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The Theory of Stable Metrics

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In the present paper we exhibit a concept of the stable metric in universal algebras. By means of stable metric one can describe the properties of free topological algebras. For groups the notion of the stable metric coincides with the notion of the invariant metric. The topology induced by the stable metric is compatible with the algebraical structure.

0. Introduction

The present paper is connected with the results of A. V. Arhangel'skiĭ [3, 4], A. I. Malčev [16, 17], M. M. Čoban [5-9], M. M. Čoban and S. S. Dimitraşcu [11], M. I. Graev [14], S. A. Morris and P. Nicolas [19], A. A. Marcov [18], S. Świerczkowski [23], W. Taylor [24, 25] and of other authors.

In Section 1 we define the introductory notions. The terms, the polynomials and the identities are discussed in Section 2. Section 3 is a continuation of the Section 2. In Sections 4 and 5 we discuss the notion of stable symmetrics, pseudometrics and metrics.

In section 6 by Marcov – Graev – Świerczkowski constructions every normed metric d on X is extending to a stable metric on the free algebra $F(X, K)$. For groups the constructions of M. I. Graev and S. Świerczkowski are studied in [19]. By Marcov – Graev – Świerczkowski constructions W. Taylor [25] has proved: every metric d_0 on X is extendible to a metric d on $F(X, K)$ and all operations of $F(X, K)$ are d-continuous. It is necessary to note that Theorem 6.9 is a much more general assertion than Theorem of S. Świerczkowski [23] and Theorem 2.1 of W. Taylor [25]. In fact the Lemmas 6.1-6.8 are proved by S. Świerczkowski [23], but we consider them for pseudometrics but not for functions in Euclidian spaces.

In Sections 7-9 by means of stable metric one can describe the properties of free topological algebras. In the case of groups this results were obtained by A. V. Arhangel'skiĭ [3, 4], M. M. Čoban [7, 8], V. K. Belnov [12], M. I. Graev [14].

In Section 10 we study the topological quasygroups.

In Sections 11 and 12 some applications are given of the theory of stable metrics.

The space X is zero-dimensional if $\dim X = 0$. Every space to be considered is non-empty. Below $|X|$ is the cardinality of X, $w(X)$ is the weight of the space X, $uw(X)$ is the uniform weight of the uniform space X.

The paper uses the terminology from [10, 13, 15].

1. Introductory definitions

The disjoint sum of the discrete spaces $\{E_n : n \in N = \{0, 1, 2, ..\}\}$ is denoted by E and it is called a signature or a set of fundamental operations. For every $n \in N$ the set E_n is a set of operations of type n. We say that an E-algebra or an algebra G of a signature E is given if the set G is non-empty and there are maps $\{P_{nG} : E_n \times G^n \to G : n \in N\}$. The maps $\{P_{nG} : n \in N\}$ are called the structure of the E-algebra in the set G. Distinct systems of maps, even if G is the same, are considered as distinct algebras with the same support G. Subalgebras, homomorphisms, isomorphisms and Cartesian products of E-algebras are defined as in [10, 15].

An E-algebra G together with a given topology on it is called a topological E-algebra if all the maps P_{nG} are continuous.

Tychonoff product of topological E-algebras is a topological E-algebra.

By a variety of E-algebras we mean a class of E-algebras closed under the formation of subalgebras, Cartesian products and homomorphism images (sf. [10, 15]).

Any topological space is called T_{-1}-space. By $K(E)$ we denote the class of all topological E-algebras and $K_i(E) = \{G \in K(E) : G \text{ is a } T_i\text{-space}\}$ where $i \in \{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$.

Let an operator J be defined in a class of E-algebras. If K is a class of E-algebras, we denote by $J(K)$ the class obtained by applying the operator J to E-algebras of families of E-algebras of class K. We denote by C the operator of Cartesian products, by S, the operator of taking of E-subalgebras, by P, the operator of Tychonoff products, by Q, $\hat{H}$, H, the operators of taking of factor-homomorphic images, of continuous homomorphic images and respectively of homomorphic images. A homomorphism $f : X \to Y$ of topological E-algebras X and Y is called factor-homomorphism if the mapping f is quotient.

A class L of E-algebras is called a quasivariety if $L = S(L) = C(L)$.

1.1. Definitions. *Let* $i \in \{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$. *A class* K *of topological* E*-algebras is called*:

— *a* T_i*-quasivariety if* $K = P(K) = S(K) \subset K_i(E)$;

— *a* T_i*-variety if* $K = P(K) = S(K) = Q(K) \cap K_i(E)$;

— *a complete* T_i*-variety if* $K = P(K) = S(K) = H(K) \cap K_i(E)$;

— *non-trivial if a topological* E*-algebra* G, *containing at least three different open sets, exists in* K.

Let G be a E-algebra, $X \subset G$ and $X \neq \emptyset$. By $s(X, G)$ we denote the subalgebra of the algebra G generated by a set X.

We fix a T_i-quasivariety K of topological E-algebras and a space X. A couple $(F(X, K), i_X)$, where $F(X, K) \in K$ and $i_X : X \to F(X, K)$ is a continuous map, is called a free topological E-algebra of a space X in the class K if $i_X(X)$ algebraically generates $F(X, K)$ and for any continuous map $f : X \to G$ with $G \in K$ there exists a continuous homomorphism $\hat{f} : F(X, K) \to G$ with $f = \hat{f} \circ i_X$. A couple $(F^a(X, K), j_X)$, where $F^a(X, K) \in K$ and $j_X : X \to F^a(X, K)$ is a map, is called algebraically or abstractly free E-algebra of X in K if $F^a(X, K) = s(j_X(X), F^a(X, K))$ and for any map $g : X \to G$ with $G \in K$ there exists a continuous homomorphism

$\bar{g}: F^a(X, K) \to G$ with $g = \bar{g} \circ j_X$. The algebras $F(X, K)$ and $F^a(X, K)$ exist and are unique up to topological isomorphism [5, 11, 17]. We denote by $t(X, K)$ and $t_a(X, K)$ the topologies of spaces $F(X, K)$, and $F^a(X, K)$ respectively. For any space X there exists a unique continuous homomorphism $p_X : F^a(X, K) \xrightarrow{\text{onto}} F(X, K)$ with $i_X = p_X \circ j_X$. If p_X is an algebraic isomorphism then $F(X, K)$ is said to be algebraically free in K.

2. Terms, polynomials and identities

Fix a signature E.

For any integers j and n with $1 \leqq j \leqq n$ we denote by b^j_n the operation of type n defined by $b^j_n(x_1, \ldots, x_j, \ldots, x_n) = x_j$.

For any E-algebra G, $n \geqq 1$, $e \in E_n$ and $x_1, \ldots, x_n \in G$ we denote $P_{nG}(e, x_1, \ldots, x_n) = e(x_1, \ldots, x_n)$. Then $e : G^n \to G$ is a map of the set G^n in G.

If $L \subset E$ then $T(L)$ is, by definition, the smallest class of operations such that:

1. $L \cup \{b^j_n : 1 \leqq j \leqq n,\ n \in N \setminus \{0\}\} \subset T(L)$;
2. If $e \in L \cap E_n$ where $n \geqq 1$ and $u_1, \ldots, u_n \in T(L)$ then $e(u_1, \ldots, u_n) \in T(L)$. The type of the operation $e(U_1, \ldots, U_n)$ is equal with the sum of types of operations $u_1, \ldots, u_n$.

