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On Separable Type I C*-Algebras

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It is well known that if $C(X)$, where X is compact Hausdorff space, is separable, then, X is metrisable. We generalize the above result for separable type I C^*-algebras. Also, we prove similar results for C^*-algebras, completion of tensor product of type I C^*-algebras.

1. Introduction

All the C^*-algebras that we are concerned here are of **type I.** In particular, we are concerned with the spectrum of separable homogeneous C^*-algebras. We prove (§3) that the spectrum of a unital n-homogeneous separable C^*-algebra is compact and metrisable as a corrollary of an analogous theorem on (post) liminal C^*-algebras, with inner derivations (Theor. 3.1). We state the "local" structure of an n-homogeneous C^*-algebras, and we prove some sort of converse of the above theorem (Corr. 3.7). In the sequel, we generalize the above result on the C^*-algebra, completion of tensor product of n, m-homogeneous (resp. post liminal) C*-algebras and we prove similar result on the spectrum of the completion of tensor product of C^*-algebras if one of them is type I (Corr. 3.8, Prop. 3.9).

2. Notations and terminology

For general results on C^*-algebras we refer the reader to [4]. We consider only unital C^*-algebras.

A derivation on an algebra E is a linear map $\delta : E \to E$ such that

$$\delta(xy) = (\delta x)y + x(\delta y) \quad (x,\ y \in E).$$

An element α of some possible larger algebra is said to implement δ if

$$\delta x = \alpha x - x\alpha \quad (x \in E)$$

and δ is said to be inner if such an α can be found in E. Otherwise δ is said to be outer ([13]).

A C^*-algebra is said to be n-homogeneous iff all its irreducible *-representations are of the same finite dimension n.

A C^*-algebra is said to be liminal if, for every irreducible representation π of E and each $x \in E$, $\pi(x)$ is compact. The C^*-algebra E is said to be postliminal is every non-zero quotient C^*-algebra of E possesses a non-zero liminal closed two-sided ideal ([4, 4.2.1, 4.3.1]).

A topological space E is said to be p o l i s h, if it is separable and if there exists a metric on E for which the topology is τ and $E[\tau]$ is complete. A Hausdorff space $E[\tau]$ is said to be L u s i n (resp. S o u s l i n) if it is the injective continuous (resp. continuous) image of a polish space. For general results of A n a l y t i c S e t s see: [10], [12], [14].

3. On the spectrum

Theorem 3.1. *Let E be a unital separable (post) liminal C^*-algebra with all derivations inner. Then, its spectrum $\hat{E}$ is a compact and metrisable space.*

1 s t p r o o f. Since E is a unital separable (post) liminal C^*-algebra with all derivations to be inner, the pure states set $P(E)$ is w^*-compact ([1]), and $\hat{E}'_s$ is a Lusin space ([12, p. 115]). Now, the canonical map $h: P(E) \to \hat{E}$ is continuous onto (and open) and thus, $\hat{E}$ is compact and metrisable as the continuous image of a compact and metrisable set in a Hausdorff ([1, Corr. 5.5, ex. 4.5, Th. 4,2]) space.

2 n d p r o o f. Since E is unital and liminal, every irreducible *-representation is of finite dimension and so $\hat{E} = \bigcup_n \hat{E}_n$ $n = 1, 2, \ldots$

Let $\Phi: \hat{E} \to \mathrm{N}: \pi \to \dim \pi$, Φ is continuous on $\hat{E}$, $(\hat{E}_n)$ $n \in \mathrm{N}$ are closed and open. Furthermore, E is quasi-compact ([4, 3.1.8]) and thus $(\hat{E}_n)$ $n \in \mathrm{N}$ are empties except for finitely many n. By [12, ch. II] and [4, 3.7.4], $\bigcup_{k=1}^{n} E_k$ is a Souslin space. That is, $\hat{E}$ is compact and metrisable.

