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An Asymptotic Formula Having no Remainder Term for the Orthogonal Hahn Polynomials of Discrete Variable

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Presented by Bl. Sendov

The classical orthogonal Hahn polynomials may be defined by the following equality

$$\begin{aligned}(1)\qquad &\binom{x+\alpha}{x}\binom{N-1-x+\beta}{N-1-x}\binom{N-1}{n} Q_n(x;\ \alpha,\ \beta,\ N)\\ &=\binom{n+\beta}{n}\Delta^n\left[\binom{x+\alpha}{\alpha+n}\binom{N-1-x+\beta+n}{\beta+n}\right],\end{aligned}$$

where $\Delta f(x)=f(x+1)-f(x)$, $\Delta^n f(x)=\Delta(\Delta^{n-1}f(x))$ $(n\geqq 2)$.

Let $0\leqq\alpha$, β be integers

$$(2)\qquad \sum_{k=0}^{m} S(m,\ k,\ \gamma)\,x^k=(x+\gamma)\dots(x+\gamma-m+1),$$

in particular, at $\gamma=0$ $S(m,k,0)=S(m,k)$ are the Stirling numbers of the first kind;

$$(3)\qquad \sum_{l=0}^{r}\sigma(r,\ l)x\dots(x-l+1)=x^r\qquad (x\dots(x-l+1)\equiv 0\qquad (l=0)),$$

that is $\sigma(r,l)$ $(0\leqq l\leqq r,\ r=0,\ 1,\ 2,\ \dots)$ are the Stirling numbers of the second kind;

$$F_{r,k,j}(t)=S(n+\alpha,\ k,\ \alpha)S(n+\beta,\ j)\sigma(r,\ n)\,N^{k+j-r-n}$$

$$\times\begin{cases}(-1)^{r-j}2^{2r-k-j}(1+t)^{k-r}(1-t)^{j-r}P_r^{(j-r,\,k-r)}(t) & (j,\ k\geqq r),\\[2mm] \dfrac{k!\,j!}{r!\,(k+j-r)!}\left(\dfrac{1+t}{2}\right)^{k-r}P_j^{(r-j,\,k-r)}(t) & (j<r\leqq k),\\[2mm] \dfrac{k!\,j!}{r!\,(k+j-r)!}\left(\dfrac{t-1}{2}\right)^{j-r}P_k^{(j-r,\,r-k)}(t) & (k<r\leqq j),\\[2mm] P_{j+k-r}^{(r-j,\,r-k)}(t) & (k,\ j<r\leqq j+k),\\[2mm] 0 & (j+k<r),\end{cases}$$

where

$$(4)\qquad P_n^{(\gamma,\nu)}(x)=\frac{(-1)^n}{2^n n!}(1-x)^{-\gamma}(1+x)^{-\nu}\frac{d^n}{dx^n}\{(1-x)^{n+\gamma}(1+x)^{n+\nu}\}$$

is the Jacoby polynomial.

Theorem. *The following asymptotic formula concerning N is true for the integers* $\alpha,\ \beta\geqq 0$

$$(5)\qquad \frac{(n+\alpha)!\,(N-1)!\,[N(1+t)/2+1]_\alpha[N(1-t)/2]_\beta}{(-1)^{n+\beta}n!\,(N-n-1)!\,N!\alpha!}Q_n[\tfrac{N}{2}(1+t);\ \alpha,\ \beta,\ N] = \sum_{\substack{n\leqq r\leqq 2n+\alpha+\beta\\ 0\leqq k\leqq n+\alpha\\ 0\leqq j\leqq n+\beta,}} F_{r,k,j}(t),$$

where $(q)_k=q(q+1)\dots(q+k-1)(k\geqq 1)$, $(q)_0=1$.

Proof. Let

$$\mu(x)=\binom{x+\alpha}{x}\binom{N-1-x+\beta}{N-1-x},\quad S(x)=\binom{x+\alpha}{\alpha+n}\binom{N-1-x+\beta+n}{\beta+n}.$$