We denote $\bar{E} = T(E)$. The set $\bar{E}$ is called the set of terms. We have $\bar{E} = U\{\bar{E}_n : n \in N\}$ where $\bar{E}_n$ is the set of terms of type n. The set $\bar{E}_0$ is called the set of constant terms. For any term the rank is determined. The elements of the set $E^1 = E \cup \{b^j_n : 1 \leqq j \leqq n \in N\}$ are the terms of the first rank. Let E^n be the set of the terms of a rank $\leqq n$. If $m \geqq 1$, $g \in E_m$ and $u_1, \ldots, {}_m \in E^n$ then $g(u_1, \ldots, u_m)$ is a term of a rank $\leqq m+1$. Let E^m_n be the set of the terms of a rank m and of a type n.

Let $N_m = \{1, 2, \ldots, m\}$.

2.1. Definition. *Fix the integers m and n with* $1 \leqq m \leqq n$, *the map* $h : N_n \xrightarrow{\text{onto}} N_m$ *and the operation* $f : G^n \to G$. *The operation* $g : G^m \to G$, *where* $g(x_1, \ldots, x_m) = f(x_{h(1)}, \ldots, x_{h(n)})$, *is called a h-permutation of the operation f. For every* $i \in N_m$ *the number* $m_i(g) = |h^{-1}(i)|$ *is called a multiplicity of the variable* x_i.

The set $P(E)$ of the polynomials or of the derived operations is the smallest class of operations such that:

1. $T(E) \subseteq P(E)$;
2. If $f \in P(E)$ and g is a h-permutation of the operation f, then $g \in P(E)$.

For any polynomial the rank and the type are determined. If G is a topological E-algebra and $g \in P(E)$ is a polynomial of a type m then $g : G^m \to G$ is a continuous map.

If $f, g \in P(E)$ then the form $f(x_{i_1}, \ldots, x_{i_n}) = g(x_{j_1}, \ldots, x_{j_m})$ is called an identity. The integer $|\{x_{i_1}, \ldots, x_{i_n}\} \cap \{x_{j_1}, \ldots, x_{j_m}\}|$ is called an exponent of the identity. The exponent indicates the number of the variables determining identity.

For quasivariety K of the E-algebras by $I(K)$ we denote the class of all identities such that every algebra $G \in K$ satisfies identites $I(K)$. The class $I(K)$ is a set.

Every element $i \in \{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$ and the set J of identities determine the class $V_i(J)$ of topological algebras $G \in K_i(E)$ which obey each identity from J. The class $V_i(J)$ is a complete T_i-variety and $J = I(V_i(J))$. If K is a complete T_i-variety then $K = V_i(I(K))$. For any quasivariety K and space X algebraically we have $F^a(X, K) = F^a(X, V_i(I(K)))$ (sf. [5]).

3. The support of elements

Fix a non-trivial T_i-quasivariety K of topological E-algebras and the space X. If $e \in T(E)$ is a term of zero rank then $e(G) = 1_{eG}$ for every algebra $G \in K$. Let $1_e = 1_{eG}$. The map $j_X : X \to F^a(X, K)$ is one-to-one. For every $x \in X$ we identify the elements x and $j_X(x)$. Then $X_{\#}$is a subset of algebra $F^a(X, K)$ and $j_X(x) = x$. Let $L \subset F^a(X, K)$. Then the set $L^s = \{x \in X : L \setminus s(X \setminus \{x\}, F^a(X, K)) \neq \emptyset\}$ is called a support of the set L. If $b \in F^a(X, K)$ then $X_b = \{b\}^s$. Always $L \cap X \subset L^s$.

3.1. Lemma. *If $b \in F^a(X, K)$ and $X_b = \emptyset$ then $|X| > 1$.*

Proof. If $|X| = 1$, $L \subset F^a(X, K)$ and $L \neq \emptyset$, then $L^s = X$.

3.2. Lemma. *Let $b \in F^a(X, K)$ and $b = f(b_1, \dots, b_n)$ where $f \in P(E)$, $1 \leqq n = |\{b_1, \dots, b_n\}|$ and $b_1, \dots, b_n \in X$. If $b_i \notin X_b$ for some $i \leqq n$ then $b = f(b_1, \dots, b_{i-1}, x, b_{i+1}, \dots, b_n)$ for every $x \in X$. Moreover, the equation $f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) = f(x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$ is an identity from $I(K)$.*

Proof. Fix an element $c \in X$. Let us consider the map $h : X \to X$ where $h(b_i) = c$, $h(c) = b_i$ and $h(x) = x$ for every $x \in X \setminus \{c, b_i\}$. By condition, $b \in G = s(X \setminus \{b_i\}, F^a(X, K))$. Hence there exists an isomorphism $\bar{h} : F^a(X, K) \to F^a(X, K)$ such that $h = \bar{h}|X$ and $\bar{h}(G) = G$. If $x_1, \dots, x_n \in X$ then $\bar{h}(f(x_1, \dots, x_n)) = f(h(x_1), \dots, h(x_n))$. In particular, $b = \bar{h}(b)$ and $f(b_1, \dots, b_n) = b = \bar{h}(b) = f(h(b_1), \dots, h(b_n)) = f(b_1, \dots, b_{i-1}, c, b_{i+1}, \dots, b_n)$.

3.3. Corollary. *Let $|X| \geqq 2$. Then:*

1. *If $b \in \{1_t : t$ is a term of zero type$\} = F^a_0(X, K)$ then $X_b = \emptyset$.*
2. *If $X_b = \emptyset$ then $b \in F^a_0(X, K)$.*
3. *If $b = h(b_1, \dots, b_m) \in F^a_0(X, K)$, $h \in P(E)$ and $|\{b_1, \dots, b_m\}| = m \geqq 1$ then $h(x_1, \dots, x_m) = h(y_1, \dots, y_m)$ is an identity of zero exponent in K.*

3.4. Corollary. *Let $|X| > 1$, $a \in F^a(X, K)$ and $X_a = \{a_1, \dots, a_m\}$ where $m \geqq 1$. If $a = f(b_1, \dots, b_n)$ and $f \in P(E)$ then:*

1. *$X_a \subset \{b_1, \dots, b_n\}$ and $m \leqq n$.*
2. *For some map $h : N_n \xrightarrow{\text{onto}} N_m$ we have $a = f(a_{h(1)}, \dots, a_{h(n)})$.*
3. *There exists a h-permutation g of the operation f such that $a = g(a_1, \dots, a_m)$.*

4. Operations on pseudometrics

A map $d : X \times X \to R^+$ of the set $X \times X$ into the set R^+ of non-negative real numbres is a pseudo-o-metric if $d(x, x) = 0$ for every $x \in X$. The pseudo-o-metric d is called:

— a o-metric if $d(x, y)+d(y, x)=0$ implies that $x=y$;
— a pseudosymmetric if $d(x, y)=d(y, x)$;
— a pseudo-Δ-metric if $d(x, z)\leqq d(x, y)+d(y, z)$;
— a pseudometric if d is a pseudo-Δ-metric and a pseudosymmetric;
— a symmetric if d is a pseudosymmetric and a o-metric;
— a metric if d is a symmetric and a pseudometric;
— normed if $d(x, y)\leqq 1$;
— totally bounded if for every $r>0$ there exists a finite set $A\subset X$ where for every $x\in X$ there exists an $a\in A$ such that $d(x, a)+d(a, x)<r$.

