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An Asymptotic Formula Having no Remainder Term for the Orthogonal Hahn Polynomials of Discrete Variable

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Presented by Bl. Sendov

The classical orthogonal Hahn polynomials may be defined by the following equality

$$(1) \quad \begin{aligned} & \binom{x+\alpha}{x} \binom{N-1-x+\beta}{N-1-x} \binom{N-1}{n} Q_n(x; \alpha, \beta, N) \\ & = \binom{n+\beta}{n} \Delta^n \left[\binom{x+\alpha}{\alpha+n} \binom{N-1-x+\beta+n}{\beta+n} \right]. \end{aligned}$$

where $\Delta f(x) = f(x+1) - f(x)$, $\Delta^n f(x) = \Delta(\Delta^{n-1} f(x))$ ($n \geq 2$).

Let $0 \leq \alpha, \beta$ be integers

$$(2) \quad \sum_{k=0}^m S(m, k, \gamma) x^k = (x+\gamma) \dots (x+\gamma-m+1),$$

in particular, at $\gamma=0$ $S(m, k, 0) = S(m, k)$ are the Stirling numbers of the first kind;

$$(3) \quad \sum_{l=0}^r \sigma(r, l) x \dots (x-l+1) = x^r \quad (x \dots (x-l+1) \equiv 0 \quad (l=0)),$$

that is $\sigma(r, l)$ ($0 \leq l \leq r$, $r=0, 1, 2, \dots$) are the Stirling numbers of the second kind;

$$\begin{aligned} F_{r,k,j}(t) &= S(n+\alpha, k, \alpha) S(n+\beta, j) \sigma(r, n) N^{k+j-r-n} \\ &\times (-1)^{r-j} 2^{2r-k-j} (1+t)^{k-r} (1-t)^{j-r} P_r^{(j-r, k-r)}(t) \quad (j, k \geq r), \\ &\frac{k! j!}{r! (k+j-r)!} \left(\frac{1+t}{2} \right)^{k-r} P_j^{(r-j, k-r)}(t) \quad (j < r \leq k), \\ &\frac{k! j!}{r! (k+j-r)!} \left(\frac{t-1}{2} \right)^{j-r} P_k^{(j-r, r-k)}(t) \quad (k < r \leq j), \\ &P_{j+k-r}^{(r-j, r-k)}(t) \quad (k, j < r \leq j+k), \\ &0 \quad (j+k < r), \end{aligned}$$

where

$$(4) \quad P_n^{(\gamma, \nu)}(x) = \frac{(-1)^n}{2^n n!} (1-x)^{-\gamma} (1+x)^{-\nu} \frac{d^n}{dx^n} \{(1-x)^{n+\gamma} (1+x)^{n+\nu}\}$$

is the Jacoby polynomial.

Theorem. *The following asymptotic formula concerning N is true for the integers $\alpha, \beta \geq 0$*

$$(5) \quad \begin{aligned} & \frac{(n+\alpha)! (N-1)! [N(1+t)/2+1]_\alpha [N(1-t)/2]_\beta}{(-1)^{n+\beta} n! (N-n-1)! N! \alpha!} Q_n \left[\frac{N}{2}(1+t); \alpha, \beta, N \right] \\ &= \sum_{\substack{n \leq r \leq 2n+\alpha+\beta \\ 0 \leq k \leq n+\alpha \\ 0 \leq j \leq n+\beta}} F_{r,k,j}(t), \end{aligned}$$

where $(q)_k = q(q+1)\dots(q+k-1)$ ($k \geq 1$), $(q)_0 = 1$.

Proof. Let

$$\mu(x) = \binom{x+\alpha}{x} \binom{N-1-x+\beta}{N-1-x}, \quad S(x) = \binom{x+\alpha}{\alpha+n} \binom{N-1-x+\beta+n}{\beta+n}.$$

