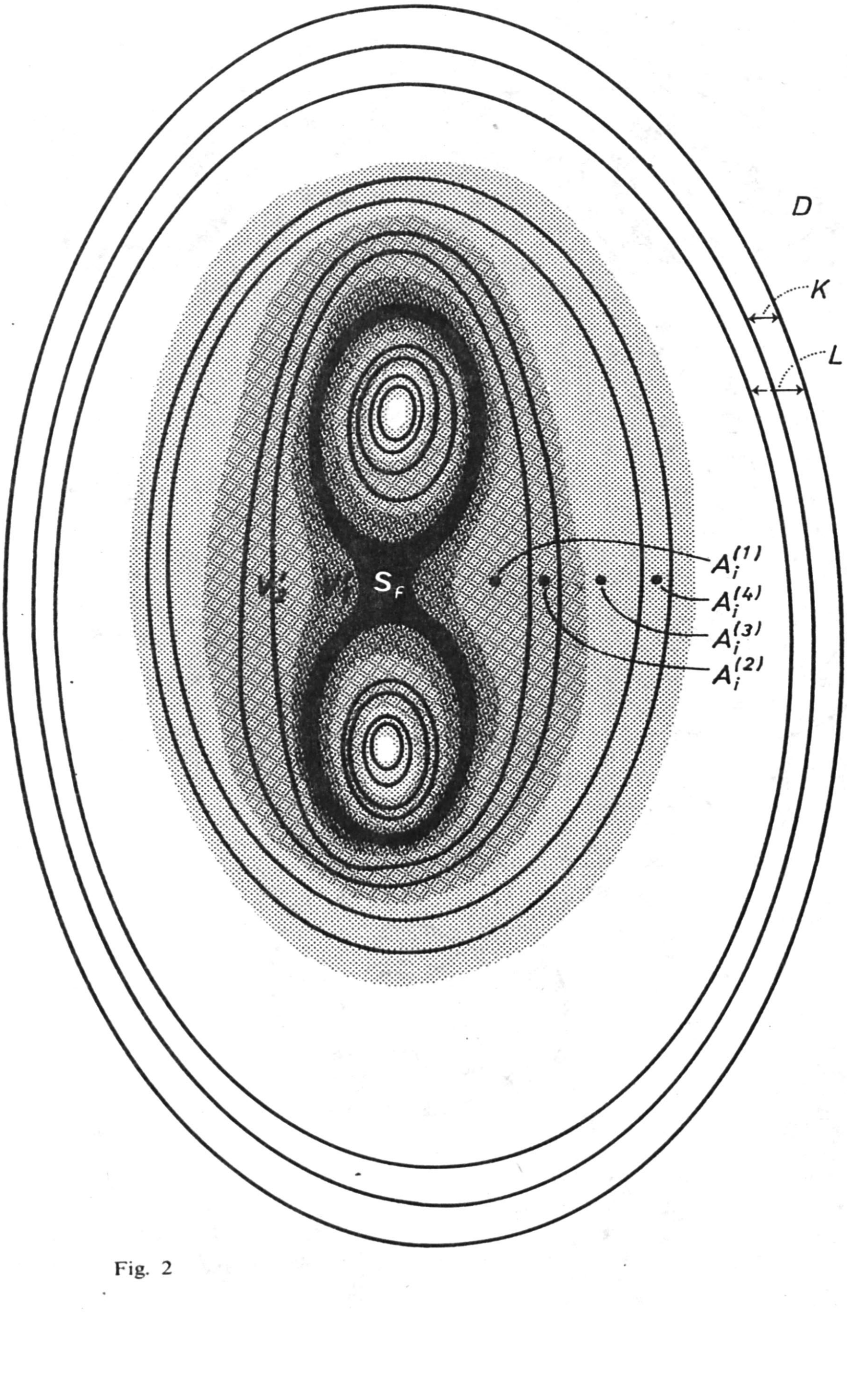

Fig. 2

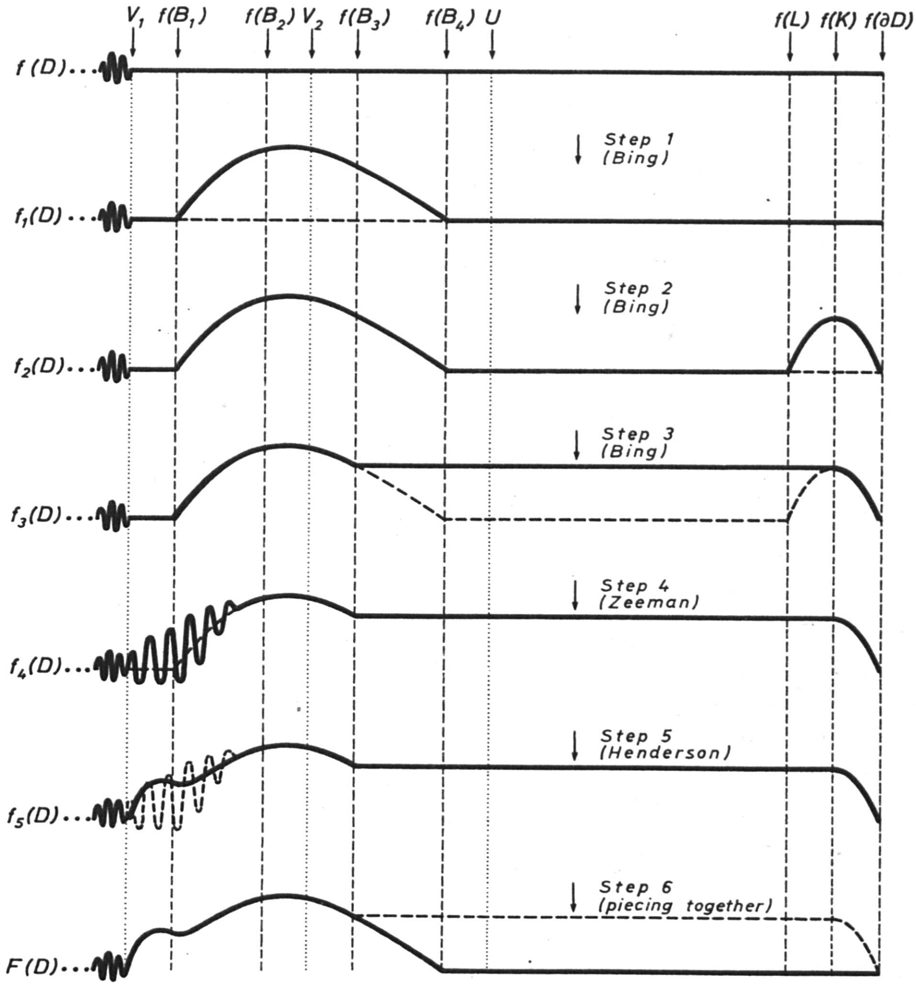

Fig. 3

(1) for each i, j, $A_i^{(j)} \subset \operatorname{int} A_i^{(j+1)}$; and
(2) $S_f \subset \operatorname{int} B_1$;

where $B_j = \bigcup_{i=1}^{t} A_i^{(j)}$. Let $k=1, 2$. By (1) and (2), $f|(D\text{-int } B_{2k-1})$ is an embedding hence $f(D\text{-int } B_{2k-1})$ is closed in M thus $V_k = U - f(D\text{-int } B_{2k-1})$ is open in M and $V_1 \subset V_2 \subset U$. Let $V_k' = f^{-1}(V_k)$. Then:

(3) $S_f \subset V_1' \subset \operatorname{int} B_1$; and

(4) $B_2 \subset V_2' \subset \operatorname{int} B_3$.

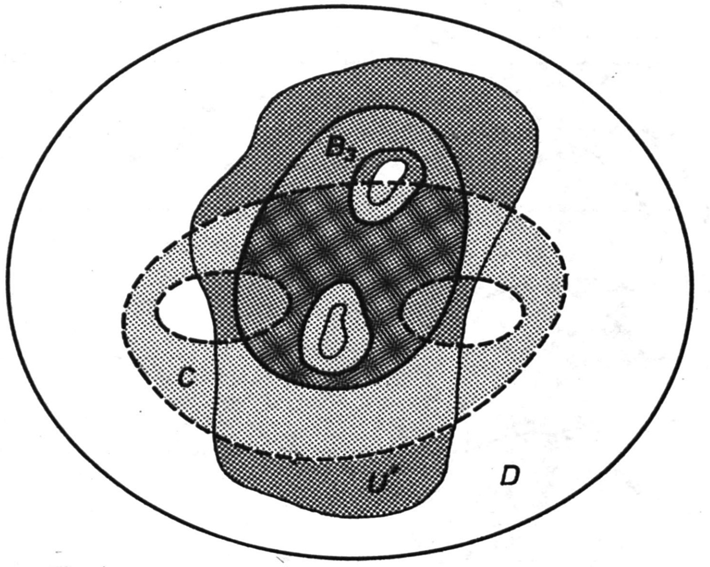

Fig. 4

Let $K \subset L \subset \dot{D} - U'$ be PL annuli such that $\partial D = \partial L \cap \partial K$. (See Figure 2.)