Let d be a pseudo-o-metric on the set X. The set $B(x, r, d)=\{y\in X : d(x, y)<r\}$ is called the r-ball about $x\in X$. The set $W\subset X$ is called d-open subset if for every $x\in W$ there exists $r>0$ such that $B(x, r, d)\subset W$. The family $\mathcal{J}_d$ of all d-open sets will be called the topology induced by the pseudo-o-metric d. If $x\in \operatorname{Int} B(x, r, d)$ for all $x\in X$ and $r>0$ then d is called a strong pseudo-o-metric. Every pseudo-Δ-metric is a strong pseudo-o-metric.

The pseudo-o-metric d induces a pseudometric $sd(x, y)=\min\{d(x, y), d(y, x)\}$, a pseudo-Δ-metric $\Delta d(x, y)=\min\{d(x, z_0)+d(z_0, z_1)+\dots+d(z_{n-1}, z_n)+d(z_n, y) : z_i\in X, i\leqq n, n\in N\}$ and a pseudometric $md=\Delta sd$.

It is easy to prove the following lemmas.

4.1. Lemma. *If d_1 is a pseudometric and $d_1(x, y)\leqq d(x, y)$ then $d_1(x, y)\leqq sd(x, y)$.*

4.2. Lemma. *If d_1 is a pseudometric and $d_1(x, y)\leqq d(x, y)$ then $d_1(x, y)\leqq md(x, y)$.*

4.3. Lemma. $md\leqq s\Delta d$.

4.4. Lemma. $\mathcal{J}_{sd}\cup\mathcal{J}_{\Delta d}\cup\mathcal{J}_{md}\subset\mathcal{J}_d$.

The family $\mathcal{L}=\{d_a : a\in A\}$ of pseudo-o-metrics induces a topology $\mathcal{J}_{\mathcal{L}}=\sup\{\mathcal{J}_{d_a} : a\in A\}$. Every topology is induced by a family of totally bounded pseudo-Δ-metrics (sf. [5, 26]). For every family $\mathcal{L}$ of pseudometrics there exists a family $b\mathcal{L}$ of totally bounded pseudometrics such that $\mathcal{J}_{\mathcal{L}}=\mathcal{J}_{b\mathcal{L}}$ and for every $d\in b\mathcal{L}$ there exists $d_1\in\mathcal{L}$ such that $d(x, y)\leqq d_1(x, y)$ for any $x, y\in X$.

5. Stable metrics

Fix a signature E.

5.1. Definition. *A pseudo-o-metric d on the E-algebra G is called stable if $d(P_{nG}(e, x_1,\dots, x_n), P_{nG}(e, y_1,\dots, y_n))\leqq\Sigma\{d(x_i, y_i) : i\leqq n\}$, i.e. $d(e(x_1,\dots, x_n), e(y_1,\dots, y_n))\leqq\Sigma\{d(x_i, y_i) : i\leqq n\}$ for every $n\geqq 1$, $e\in E_n$ and $x_1, y_1,\dots, x_n, y_n\in G$.*

5.2. Lemma. *If the pseudo-o-metric d on the E-algebra G is stable then and the pseudo-Δ-metric Δd is stable.*

Proof. We fix $n\geqq 1$, $e\in E_n$, $r>0$ and $x_1, y_1,\dots, x_n, y_n\in G$. There exist elements $\{z_{ij}\in G : i\leqq n, j\leqq m\}$, such that $d(x_i, z_{i1})+\dots+d(z_{im}, y_i)\leqq\Delta d(x_i, y_i)+r$ for every $i\leqq n$. Thus $\Delta d(e(x_1,\dots, x_n), e(y_1,\dots, y_n))\leqq d(e(x_1,\dots, x_n), e(z_{11},\dots, z_{n1}))+d(e,(z_{11},\dots, z_{n1}), e(z_{12},\dots, z_{n2}))+\dots+d(e(z_{1m},\dots, z_{nm}), e(y_1,\dots, y_n))\leqq\Sigma\{\Delta d(x_i, y_i) : i\leqq n\}+n\cdot r$.

5.3. Lemma. *The stable pseudo-o-metric d on the E-algebra G has the following properties:*

1. *$d(t(x_1,\dots, x_n), t(y_1,\dots, y_n))\leqq\Sigma\{d(x_i, y_i):i\leqq n\}$ for every term t of a type $n\geqq 1$.*

2. *$d(p(x_1,\dots, x_n), p(y_1,\dots, y_n))\leqq\Sigma\{m_i\cdot d(x_i, y_i):i\leqq n\}$ for every polynomial p of a type $n\geqq 1$. The number $m_i=m_i(p)$ is a multiplicity of the variable x_i.*

Proof. Let the Lemma be true for terms of the rank $m\geqq 1$. Fix $n\geqq 1$, $e\in E_n$ and terms $t_1,\dots, t_n$ of the rank m. Assume that r_i is a type of term t_i. Then $d(e(t_1(x_{11},\dots, x_{1r_1}),\dots, t_n(x_{n1},\dots, x_{nr_n})), e(t_1(y_{11},\dots, y_{1r_1}),\dots, t_n(y_{n1},\dots,y_{nr_n})))$ $\leqq\Sigma\{d(t_i(x_{i1},\dots, x_{ir_i}), t_i(y_{i1},\dots, y_{ir_i})):i\leqq n\}\leqq\Sigma\{\Sigma\{d(x_{ij}, y_{ij}):j\leqq r_i\}:i\leqq n\}$. This completes the proof.

5.4. Lemma. *If d is a stable strong pseudo-o-metric on the E-algebra G then $(G, \mathcal{J}_d)$ is a topological E-algebra.*

Proof. Let $d^n((x_1,\dots, x_n), (y_1,\dots, y_n))=\Sigma\{d(x_i, y_i):i\leqq n\}$. Then d^n is a strong pseudo-o-metric. By condition the map $e:G^n\to G$ is uniformly continuous and $(G^n, \mathcal{J}_{d^n})=(G, \mathcal{J}_d)^n$ for all $n\geqq 1$ and $e\in E_n$.

5.5. Corollary. *The topology induced by a family of stable pseudometrics or pseudo-Δ-metrics is compatible with the structure of the E-algebra.*

5.6. Corollary. *The topology induced by a stable metric is compatible with the structure of the E-algebra.*

5.7. Theorem. *Let $i\in\{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$ and V be a complete T_i-variety of topological E-algebras. Suppose further that d is a stable metric on the algebra $G\in V$ and $(\tilde{G}, \tilde{d})$ is a Hausdorff completion of the metric space (G, d). Then $\tilde{d}$ is a stable metric on the algebra $\tilde{G}\in V$ and G is a subalgebra of the algebra $\tilde{G}$.*

Proof. The map $P_{nG}:E_n\times G^n\to G$ is uniformly continuous for every $n\in N$ with respect to the discrete normed metric in E and metric d^n in G^n. Hence the map P_{nG} is extendible to a uniformly continuous map $P_{n\tilde{G}}:E_n\times\tilde{G}^n\to\tilde{G}$. Then $\tilde{G}$ is an E-algebra. Fix a Cauchy sequences $\{\{x^m_k\in G:k=1, 2,\dots\}, \{y^m_k\in G:k=1, 2,\dots\}:m=1, 2,\dots\}$. Let $x^m_0=\lim x^m_k$ and $y^m_0=\lim y^m_0$. Then $\tilde{d}(P_{n\tilde{G}}(e, x^1_0,\dots, x^n_0), P_{n\tilde{G}}(e, y^1_0,\dots, y^n_0))=\lim d(P_{nG}(e, x^1_k,\dots, x^n_k), P_{nG}(e, y^1_k,\dots, y^n_k))\leqq\Sigma\{\lim d(x^j_k, y^j_k):j\leqq n\}=\Sigma\{\tilde{d}(x^j_0, y^j_0):j\leqq n\}$. Hence the metric $\tilde{d}$ is stable. If the equation $f(x_1,\dots, x_s)=g(y_1,\dots, y_q)$ is an identity in V then $f(x^1_0,\dots, x^s_0)=\lim f(x^1_k,\dots, x^s_k)=\lim g(y^1_k,\dots, y^q_k)=g(y^1_0,\dots, y^q_0)$. Therefore $\tilde{G}\in V$.