Then from (1) we deduce

$$(6)\qquad \begin{aligned}\binom{N-1}{n}\mu(x)Q_n(x;\ \alpha,\ \beta,\ N)&=\binom{n+\beta}{n}\Delta^n S(x)\\ &=(-1)^n\binom{n+\beta}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}S(x+k)\\ &=(-1)^n\binom{n+\beta}{n}\sum_{k=0}^{n}\frac{S(k+x)}{\cos(k\pi)}\frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)}\\ &=\binom{n+\beta}{n}\frac{(-1)^n n!}{2i}\int_0\frac{S(x+z)dz}{\sin(\pi z)\Gamma(n-z+1)\Gamma(z+1)},\end{aligned}$$

where 0 is the closed loop including the point $z=0,\ 1,\dots,\ n$. Since $\sin(\pi z)\Gamma(n+1-z)=(-1)^n\pi/\Gamma(z-n)$, from (6) we find

$$(7)\qquad \binom{N-1}{n}\mu(x)Q_n(x;\ \alpha,\ \beta,\ N)=\binom{n+\beta}{n}\frac{n!}{2\pi i}\int_0\frac{S(x+z)\Gamma(z-n)}{\Gamma(z+1)}dz.$$

It is known that the following expansion is true for the Stirling numbers of the second kind

$$(8)\qquad \frac{\Gamma(z-n)}{\Gamma(z+1)}=\frac{1}{z(z-1)\dots(z-n)}=\sum_{r=n}^{\infty}\frac{\sigma(r,n)}{z^{r+1}}\quad(|z|>n).$$

Since $S(t)$ is the polinomial of the $2n+\alpha+\beta$ power, from (7) and (8) we deduce

(9)
$$\binom{N-1}{n}\mu(x)Q_n(x;\ \alpha,\ \beta,\ N) = \binom{n+\beta}{n}\sum_{r=n}^{2n+\alpha+\beta}\frac{n!\,\sigma(r,n)}{2\pi i}\int_0\frac{S(z)dz}{(z-x)^{r+1}}.$$

It follows from (2) that

(10)
$$S(z)=\frac{(-1)^{n+\beta}}{(n+\alpha)!\,(n+\beta)!}\sum_{k=0}^{n+\alpha}\sum_{j=0}^{n+\beta}S(n+\alpha,\ k,\ \alpha)\,S(n+\beta,\ j)z^k(z-N)^j.$$

The equalities (9) and (10) give

(11)
$$\binom{N-1}{n}\mu(x)Q_n(x;\ \alpha,\ \beta,\ N) = \binom{n+\beta}{n}\sum_{r=n}^{2n+\alpha+\beta}\frac{\sigma(r,n)n!}{r!}\sum_{\substack{0\leq k\leq n+\alpha\\0\leq j\leq n+\beta}}\frac{S(n+\alpha,k,\alpha)\,S(n+\beta,j)}{(n+\alpha)!\,(n+\beta)!\,(-1)^{n+\beta}}\frac{d^r}{dx^r}\{x^k(x-N)^j\}.$$

Let $x=N(1+t)/2$, $\mu_N(t)=\mu(N(1+t)/2)$. From (4) and (11) we find

(12)
$$\binom{N-1}{n}\mu_N(t)\,Q_n\Big(\frac{N}{2}(1+t)\Big)=\frac{(-1)^{n+\beta}}{(n+\alpha)!\,\beta!}\sum_{r=n}^{2n+\alpha+\beta}$$
$$\times\sum_{\substack{0\leq k\leq n+\alpha\\0\leq j\leq n+\beta\\k+j\geq r}}\frac{\sigma(r,n)S(n+\alpha,k,\alpha)\,S(n+\beta,j)}{(N/2)^{r-k-j}2^{-r}(-1)^{r-j}}(1+t)^{k-r}(1-t)^{j-r}P_r^{(j-r,\,k-r)}(t).$$

For the Jacoby polynomials we know the following formula

$$\binom{n}{l}P_n^{(-l,\,v)}(x)=\binom{n+v}{l}\Big(\frac{x-1}{2}\Big)^l P_{n-l}^{(l,\,v)}(x),$$

where l — the integer, $1\leq l\leq n$. It gives

(13) $$(1-t)^{j-r}P_r^{(j-r,\,k-r)}(t)=\binom{k}{r-j}\binom{r}{r-j}^{-1}(-2)^{j-r}P_j^{(r-j,\,k-r)}(t)\quad (j<r),$$

(14)
$$(1+t)^{k-r}P_r^{(j-r,\,k-r)}(t)=(1+t)^{k-r}P_r^{(k-r,\,j-r)}(-t)(-1)^r = 2^{k-r}\binom{j}{r-k}\binom{r}{r-k}^{-1}P_k^{(r-k,\,j-r)}(-t)(-1)^k\quad (k<r),$$

(15) $$(1+t)^{k-r}(1-t)^{j-r}P_r^{(j-r,\,k-r)}(t)=(-1)^{r-j}2^{j+k-2r}P_{j+k-r}^{(r-j,\,r-k)}(t).$$

Since

$$\mu_N(t)=\mu(\frac{N}{2}(1+t))=\frac{1}{\alpha!\,\beta!}\,[N(1+t)/2+1]_\alpha[\frac{N}{2}(1-t)]_\beta,$$

then from (12)-(15) we proved the theorem.