3.2. Let E be a C^*-algebra with c o n t i n u o u s t r a c e ([4, 4.5]). If E is unital (resp. if it has paracompact spectrum) then every derivation on E is inner (resp. is determined by a multiplier) [1, Th. 3.2]. For the class of C^*-algebras with continuous trace the theorem 3.1 is a sort of converse.

Corrollary 3.3. *Let E be a unital separable n-homogeneous C^*-algebra. Then, its spectrum is compact and metrisable.*

P r o o f: Obvious by [13].

Corrollary 3.4. *Let E be a unital separable type I C^*-algebra with all derivations inner. Then its spectrum is compact and metrisable.*

P r o o f: Obvious by [4, IX, Th. 9.1, (i), (iii)].

3.5. Every unital n-homogeneous C^*-algebra or a unital separable (post) liminal C^*-algebra with inner derivations is a c e n t r a l C^*-algebra. See, for example, [7, Th. 4.2], [4, 3.1.6, 4.3.7], [3, Corr. of Prop. 3, p. 109], [6, p. 414].

3.6. If E is a unital n-homogeneous C^*-algebra it is known ([2, p. 345], [13, p. 524]) that to each $t \in T = \hat{E}$ there corresponds a closed nbhd V, such that the algebra E/V, of restrictions to V of the elements of E (E is isometrically *-isomorphic to a suitable m a x i m a l f u l l a l g e b r a of operator fields, [4, ch. 10]) is isomorphic to $C(V, M_n) = C(V) \otimes M_n$ where M_n is the full matrix algebra of $n \times n$ matrices.
The "local" structure of E is thus the algebra $C(V, M_n)$.

Corrollary 3.7. *Let E be a unital n-homogeneous C^*-algebra. We suppose that its spectrum $\hat{E}$ is metrisable. Then, the "local" structure of E is separable.*

P r o o f. By 3.5 $E/V \simeq C(V, M_n)$ where V is a closed nbhd of any element of $\hat{E}$. Now for any cross-norm α we have

$$C(V, M_n) = C(V) \underset{\alpha}{\otimes} M_n$$

([8, p. 159], [14, p. 254]).
Obvious $C(V)$ and M_n are separable Banach algebras and by [11, Lem. 2.3, 2.4] we have the result.

Corrollary 3.8. *Let E_i, $i=1, 2$ unital separable n, m-homogeneous* (resp. *postliminal*) *C^*-algebras. Then, the spectrum $E_1 \underset{\alpha}{\otimes} E_2$ is metrisable and compact, where α is any C^*-crossnorm.*

Proof: The completion of the tensor product of two, n, m-homogeneous, C^*-algebras is an nm-homogeneous C^*-algebra ([5, Prop. 2]) in respect with any C^*-crossnorm ([8, p. 159]). By [11, Lem. 2.4, p. 28] the nm-homogeneous C^*-algebra is separable and we have the result, for homogeneous C^*-algebras. Also, it is obvious by [15, Th. 4, p. 26], [16, Th. 1] and [8, p. 159] that we have the result for postliminal algebras.

Proposition 3.9. *Let E and F be separable* (*unital*) *C^*-algebras with Hausdorff spectra. We suppose that E is of type I. Then, $(E \underset{\alpha}{\otimes} F)^\wedge$ is a Souslin space.*

Proof: It is known that $P(E)$ and $P(F)$ are polish sets as extreme sets of the corresponding state spaces ([2, 4.1], [12, p. 115] see also [10, p. 101]). There is a canonical map

$$P(E) \times P(F) \to E^\wedge \times F^\wedge = (E \underset{\alpha}{\otimes} F)^\wedge$$

continuous, onto (and open) ([4, 2.5.4, 3.4.11], [9, p. 476], [8, p. 159], [15, Th. 4.1]). Thus, $(E \underset{\alpha}{\otimes} F)^\wedge$ is a Souslin space by well known definitions.