5.8. Remark. For groups the notion of the stable metric coincides with the notion of the invariant metric. Let E be a signature of groups, where $E_0=\{0\}$, $E_1=\{-\}$, $E_2=\{+\}$ and $E=E_0\cup E_1\cup E_2$. If d is a stable metric on the group G then $d(x, y)=d((x+a)-a, (y+a)-a)\leqq d(x+a, y+a)+d(-a, -a)$ $=d(x+a, y+a)\leqq d(x, y)$, $d(x, y)\leqq d(-x, -y)\leqq d(x, y)$ and $d(x, y)\leqq d(b+x, b+y)\leqq d(x, y)$. Therefore the metric d is invariant. If d is invariant metric then $d(a+x, b+y)\leqq d(a+x, a+y)+d(a+y, b+y)=d(x, y)+d(a, b)$.

5.9. Remark. In a ring with a unit every stable metric is bounded. In this case we have the signature $E=E_0\cup E_1\cup E_2$, where $E_0=\{0, 1\}$, $E_1=\{-\}$, $E_2=\{+, \cdot\}$. If d is a stable metric in the ring G then $d(x, y)\leqq d(x, 0)+d(0, y)=d(1\cdot x, 0\cdot x)+d(0\cdot y, 1\cdot y)\leqq d(1, 0)+d(x, x)+d(0, 1)+d(y, y)=2d(0, 1)$.

6. Construction of stable metrics in free algebras

We fix a signature E, element $i\in\{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$, a complete non-trivial T_i-variety K of topological E-algebras and a non-empty set X. Let $A=F^a(X, K)$ and $\Delta(A)=\{(x, x): x\in A\}$. If $L\subset A^2$ then $L^{-1}=\{(a, b):(b, a)\in L\}$.

Consider a normed pseudometric d on the set X. We put $D(1)=A^2$ and $D(r)=\Delta(A)\cup\{(a, b)\in A^2$: there exist a term t of type $n\geqq 1$ and elements $a_1, b_1,\ldots, a_n, b_n\in X$ such that $a=t(a_1,\ldots, a_n)$, $b=t(b_1,\ldots, b_n)$ and $\Sigma\{d(a_j, b_j): j\leqq n\}<r\}$ for every $0<r<1$. For all $a\in A$ and $0<r<1$ we denote $B(a, 1)=A$ and $B(a, r)=\{b\in A:(a, c_1)\in D(r_1), (c_1, c_2)\in D(r_2),\ldots, (c_n, b)\in D(r_{n+1})$ for some $c_1, c_2,\ldots, c_n\in A$ and positive numbers $r_1, r_2,\ldots, r_{n+1}$ such that $r_1+r_2+\ldots+r_{n+1}<r\}$.

Immediately from our definitions we have the following lemmas.

6.1. Lemma. *$D(r')\subset D(r)=D(r)^{-1}$ for any $0<r'\leqq r\leqq 1$.*

6.2. Lemma. *If $b\in B(a, r)$ then $a\in B(b, r)$.*

6.3. Lemma. *If $b\in B(a, r_1)$, $c\in B(b, r_2)$ and $r=\min\{1, r_1+r_2\}$ then $c\in B(a, r)$.*

By $h^m: Y^m\to Y^m$ we denote the m-th power of the map $h: Y\to Y$ where $h^m(y_1,\ldots, y_m)=(h(y_1),\ldots, h(y_m))$.

Let $L\subset X$. If $|L|\leqq 1$ then $d(L)=1$. If $|L|\geqq 2$ then $d(L)=\inf\{d(x, y): x, y\in L$ and $x\neq y\}$.

The symbol $t^{(n)}$ signifies that t is a term of type n.

6.4. Lemma. *Let $a\in A$, $x\in X^n$, be a term of type n and $h: X\to X$ be a map for which $h(y)=y$ for every $y\in X_a$. If $a=t(x)$ then $a=t(h^n(x))$.*

Proof. Follows immediately from the Lemma 3.2.

6.5. Definition (S. Świerczkowski [23]). *The system $\mathscr{L}=\{f_j^{(n_j)}, x_j, y_j: 1\leqq j\leqq m\}$ is called a linked system iff:*

1. *$f_j^{(n_j)}$ is a term of type $n_j\geqq 1$;*
2. *x_j, $y_j\in X^{n_j}$ and $f_j^{(n_j)}(y_j)=f_{j+1}^{(n_{j+1})}f_{j+1}^{(n_{j+1})}$.*

Fix a linked system $\mathscr{L}=\{f_j^{(n_j)}, x_j=(x_1^j,\ldots, x_{n_j}^j), y_j=(y_1^j,\ldots, y_{n_j}^j): 1\leqq j\leqq m\}$. We call two elements $x, y\in X$ associated with respect to the sysem $\mathscr{L}$, writing $x\sim y$, provided that, for a certain j, with $1\leqq j\leqq m$, and a certain k, with $1\leqq k\leqq n_j$, we have $(x, y)=(x_k^j, y_k^j)$, or $(x, y)=(y_k^j, x_k^j)$. We denote by $\approx$ the equivalence relation in X generated by $\sim$, i.e. $x\approx y$ iff $x=z_0\sim z_1\ldots, z_k=y$ for some $z_0, z_1,\ldots, z_k\in X$ and $K\geqq 0$.

6.6. Lemma. *Let $\mathscr{L}=\{f_j^{(n_j)}, x_j, y_j: 1\leqq j\leqq m\}$ be a linked system and $a=f_1^{(n_1)}(x_1)\in A$. We denote by $\approx$ the equivalence associated with the system $\mathscr{L}$. Then, for every map $h: X\to X$, such that*

(i) *if $x\approx y$ then $h(x)=h(y)$,*

(ii) *if $x\in X_a$ then $h(x)=x$,*

we have $a=f_m^{(n_m)}(h^{n_m}(y_m))$.

Proof. By conditions we have $x_k^j\approx y_k^j$ and $h^{n_j}(x_j)=h^{n_j}(y_j)$. This implies that $f_j^{(n_j)}(h^{n_j}(x_j))=f_j^{(n_j)}(h^{n_j}(y_j))$ and $f_j^{(n_j)}(h^{n_j}(y_j))=f_{j+1}^{(n_{j+1})}(h^{n_{j+1}}(x_{j+1}))$. By Lemma 6.4 we have $a=f_1^{(n_1)}(h^{n_1}(x_1))=f_m^{(n_m)}(h^{n_m}(y_m))$. This completes the proof.