Apply [1] to replace f by a Dehn disk $f_1 : D \to M$ with the following properties:

(5) $f_1|(D - D_1) = f|(D - D_1)$;

(6) $f_1|D_1$ is locally PL; and

(7) $S_{f_1} = S_f$,

where $D_1 = \text{int}(B_4 - B_1)$. Apply [1] again to get a Dehn disk $f_2 : D \to M$ such that:

(8) $f_2|(D\text{-int } L) = f_1|(D\text{-int } L)$;

(9) $f_2|\text{int } L$ is locally PL; and

(10) $S_{f_2} = S_{f_1}$.

Remark. We could have gotten the map f_2 from f in just one step rather than going via f_1. However, we shall need f_1 in assembling the final map F (See Figure 4.)

Another application of [1] yields a Dehn disk $f_3 : D \to M$ such that:

(11) $f_3|D_2 = f_2|D_2$;

(12) $f_3|(D - D_2)$ is locally PL; and

(13) $S_{f_3} = S_{f_2}$;

where $D_2 = \dot{K} \cup \dot{B}_3$.

Remark. If for some $j \in \{1, 2, 3, 4\}$ the simple closed curves $f(\partial B_j) \subset M$ and $f(\partial K) \subset M$ are nicely embedded in M we can skip f_1 and f_2 and just apply [1] to $f|(D\text{-int } B_j)$ to get f_3. However, if this isn't the case then we must get f_1 and f_2 first to make certain that $f(\partial(\overline{D - D_2}))$ is nicely embedded in M.

By [7] there is a Dehn disk $f_4 : D \to M$ such that:

(14) $f_4|(D\text{-int } B_2) = f_3|(D\text{-int } B_2)$;

(15) $f_4|(D\text{-}(\text{int } K \cup \partial D))$ is PL; and

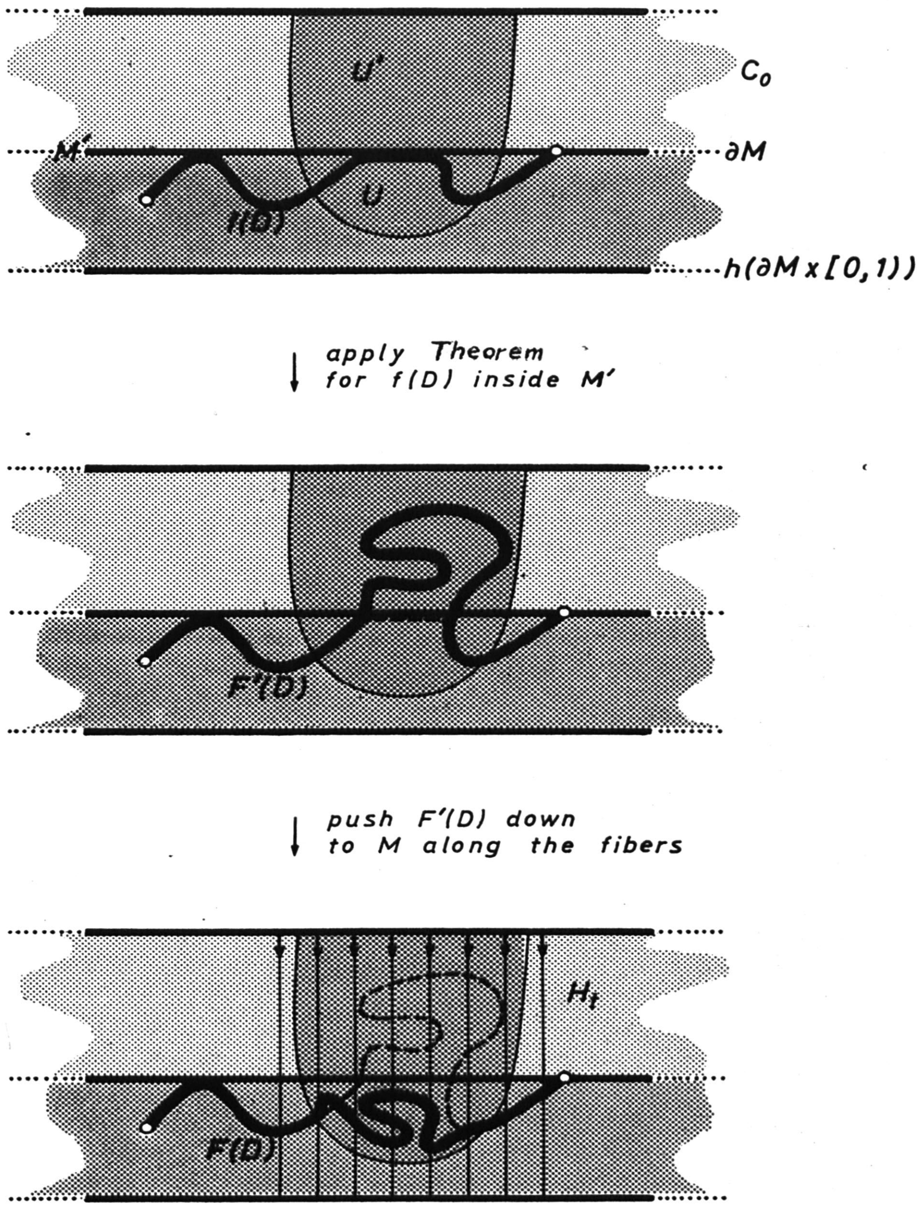

Fig. 5

(16) $S_{f_4} \subset V'_2$.