Proposition 3.10. *Let $(E_n)_{n \in \mathbb{N}}$ sequence of separable C^*-algebras with Hausdorff spectra. Then, the strict inductive limit, $\underset{\underset{s}{\longrightarrow}}{\lim}\ (\hat{E}_n)$ is a Souslin space.*

Proof: $P(E_n)$, $n \in \mathbb{N}$ are polich sets. The canonical maps

$$h_n : P(E_n) \to \hat{E}_n$$

are continuous and onto. We consider the inductive limit of h_n

$$\underset{\underset{s}{\longrightarrow}}{\lim}\ h_n : \underset{\underset{s}{\longrightarrow}}{\lim}\ P(E_n) \to \underset{\underset{s}{\longrightarrow}}{\lim}\ \hat{E}_n$$

By [12, II] $\underset{\longrightarrow}{\lim}\ (\hat{E}_n)$ is a Souslin space.

Let E Banach *-algebra. The canonical map $h : P(E) \to \hat{E}$ is onto and $P(E)$ is a subset of unital ball in the dual space of E. $\hat{E}$ is endowed with the strongest topology, within h is continuous. If E is a C^*-algebra the above topology coincides with Jacobson topology [4, § 3.4].

Proposition 3.11. *Let E be Banach algebra with approximate identity, separable with Hausdorff spectrum. Then $\hat{E}$, is a Souslin space.*

P r o o f: Let $h : P(E) \to \hat{E}$ the canonical map (continuous, onto and open). It is known,

$$P(E) \subseteq S(E) \subseteq E'_s.$$

The space E'_s is a Lusin space, the state space $S(E)$ is a weakly *-compact subset of E'_s and thus metrisable and weakly *-compact, that is polish. $P(E)$ is a G_δ-set of a polish set and we have obvious that $\hat{E}$ is Souslin.

References

1. C. A k e m a n n, G. A. E l l i o t, G. K. P e d e r s e n, J. T o m i y a m a. Derivations and multipliers of C*-algebras. *Amer. J. Math.*, **98**, 1976.
2. O. B r a t t e l i, D. W. R o b i n s o n. Operator Algebras and Quantum Statistical Mechanics. Springer-Verlag, 1979.
3. C. D e l a r o c h e. Sur les centres de C*-algèbres. *Bull. Sci. Math.* **91**, 1967, 105-112.
4. J. D i x m i e r. C*-algebras. North Holland. Publ. Comp., 1982.
5. A. G u i c h a r d e t. Tensor Products of C*-algebras, I. Aarhus Univ. *Lecture Notes*, Series No **12**, 1969.
6. I. K a p l a n s k y. Normed Algebras. *Duke Math. J.*, **16**, 1969.
7. I. K a p l a n s k y. The structure of certain operator algebras. *Trans. Amer. Math. Soc.*, **70**, 1951, 219-255.
8. C. L a n c e. On Nuclear C*-algebras. *Journ. of Funct. Anal.*, **12**, 1973, 157-176.
9. K. B. L a u r s e n. Tensor Products of Banach algebras with involution. *Trans. Amer. Math. Soc.*, **136**, 1969, 467-487.
10. G. K. P e d e r s e n. C*-algebras and their automorphisms. Academic Press, 1979.
11. R. S c h a t t e n. A theory of cross-spaces. Princeton Univ. Press, 1950.
12. L. S c h w a r t z. Radon Measures. Oxford Univ. Press, 1973.
13. J. P. S p r o s t o n. Derivations and automorphisms of homoheneous C*-algebras. *Proc. London Math. Soc.* (3), **32**, 1976, 521-536.
14. M. T a k e s a k i. Operator Algebras I. Springer-Verlag, 1979.
15. A. W u l f s o h n. Produit tensoriel de C*-algebras. *Bull. Sci. Math.*, **87**, 1963, 13-27.
16. A. W u l f s o h n. Le produit tensoriel de certaines C*-algèbres. *C. R. Acad. Sci. Paris*, 258, 1964, 6052-6054.

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