6.7. Lemma. *Let $b \in B(a, r)$ and $0<r<1$. Then there exists a linked system $\mathscr{L}=\{f_j^{(n_j)}, x_j, y_j : 1 \leqq j \leqq m\}$ such that $a=f_1^{(n_1)}(x_1)$, $b=f_m^{(n_m)}(y_m)$ and the equivalence $x \approx y$ implies that $d(x, y)<r$.*

Proof. If $x=(x_1, \ldots, x_n) \in X^n$ and $y=(y_1, \ldots, y_n) \in X^n$ then $d^n(x, y)=\Sigma\{d(x_j, y_j) : j \leqq n\}$. Since $b \in B(a, r)$ there are elements $a_0, a_1, \ldots, a_m \in A$ and positive numbers $r_1, \ldots, r_m$ such that $a=a_0$, $b=a_m$, $r_1+\ldots+r_m<r$ and $(a_{j-1}, a_j) \in D(r_j)$. For every $j \leqq m$ there are term $f_j^{(n_j)}$ and points $x_j=(x_1^j, \ldots, x_{n_j}^j) \in X^{n_j}$ and $y_j=(y_1^j, \ldots, y_{n_j}^j) \in X^{n_j}$ such that $a_{j-1}=f_j^{(n_j)}(x_j)$, $a_j=f_j^{(n_j)}(y_j)$ and $d^{n_j}(x_j, y_j)<r_j$. Let $x, y \in X$ and $x \approx y$. Then we have $x=z_0 \sim z_1 \sim \ldots \sim z_s=y$ for some $z_0, z_1, \ldots, z_s \in X$. We can assume that the sequence $z_0, z_1, \ldots, z_s$ is of minimal length. We note now that every pair (z_{j-1}, z_j) is of one of the forms (x_q^k, y_q^k), (y_q^k, x_q^k) for suitable indices q, k. Since for every pair $\{x_q^k, y_q^k\}$ there is at most one j such that $\{z_{j-1}, z_j\}=\{x_q^k, y_q^k\}$ we conclude that $\Sigma\{d(z_{j-1}, z_j) : j \leqq s\} \leqq \Sigma\{d(x_q^k, y_q^k) : q \leqq n_k, k \leqq m\}<r$. Hence $d(x, y)<r$.

6.8. Lemma. *If $0<r'<r=d(X_a \cup X_b)$ and $b \in B(a, r')$ then $a=b$.*

Proof. By Lemma 6.7 there exists a linked system $\mathscr{L}=\{f_j^{(n_j)}, x_j, y_j : 1 \leqq j \leqq m\}$ such that $a=f_1^{(n_1)}(x_1)$, $b=f_m^{(n_m)}(y_m)$ and the equivalence $x \approx y$ implies that $d(x, y)<r'$. There exists a map $h: X \to X$ such that $h(x)=x$ for $x \in X_a \cup X_b$ and if $x \approx y$ then $h(x)=h(y)$. By Lemma 6.6 we have $a=f_m^{(n_m)}(h^{n_m}(y_m))$ and by Lemma 6.4 we have $b=f_m^{(n_m)}(h^{n_m}(y_m))$. We have obtained $a=b$. This completes the proof.

6.9. Theorem. *For every normed pseudometric d on the non-empty set X there exists exactly one stable normed pseudosymmetric $\hat{d}$ and one stable normed maximal pseudometric $\bar{d}$ on $F^a(X, K)$, such that:*

1. *$d(x, y)=\bar{d}(x, y)=\hat{d}(x, y)$ for every $x, y \in X$;*
2. *$\bar{d}=md=\Delta\hat{d}$.*
3. *If the normed pseudometric p is stable on $F^a(X, K)$ and $p(x, y) \leqq d(x, y)$ for $x, y \in X$ then $p(x, y) \leqq \bar{d}(x, y)$ for any $x, y \in F^a(X, K)$.*
4. *If $|X_a \cup X_b| \leqq 1$ and $a \neq b$ then $\bar{d}(a, b)=1$.*
5. *If $a, b \in F^a(X, K)$ and $\hat{d}(x, y)<1$ then for every $r>0$ there exist a term t of type $n \geqq 1$ and elements $x_1, y_1, \ldots, x_n, y_n \in X$, such that $a=t(x_1, \ldots, x_n)$, $b=(y_1, \ldots, y_n)$ and $\hat{d}(a, b) \leqq \Sigma\{d(x_j, y_j) : j \leqq n\}<\hat{d}(a, b)+r$.*
6. *If d is a metric then $\hat{d}$ is a symmetric and $\bar{d}$ is a metric.*

Proof. We put $A=F^a(X, K)$ and $\hat{d}(a, b)=\inf\{r : (a, b) \in D(r), 0<r \leqq 1\}$. Hence for $\bar{d}=m\hat{d}$ we have $\bar{d}(a, b)=\inf\{r : b \in B(a, r), 0<r \leqq 1\}$. By definition of sets $D(r)$ we have $\hat{d}(a, b)=\hat{d}(b, a)$ and $\hat{d}$ satisfies the property 5.

Let $a=t(a_1, \ldots, a_n)$ and $b=t(b_1, \ldots, b_n)$ for some elements $a_1, b_1, \ldots, a_n, b_n \in A$ and for a term t of type $n \geqq 1$. If $\Sigma\{\hat{d}(a_j, b_j) : j \leqq n\} \geqq 1$ then we have $\hat{d}(a, b) \leqq \Sigma\{\hat{d}(a_j, b_j) : j \leqq n\}$. Let $\Sigma\{\hat{d}(a_j, b_j) : j \leqq n\}=r<1$. Then for any $j \leqq n$ and $r'>0$ there exist a term t_j of type $m_j \geqq 1$ and elements $x_1^j, y_1^j, \ldots, x_{m_j}^j, y_{m_j}^j \in X$, such that $a_j=t_j(x_1^j, \ldots, x_{m_j}^j)$, $b_j=t_j(y_1^j, \ldots, y_{m_j}^j)$ and $\Sigma\{d(x_s^j, y_s^j) : s \leqq m_j\}<\hat{d}(a_j, b_j)+r'$. Then for a term $t'=t(t_1, \ldots, t_n)$ we have $a=t'(x_1^1, \ldots, x_{m_1}^1, \ldots, x_1^n, \ldots, x_{m_n}^n)$, $b=t'(y_1^1, \ldots, y_{m_1}^1, \ldots, y_1^n, \ldots, y_{m_n}^n)$ and $\hat{d}(a, b) \leqq \Sigma\{d(x_s^j, y_s^j) : s \leqq m_j,$

$j \leqq n\} \leqq \Sigma\{\hat{d}(a_j, b_j) : j \leqq n\} + n.r'$. Therefore the pseudosymmetric $\hat{d}$ is stable. By Lemma 5.2 the pseudometric $\bar{d} = m\hat{d}$ is stable too. By Lemma 6.8 we have $\bar{d}(a, b) \geqq d(X_a \cup X_b)$. If $a, b \in X$ and $a \neq b$ then $d(a, b) \geqq \hat{d}(a, b) \geqq \bar{d}(a, b) \geqq d(X_a \cup X_b) = d(\{a, b\}) = d(a, b)$. Properties 1 and 2 are proved.