Assuming $\alpha=\beta=0$ from the theorem for the Chebychev polynomials $Q_n(x,N)=Q_n(x;0,0,N)$, we deduce the following result.

Corollary 1. *For all t the following asymptotic formula exists*

$$\frac{(-1)^n(N-1)!}{(N-1-n)!\,N^n}Q_n(\frac{N}{2}(1+t);\ 0,\ 0,\ N)$$

(16)

$$=\sum_{r=n}^{2n}\sum_{\substack{0\leq k,j\leq n\\ r\leq k+j}}\sigma(r,n)\frac{S(n,k)\,S(n,j)}{N^{r+n-j-k}}P_{j+k-r}^{(r-j,\,r-k)}(t).$$

Since $(t-1/N<\theta_i<t)$

$$P_n^{(0,0)}(t-\frac{1}{N})=P_n^{(0,0)}(t)-\frac{n+1}{2N}P_{n-1}^{(1,1)}(t)+\frac{(n+1)(n+2)}{8N^2}P_{n-2}^{(2,2)}(\theta_1),$$

$$P_{n-1}^{(0,1)}(t-\frac{1}{N})=P_{n-1}^{(0,1)}(t)-\frac{n+1}{2N}P_{n-2}^{(1,2)}(\theta_2),$$

$$P_{n-1}^{(1,0)}(t-\frac{1}{N})=P_{n-1}^{(1,0)}(t)-\frac{n+1}{2N}P_{n-2}^{(2,1)}(\theta_3),$$

$$P_{n-1}^{(1,1)}(t-\frac{1}{N})=P_{n-1}^{(1,1)}(t)-\frac{n+2}{2N}P_{n-2}^{(2,2)}(\theta_4),$$

$$\sigma(n,n)=1,\quad S(n,n)=1,\quad \sigma(n+1,n)=\binom{n+1}{2},\quad S(n,n-1)=-\binom{n}{2},$$

$$-\frac{n+1}{2}P_{n-1}^{(1,1)}(t)+S(n,n-1)[P_{n-1}^{(1,0)}(t)+P_{n-1}^{(0,1)}(t)]+\sigma(n+1,n)P_{n-1}^{(1,1)}(t)\equiv 0,$$

then from (16) we find

$$\frac{(-1)^n(N-1)!}{N^n(N-1-n)!}Q_n(\frac{N}{2}(1+t-\frac{1}{N});\ 0,\ 0,\ N)=P_n^{(0,0)}(t)$$

(17)

$$+\frac{1}{N^2}\sum_{\substack{n\leq r\leq 2n\\ 0\leq k,j\leq n\\ r\leq j+k\\ r+n-j-k\geq 2}}\frac{\sigma(r,n)\,S(n,k)\,S(n,j)}{N^{r+n-j-k-2}}P_{j+k-r}^{(r-j,\,r-k)}(t-\frac{1}{N})+R_{n,N}(t),$$

where

$$R_{n,N}(t)=\frac{1}{N^2}\Big[\frac{(n+1)(n+2)}{8}P_{n-2}^{(2,2)}(\theta_1)$$

(18)

$$-\frac{n+1}{2}S(n,\ n-1)\big(P_{n-2}^{(1,2)}(\theta_2)+P_{n-2}^{(2,1)}(\theta_3)\big)-\frac{n+2}{2}\sigma(n+1,\ n)P_{n-2}^{(2,2)}(\theta_4)\Big].$$

From (17) and (18) we deduce

Corollary 2. *For fixed n and* $N\to\infty$

$$(-1)^n\frac{(N-1)!}{N^n(N-1-n)!}Q_n\Big(\frac{N}{2}\Big(1+t-\frac{1}{N}\Big);\ 0,\ 0,\ N\Big)=P_n^{(0,0)}(t)+O\Big(\frac{1}{N^2}\Big).$$

This result was obtained in another way in [2].

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