By [4], there is an embedding $f_5 : D \to M$ such that:

(17) $f_5 | \operatorname{int} D$ is locally PL;

(18) $f_5|K=f_4|K$; and

(19) $f_5(D)-V_2=f_4(D)-V_2$.

In particular, by (4), (5), (8), (11), (14), (18), and (19):

(20) $f_4(D\text{-int } B_3)\subset f_5(D)\subset f_4(D)\cup V_2$.

Note, however, that in general, f_4 and f_5 do not agree pointwisely, not even on D-int B_3.

We wish to know what regions of D are mapped by f_5 onto $f_4(D\text{-int } B_3)$. Let $C=f_5^{-1}f_4(D\text{-int } B_3)$. By (20), C is well-defined and non-empty. There exist pairwise disjoint PL disks with holes $\{E_i|\, 1\leqslant i\leqslant r\}$ such that

(21) $D-\text{int } B_3=\bigcup_{i=1}^{r} F_i$.

By (16), $f_4(D\text{-int } B_3)$ is a collection of disks with holes, namely $f_4(E_i)$'s hence by (20) so is $C=\bigcup_{i=1}^{r} f_5^{-1}f_4(E_i)$. Define $F: D\to M$ by

$$(22) \qquad F(x)=\begin{cases} f_1\cdot(f_4|_{(D-\text{int } B_3)})^{-1}\cdot f_5(x); & x\in C\cdot \\ f_5(x) & ; \quad x\in D-(\text{int } C\cup\partial D)\end{cases}$$

The map F is well-defined: each $x\in C$ lies in precisely one disk with holes $f_5^{-1}f_4(E_i)$, so $f_5(x)$ lies in $f_4(E_i)$. Now, by (16), $f_4|(D-\text{int } B_3)$ is an embedding, therefore f_4^{-1} is well-defined over $f_4(D-\text{int } B_3)$. Also, by (8), (11), (14), and (21):

(23) $f_1|\partial B_3=f_4|\partial B_3$

hence for every $x\in\partial C-\partial D$, $f_1\circ(f_4|_{D-\text{int } B_3)})^{-1}\circ f_5(x)=f_1\circ(f_4|_{\partial B_3})^{-1}\circ f_5(x)$ $=id\circ f_5(x)=f_5(x)$ so F is well-defined. By (3), (7), and (20)-(23), F is an embedding and by (5), (8), (11), (14), (19), and (20)-(23), $F(D)-U=f(D)-U$ as desired. (See Figure 3.)

Remark. The disk $F(D)$ is thus obtained from $f_5(D)$ by glueing together the pieces $f_5(D-\text{int } C\cup\partial D)$ and $f_1(D-\text{int } B_3)$ using the homeomorphism $f_4^{-1}\circ f_5$ on $\partial C-\partial D$. (See Figure 4.)

It remains to consider the case when $f(D)\cap\partial M\neq\emptyset$. Attach a collar $C_0=\partial M\times[0,\,1]$ to ∂M and extend the neighborhood U over C_0 in the obvious way -- let $U'=U\cup((U\cap\partial M)\times[0,\,1])$. Let $M'=M\cup_{\partial M}C_0$. Apply the preceding case to the 3-manifold M' to get an embedding $F': D\to M$: such that $F'(D)-U'=f(D)-U'$ and $F'|\partial D=f|\partial D$. The disk $F'(D)$ may now hit $M'-M$ so we wish to push it in M by a nice ambient PL isotopy with support in U'. Note that by taking a PL collar $h: \partial M\times[0,\,1]\to M$ of ∂M in M we get a "product structure" in M' close to ∂M, i.e., $C_0\cup h(\partial M\times[0,\,1])$ is PL homeomorphic to $\partial M\times[-1,\,1]$ where we identify ∂M with $\partial M\times\{0\}$. We can now construct the desired ambient PL isotopy $H_t: M'\times[0,\,1]\to M'$ by pushing $F'(D)$ from $M'-M$ down to M by means of stretching down the fibers of the product $\partial M\times[-1,\,1]$. Finally, let $F: D\to f(D)\cup U$ be given by $F=H_1F'$. (See Figure 5.)

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