Let a normed pseudometric p be stable and $p(x, y) \leqq d(x, y)$ for any $x, y \in X$. We fix $a, b \in A$. If $\hat{d}(a, b) = 1$ then $p(a, b) \leqq \hat{d}(a, b)$. We suppose that $\hat{d}(a, b) < 1$. Then for every $r > 0$ there exists a term t of type $n \geqq 1$ and elements $x_1, y_1, \ldots x_n, y_n \in X$ such that $a = t(x_1, \ldots, x_n)$, $b = t(y_1, \ldots, y_n)$ and $\Sigma\{d(x_j, y_j) : j \leqq n\} < \hat{d}(a, b) + r$. Then $p(a, b) \leqq \Sigma\{p(x_j, y_j) : j \leqq n\} \leqq \Sigma\{d(x_j, y_j) : j \leqq n\} < \hat{d}(a, b) + r$. Hence $p(a, b) \leqq \hat{d}(a, b)$. By Lemma 4.2 we have $p(a, b) \leqq \bar{d}(a, b)$. Property 3 is proved.

If $|X_a \cup X_b| \leqq 1$ and $a \neq b$ then by Lemma 6.8 we have $\bar{d}(a, b) \geqq d(X_a \cup X_b) = 1$.

If d is a metric, $a, b \in A$ and $a \neq b$ then $d(X_a \cup X_b) > 0$ and by Lemma 6.8 we have $\bar{d}(a, b) > 0$. This completes the proof.

6.10. Corollary. (S. Świerzkowski [23]). *For every Tychonoff space X the map $i_X : X \to F(X, K)$ is a topological embedding, $F(X, K)$ is a Hausdorff space and $p_X : F^a(X, K) \to F(X, K)$ is an algebraic isomorphism.*

6.11. Remark. If $x \in X \subset F^a(X, K)$ then $\bar{d}(x, 1_e) = 1$ for every $e \in \bar{E}_0$. If $e, g \in \bar{E}_0$ and $1_e \neq 1_g$ then $\bar{d}(1_e, 1_g) = 1$.

6.12. Corollary. *Every bounded metric d on X is extendable to a bounded stable metric d^* on $F^a(X, K)$.*

6.13. Remark. If V is a non-trivial complete T_i-variety of rings with a unit, then any non bounded metric d on X is not extendable to a stable metric on $F^a(X, V)$. This is a consequence of Remark 5.9.

6.14. Remark. Let us consider the following property of $K : (L)$. For every a, b $F^a(X, K)$ there exists a linked system $\{f_j^{(n_j)}, x_j, y_j : 1 \leqq j \leqq m\}$ such that $a = f_1^{(n_1)}(x_1)$ and $b = f_m^{(n_m)}(y_m)$. Then every metric d on X is extendable to a stable metric d^* on $F^a(X, K)$.

6.15. Questions. Let d be a normed metric on X. By $\hat{d}$ and $\bar{d}$ we denote the symmetric and the metric on $F^a(X, K)$ described in Theorem 6.9. Is $\mathcal{J}_{\hat{d}} = \mathcal{J}_{\bar{d}}$? Is $\hat{d}$ a strong symmetric? Is $(F^a(X, K), \mathcal{J}_{\hat{d}})$ a topological algebra?

6.16. Question. Suppose that K is a variety with property (L) and $a, b \in F^a(X, K)$. Are there a term t of type $n \geqq 1$ and elements $x_1, y_1, \ldots, x_n, y_n \in X$ such that $a = t(x_1, \ldots, x_n)$ and $b = t(y_1, \ldots, y_n)$?

7. Topologies on free algebras

We fix a signature E, an element $i \in \{-1, 0, 1, 2, 3, 3\frac{1}{3}\}$, a complete non-trivial T_1-variety K of topological E-algebras and a space X. Then $i_X : X \to F(X, K)$ is a topological embedding and $p_X : F^a(X, K) \to F(X, K)$ is an algebraic isomorphism. We identify the set $F^a(X, K)$ and $F(X, K)$ and the elements x, $i_X(x)$, $j_X(x)$. Hence $X \subset F(X, K) = F^a(X, K)$.

7.1. Definition. *The family $\{L_a : a \in A\}$ of sets of a space Y is called strongly discrete if there exists a discrete family $\{V_a : a \in A\}$ of open subsets of Y such that $L_a \subset V_a$ for every $a \in A$.*

7.2. Lemma. *The set* $\bar{E}_{oX}=\{a\in F(X, K):X_a=\emptyset\}$ *is a strongly discrete subset of the space* $F(X, K)$.

Proof. We fix a continuous normed pseudometric d on the space X. The family $\{V_x=\{y\in F(X, K):\bar{d}(x, y)<2^{-2}\}:x\in\bar{E}_{oX}\}$ is discrete.

7.3. Definition. *For every polynomial* t *of type* $n\geqq 1$ *we put* $tX=t(X^n)\subset F(X, K)$ *and* $Xt=tX\setminus\{t(x_1,\dots, x_n):|\{x_1,\dots, x_n\}|<n\}$. *For a polynomial* t *of type* 0 *we put* $tX=Xt$. *A polinomial* t *will be called irreducible if* $Xt\neq\emptyset$ *and* $X_a\neq\emptyset$ *for every* $a\in Xt$.

By definition every polynomial of zero type is irreducible. If t is a polynomial of type $n\geqq 2$, $|X|\geqq 2$ and $Xt\neq\emptyset$ then t is irreducible.

7.4. Lemma. *For every polynomial* t *of type* n *the set* tX *is a closed subset of the space* $F(X, K)$. *If* $C^nX=\{(x_1,\dots, x_n)\in X^n:|\{x_1,\dots, x_n\}|=n\}$ *and* t *is a irreducible polynomial, then* $t(C^nX)=Xt$ *and the map* $t:C^nX\to Xt$ *is a continuous open-and-closed finite-to-one local homeomorphism. If* t *is an irreducible polynomial of type* $n\leqq 1$ *then the spaces* Xt *and* X^n *are homeomorphic.*

Proof. Let $q:X\to bX$ be a embedding of the space X into Hausdorff compactification bX. There exists a continuous isomorphism $\hat{q}:F(X, K)\to F(bX, K)$ such that $\hat{q}(X)=q(x)=x$ for every $x\in X$. The set tbX is a compact subset of the T_2-space $F(bX, K)$. Hence $tX=\hat{q}^{-1}(tbX)$ is a closed subset of the space $F(X, K)$.

Let t be an irreducible polynomial of type n. If $n=0$ then $|Xt|=|X^n|=1$ and the spaces Xt and X^n are homeomorphic. Let $n\leqq 1$. We fix $a=t(a_1,\dots, a_n)\in Xt$. Then $(a_1,\dots, a_n)\in C^nX$ and there exists a sequence $V_1,\dots, V_n$ of open subsets of bX such that $a_j\in V_j$ and $[V_j]_{bX}\cap[V_s]_{bX}=\emptyset$ for $j\neq s$. We denote $H_j=[V_j]_{bX}$. Then $\Pi\{H_j:j\leqq n\}\subset C^nbX$ and by Lemma 3.2 the map $t|\Pi\{H_j:j\leqq n\}$ is one-to-one. Hence the maps $t|\Pi\{H_j:j\leqq n\}$ and $t|\Pi\{V_j:j\leqq n\}$ are homeomorphisms and the set $t(\Pi\{V_j:j\leqq n\})$ is open in the space tX. For some $k\leqq n:=1\cdot 2\cdot\ldots\cdot n$ we have $|t^{-1}(a)|=k$ for every $a\in Xt$. If $n=1$ then $t:X\to tX=Xt$ is a one-to-one map.