Then from (1) we deduce

$$(6) \quad \begin{aligned} & \binom{N-1}{n} \mu(x) Q_n(x; \alpha, \beta, N) = \binom{n+\beta}{n} \Delta^n S(x) \\ &= (-1)^n \binom{n+\beta}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} S(x+k) \\ &= (-1)^n \binom{n+\beta}{n} \sum_{k=0}^n \frac{S(k+x)}{\cos(k\pi)} \frac{\Gamma(n+1)}{\Gamma(n-k+1)\Gamma(k+1)} \\ &= \binom{n+\beta}{n} \frac{(-1)^n n!}{2i} \int_0^\infty \frac{S(x+z) dz}{\sin(\pi z) \Gamma(n-z+1) \Gamma(z+1)}, \end{aligned}$$

where 0 is the closed loop including the point $z=0, 1, \dots, n$. Since $\sin(\pi z) \Gamma(n+1-z) = (-1)^n \pi / \Gamma(z-n)$, from (6) we find

$$(7) \quad \binom{N-1}{n} \mu(x) Q_n(x; \alpha, \beta, N) = \binom{n+\beta}{n} \frac{n!}{2\pi i} \int_0^\infty \frac{S(x+z) \Gamma(z-n)}{\Gamma(z+1)} dz.$$

It is known that the following expansion is true for the Stirling numbers of the second kind

$$(8) \quad \frac{\Gamma(z-n)}{\Gamma(z+1)} = \frac{1}{z(z-1)\dots(z-n)} = \sum_{r=n}^{\infty} \frac{\sigma(r, n)}{z^{r+1}} \quad (|z| > n).$$

Since $S(t)$ is the polynomial of the $2n+\alpha+\beta$ power, from (7) and (8) we deduce

$$(9) \quad \begin{aligned} & \binom{N-1}{n} \mu(x) Q_n(x; \alpha, \beta, N) \\ & = \binom{n+\beta}{n} \sum_{r=n}^{2n+\alpha+\beta} \frac{n! \sigma(r, n)}{2\pi i} \int_0^{\infty} \frac{S(z) dz}{(z-x)^{r+1}}. \end{aligned}$$

It follows from (2) that

$$(10) \quad S(z) = \frac{(-1)^{n+\beta}}{(n+\alpha)!(n+\beta)!} \sum_{k=0}^{n+\alpha} \sum_{j=0}^{n+\beta} S(n+\alpha, k, \alpha) S(n+\beta, j) z^k (z-N)^j.$$

The equalities (9) and (10) give

$$(11) \quad \begin{aligned} & \binom{N-1}{n} \mu(x) Q_n(x; \alpha, \beta, N) \\ & = \binom{n+\beta}{n} \sum_{r=n}^{2n+\alpha+\beta} \frac{\sigma(r, n) n!}{r!} \sum_{\substack{0 \leq k \leq n+\alpha \\ 0 \leq j \leq n+\beta \\ k+j \geq r}} \frac{S(n+\alpha, k, \alpha) S(n+\beta, j)}{(n+\alpha)!(n+\beta)! (-1)^{n+\beta}} \frac{d^r}{dx^r} \{x^k (x-N)^j\}. \end{aligned}$$

Let $x = N(1+t)/2$, $\mu_N(t) = \mu(N(1+t)/2)$. From (4) and (11) we find

$$(12) \quad \begin{aligned} & \binom{N-1}{n} \mu_N(t) Q_n\left(\frac{N}{2}(1+t)\right) = \frac{(-1)^{n+\beta}}{(n+\alpha)!\beta!} \sum_{r=n}^{2n+\alpha+\beta} \\ & \times \sum_{\substack{0 \leq k \leq n+\alpha \\ 0 \leq j \leq n+\beta \\ k+j \geq r}} \frac{\sigma(r, n) S(n+\alpha, k, \alpha) S(n+\beta, j)}{(N/2)^{r-k-j} 2^{-r} (-1)^{r-j}} (1+t)^{k-r} (1-t)^{j-r} P_r^{(j-r, k-r)}(t). \end{aligned}$$

For the Jacobi polynomials we know the following formula

$$\binom{n}{l} P_n^{(-l, v)}(x) = \binom{n+v}{l} \left(\frac{x-1}{2}\right)^l P_{n-l}^{(l, v)}(x),$$

where l – the integer, $1 \leq l \leq n$. It gives

$$(13) \quad (1-t)^{j-r} P_r^{(j-r, k-r)}(t) = \binom{k}{r-j} \binom{r}{r-j}^{-1} (-2)^{j-r} P_j^{(r-j, k-r)}(-t) (-1)^r, \quad (j < r),$$

$$(14) \quad \begin{aligned} & (1+t)^{k-r} P_r^{(k-r, j-r)}(t) = (1+t)^{k-r} P_r^{(k-r, j-r)}(-t) (-1)^r \\ & = 2^{k-r} \binom{j}{r-k} \binom{r}{r-k}^{-1} P_k^{(r-k, j-r)}(-t) (-1)^k \quad (k < r), \end{aligned}$$

$$(15) \quad (1+t)^{k-r} (1-t)^{j-r} P_r^{(j-r, k-r)}(t) = (-1)^{r-j} 2^{j+k-2r} P_{j+k-r}^{(r-j, r-k)}(t).$$

Since

$$\mu_N(t) = \mu\left(\frac{N}{2}(1+t)\right) = \frac{1}{\alpha! \beta!} [N(1+t)/2 + 1]_\alpha [\frac{N}{2}(1-t)]_\beta,$$

then from (12)-(15) we proved the theorem.