7.5. Definition. *The topology* $\mathcal{J}$ *on* $F(X, K)$ *is called admissible if*:

1. i_X *is a topological embedding of the space* X *into* $(F(X, K), \mathcal{J})$.
2. $(F(X, K), \mathcal{J})\in K$.
3. *For every term* t *the set* tX *is closed in the space* $(F(X, K), \mathcal{J})$ *and the topologies* $\mathcal{J}$ *and* $t(X, K)$ *induce the same topology on* Xt.

7.6. Definition. *The family* $L=\{d_a:a\in A\}$ *of pseudometrics on the space* X *is called permissible if*:

1. *The topology of the space* X *is induced by a family* L.
2. *If* $d_1,\dots, d_m\in L$ *then* $\frac{1}{m}(d_1+\ldots+d_m)\in L$.
3. *For every* $a\in A$ *the pseudometric* d_a *is normed.*

7.7. Theorem. *Let* $L=\{d_a:a\in A\}$ *be a permissible family of pseudometrics on the space* X. *By* $\bar{L}=\{\bar{d}_a:a\in A\}$ *denote the family of pseudometrics described in Theorem 6.9. Then* $\mathcal{J}_L=\sup\{\mathcal{J}_{\bar{d}_a}:a\in A\}$ *is admissible topology on* $F(X, K)$.

Proof. There exists a permissible family $bL=\{p_b:b\in B\}$ of totally bounded pseudometrics on the space X such that for every $b\in B$ there exists $a(b)\in A$ such that $p_b(x, y) \leqq d_{a(b)}(x, y)$ for any $x, y \in X$. Then there exists a Hausdorff compactification cX of the space X and a permissible family $M=\{q_b : b \in B\}$ of pseudometrics on the space cX such that $q_b(x, y)=p_b(x, y)$ for any $x, y \in X$ and $b \in B$. We put $\bar{M}=\{\bar{q}_b : b \in B\}$. If $y, z \in F(cX, K)$ and $y \neq z$ then $q_b(cX_y \cup cX_z)>0$ for some $b \in B$. Therefore $\mathcal{J}_{\bar{M}}$ is a Tychonoff topology on $F(cX, K)$. Hence $\mathcal{J}_{\bar{M}}$ is an admissible topology on $F(cX, K)$. Let $h : X \to cX$ be the natural embedding of X into cX. The map h is extendible to the continuous isomorphism $\hat{h} : F(X, K) \to F(cX, K)$. By Lemma 7.4 for every polynomial t the map $\hat{h}$ is a topological embedding of Xt into $(cX)t$. By construction, $\mathcal{J}_{bL} \subset \mathcal{J}_L$ and $\mathcal{J}_{bL}=\hat{h}^{-1}\mathcal{J}_{\bar{M}}$. Therefore the topologies $\mathcal{J}_{\overline{bL}}$ and $\mathcal{J}_L$ are admissible.

7.8. Corollary. *Let d be a normed metric on the space X. Then the topology $\mathcal{J}_{\bar{d}}$ is admissible.*

7.9. Remark. Let i_X be a topological embedding of the compact space X into the Hausdorff space $(F(X, K), \mathcal{J}) \in K$. Then $\mathcal{J}$ is admissible topology on $F(X, K)$.

8. Basic properties of sets tX

We fix a signature E, an element $i \in \{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$ and a complete non-trivial T_i-variety of topological E-algebras. Then for every Tychonoff space X the algebra $F(X, K)$ is algebraically free and $X \subset F(X, K)$.

Let $n \geqq 2$. We put $M_1(n)=\{h : N_n \xrightarrow{\text{onto}} N_{n-1} : h(j) \leqq j$ for every $j \leqq n\}$. Then $|M_1(n)|=\frac{n(n-1)}{2}$. For every polynomial g of type $n \geqq 2$ we denote $C_1(g)=\{f \in P(E) : f$ is a h-permutation of g and $h \in M_1(n)\}$. If f is a polynomial of type $n \leqq 1$ then $C_1(f)=\emptyset$.

For every polynomial g of type n we put $C_0(g)=\{g\}, \ldots, C_{n-1}(g)=\cup\{C_1(f) : f \in C_{n-2}(g)\}$ and $C(g)=\cup\{C_j(g) : j=0, 1, ., n-1\}$. The set $C(g)$ is called the set of natural permutations of the polynomial g. If $f \in C_j(g)$ then f is a polynomial of type $n-j$. In the set $C(g)$ we fix a well-order $<$ such that $f_1 < f_2$ for any $f_1 \in C_j(g)$, $f_2 \in C_s(g)$ and $j > s$.

8.1. Lemma. *For every polynomial g and every Tychonoff space X we have $Xg=gX \setminus \cup\{fX : f \in C_1(g)\}$.*

Proof. Let g be a polynomial of type n. If $n \leqq 1$, then $C_1(g)=\emptyset$ and $Xg=gX$. Let $n \geqq 2$. By definition, $Xg=\{g(x) : x \in C^nX\}$ and $gX=\{g(x) : x \in X^n\}$. If $x_1, \ldots, x_n \in X$ and $|\{x_1, \ldots, x_n\}|<n$, then $x_j=x_s$ for some $1 \leqq j < s \leqq n$. In $M_1(n)$ we have the map h where $h(m)=m$ for $m<s$, $h(s)=j$, $h(m)=m-1$ for $m>s$. Therefore for the h-permutation f of g we have $f(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_n)=g(x_{h(1)}, \ldots, x_{h(n)}) =g(x_1, \ldots, x_n)$, i.e. $g(x_1, \ldots, x_n)\ fX$ and $f \in C_1(g)$. This proves Lemma 8.1.

8.2. Lemma. *Fix a polynomial g of type n. There exists a map $\lambda : C(g) \to N$ such that for every Tychonoff space X and admissible topology $\mathcal{J}$ on $F(X, K)$ the family $\{Xf : f \in C(g)\}$ has the following properties:*

1. $gX = \cup\{Xf : f \in C(g)\}$ *and* $\lambda(g) \leqq n$.

2. *For every* $f \in C(g)$ *the set* $\cup\{Xp : p \in C(g)$ *and* $p < f\}$ *is closed in* $(F(X, K), \mathcal{J})$ *and* $fX \setminus \cup\{Xp : p \in C(g)$ *and* $p < f\} \subset Xf$.

3. *For every* $f \in C(g)$ *the space* $(Xf, \mathcal{J})$ *is locally homeomorphic with the space* $X^{\lambda(f)}$.

4. *If the space* $(gX, \mathcal{J})$ *is paracompact then* $\dim gX \leqq \operatorname{locdim} X^{\lambda(g)}$.

Proof. Follows immediately from the Lemmas 7.4 and 8.1.

We shall say that the set $Y \subset X$ is paracompact in the space X if any open in X covering of the set Y may be refined to some locally finite in X covering of the set Y, consisting of sets open in X. If a T_3-space X has a countable cover $\{X_n : n \in N\}$ such that for every $n \in N$ the set X_n is paracompact in X, then the space X is paracompact. If a set Y is paracompact in a Hausdorff space X, then Y is a closed subset of X.