Assuming $\alpha = \beta = 0$ from the theorem for the Chebychev polynomials $Q_n(x, N) = Q_n(x; 0, 0, N)$, we deduce the following result.

Corollary 1. *For all t the following asymptotic formula exists*

$$(16) \quad \begin{aligned} & \frac{(-1)^n (N-1)!}{(N-1-n)! N^n} Q_n\left(\frac{N}{2}(1+t); 0, 0, N\right) \\ &= \sum_{r=n}^{2n} \sum_{\substack{0 \leq k, j \leq n \\ r \leq k+j}} \sigma(r, n) \frac{S(n, k) S(n, j)}{N^{r+n-j-k}} P_{j+k-r}^{(r-j, r-k)}(t). \end{aligned}$$

Since $(t - 1/N < \theta_i < t)$

$$\begin{aligned} P_n^{(0,0)}(t - \frac{1}{N}) &= P_n^{(0,0)}(t) - \frac{n+1}{2N} P_{n-1}^{(1,1)}(t) + \frac{(n+1)(n+2)}{8N^2} P_{n-2}^{(2,2)}(\theta_1), \\ P_{n-1}^{(0,1)}(t - \frac{1}{N}) &= P_{n-1}^{(0,1)}(t) - \frac{n+1}{2N} P_{n-2}^{(1,2)}(\theta_2), \\ P_{n-1}^{(1,0)}(t - \frac{1}{N}) &= P_{n-1}^{(1,0)}(t) - \frac{n+1}{2N} P_{n-2}^{(2,1)}(\theta_3), \\ P_{n-1}^{(1,1)}(t - \frac{1}{N}) &= P_{n-1}^{(1,1)}(t) - \frac{n+2}{2N} P_{n-2}^{(2,2)}(\theta_4), \\ \sigma(n, n) &= 1, \quad S(n, n) = 1, \quad \sigma(n+1, n) = \binom{n+1}{2}, \quad S(n, n-1) = -\binom{n}{2}, \\ -\frac{n+1}{2} P_{n-1}^{(1,1)}(t) + S(n, n-1) [P_{n-1}^{(1,0)}(t) + P_{n-1}^{(0,1)}(t)] + \sigma(n+1, n) P_{n-1}^{(1,1)}(t) &\equiv 0, \end{aligned}$$

then from (16) we find

$$(17) \quad \begin{aligned} & \frac{(-1)^n (N-1)!}{N^n (N-1-n)!} Q_n\left(\frac{N}{2}(1+t - \frac{1}{N}); 0, 0, N\right) = P_n^{(0,0)}(t) \\ &+ \frac{1}{N^2} \sum_{\substack{n \leq r \leq 2n \\ 0 \leq k, j \leq n \\ r \leq j+k \\ r+n-j-k \geq 2}} \frac{\sigma(r, n) S(n, k) S(n, j)}{N^{r+n-j-k-2}} P_{j+k-r}^{(r-j, r-k)}(t - \frac{1}{N}) + R_{n,N}(t), \end{aligned}$$

where

$$(18) \quad R_{n,N}(t) = \frac{1}{N^2} \left[\frac{(n+1)(n+2)}{8} P_{n-2}^{(2,2)}(\theta_1) - \frac{n+1}{2} S(n, n-1) (P_{n-2}^{(1,2)}(\theta_2) + P_{n-2}^{(2,1)}(\theta_3)) - \frac{n+2}{2} \sigma(n+1, n) P_{n-2}^{(2,2)}(\theta_4) \right].$$

From (17) and (18) we deduce

Corollary 2. For fixed n and $N \rightarrow \infty$

$$(-1)^n \frac{(N-1)!}{N^n(N-1-n)!} Q_n\left(\frac{N}{2}(1+t-\frac{1}{N}); 0, 0, N\right) = P_n^{(0,0)}(t) + O\left(\frac{1}{N^2}\right).$$

This result was obtained in another way in [2].

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