The following lemmas are obvious.

8.3. Lemma. *Let* X *be a regular space,* $Y \subset X$, $Y = A \cup B$, *the set* A *be paracompact in* X *and the set* $B \setminus A$ *be paracompact in* $X \setminus A$. *Then the set* Y *is paracompact in* X.

8.4. Lemma. *Let* $f : X \to Y$ *be a continuous map,* $Z \subset X$ *and the map* $f|Z$ *be a homeomorphism. If the set* fZ *is paracompact in* Y *then the set* Z *is paracompact in* X.

Proof. It is obvious.

8.5. Theorem. *Let* d *be a normed metric on the space* X, *the regular topology* $\mathcal{J}$ *be admissible on* $F(X, K)$ *and* $\mathcal{J}_{\bar{d}} \subset \mathcal{J}$. *Then for every polynomial* g *the space* $(gX, \mathcal{J})$ *is a* F_σ*-metrizable paracompact in the space* $(F(X, K), \mathcal{J})$.

Proof. Denote by Y the space $(F(X, K), \mathcal{J})$ and by Z the space $(F(X, K), \bar{d})$. Then the identity map $q : Y \to Z$, where $q(y) = y$, is a continuous map. For every polynomial g of type $n \leqq 1$ the map qgX is a homeomorphism and $q(gX) = gX$ is a closed subset of the metric space Z. By Lemma 8.4 the set gX is a F_σ-metrizable paracompact in the space Y.

Suppose, that for every polynomial g of type $n < m$ (where $m \geqq 2$) the set gX is a F_σ-metrizable paracompact in Y. Fix a polynomial g of type m. Then the set $Y_1 = \cup\{fX : fC(g), f < g\}$ is a F_σ-metrizable paracompact in Y. By construction, $Xg \supset gX \setminus Y_1$. The map $q|Xg$ is a homeomorphism and the set $X_g \setminus Y_1$ is a paracompact in $Z \setminus Y_1$. By Lemma 8.4 the set $X_g \setminus Y_1$ is a paracompact in $Y \setminus Y_1$. Therefore, by Lemma 8.3, the set $gX = Y_1 \cup (Xg \setminus Y_1)$ is a E_σ-metrizable paracompact in Y.

A family S of subsets of a space X is called a network of X, if for any point x and any neighbourhood O_x in X there exists $P \in S$ such that $x \in P \subseteq O_x$ ([2]). A Hausdorff space X with a σ-discrete network is called a σ-space (cf. [21]). A Hausdorff space X with a countable network is called a cosmic space (cf. [20]).

8.6. Theorem. *Let* X *be a paracompact* σ*-space. Then there exists a continuous normed pseudometric* d *on the space* X *such that for every regular admissible topology* $\mathcal{J}$ on $F(X, K)$, where $\mathcal{J}_{\bar{d}} \subset \mathcal{J}$ *and for every polynomial* g *we have:*

1. $(gX, \mathcal{J})$ is a σ-space.

2. The set gX is paracompact in the space $(F(X, K), \mathcal{J})$.

Proof. Let $\gamma=\{\gamma_m=\{F_a^m : a\in A_m\} : m\in N\}$ be a network in X, where the families γ_m are closed and discrete in X. By Theorem of A. V. Arhangel'skiĭ (cf. [4], p. 134), there exists a continuous normed pseudometric d on X such that (X, d) is a metric space and the families γ_n are closed and discrete in (X, d). Let Y be a set X with metric $\bar{d}$. We fix an irreducible polynomial t of type $n\geqq 1$. Let $\lambda : N^n\to N$ be a one-to-one map. If $m=\lambda(m_1, m_2, \dots, m_n)$ then $\omega_m=\{H_b : b\in B_m\} = C^nX\cap(F_{a_1}^{m_1}\times \dots, \times F_{a_n}^{m_n}) : (a_1\in A_{m_1}, \dots, a_n\in A_{m_n})$ is a discrete and closed family of the spaces C^nX and C^nY. The family $\omega=\cup\{\omega_m : m\in N\}$ is a network in spaces C^nX and C^nY. There exists a continuous isomorphism $q : F(X, K)\to F(Y, K)$ such that $q(x)=x$ for $x\in X$ and $q(Xt)=Yt$. The system $\mathscr{L}=\{tH|H\in\omega\}$ is a closed σ-locally finite network in spaces Xt and Yt and $qtH=tH$ for every $H\in\omega$. The space $(F(Y, K), \bar{d})=(F(X, K), \bar{d})$ is metric. For every $m\in N$ there exists an open locally finite family $\xi_m=\{W_b : b\in B_m\}$ such that $H_b\subset W_b$ for every $b\in B_m$. Then every open cover of the set X_t in the space $Z=((F(X, K)\backslash tX)\cup Xt, \mathscr{J})$ has an open σ-locally finite refinement. Hence the set Xt is paracompact in Z. This completes the proof.

9. The case of countable signature

We fix a countable signature E, an element $i\in\{3, 3\frac{1}{2}\}$ and a complete non-trivial T_i-variety K of topological E-algebras. In this case the sets $T(E)$ and $P(E)$ are countable. Therefore, from Theorems 8.5 and 8.6 we obtain

9.1. Corollary. *The space X is a F_σ-metrizable paracompact iff $F(X, K)$ is the same one.*

9.2. Corollary. *The space X is a paracompact σ-space iff $F(X, K)$ is the same one.*

9.3. Corollary. *Let X be a Tychonoff space. If* loc dim $X^n\leqq k$ *for every $n\in N$ and $F(X, K)$ is paracompact, then* dim $F(X, K)\leqq k$.

9.4. Corollary. *Let X be a zero-dimensional metrizable space. Then* dim $F(X, K)=0$.

9.5. Corollary. *For every Tychonoff space X there exist a map $g : N\to N$ and a decomposition $F(X, K)=\cup\{F_n(X, K) : n\in N\}$ such that for any admissible topology $\mathscr{J}$ on $F(X, K)$ the set $\cup\{F_i(X, K) : i\leqq n\}$ is closed in the space $(F(X, K), \mathscr{J})$ and the space $(F_n(X, K), \mathscr{J})$ is locally homeomorphic with the space $X^{q(n)}$.*

9.6. Theorem. *Let $A\in K$ be a first countable algebra. Then there exist a first countable F_σ-metrizable paracompact algebra $B\in K$ and a factor-homomorphism $p : B\to A$ such that:*

1. $W(B)=w(A)$.
2. *If the space A is metrizable, then B is also metrizable.*
3. *If the space A is metrizable by a stable metric, then B is also metrizable by a stable metric.*

Proof. By the Theorem of V. I. Ponomarev [22], there exists a continuous open map $f : X\to A$ of a zero-dimensional metrizable space X onto A, where

$w(X)=w(A)$. We fix a normed metric d on the space X. There exists a homomorphism $p=\hat{f}:F(X, K)\to A$, where $f=\hat{f}X$. Denote by $\mathcal{J}$ the topology of the space A. We put $\mathcal{J}_1=\sup\{\mathcal{J}_d, \hat{f}^{-1}\mathcal{J}\}$ and B is an algebra $F(X, K)$ with the topology $\mathcal{J}_1$. The topology $\mathcal{J}_1$ is admissible on $F(X, K)$. Therefore B is a F_σ-metrizable paracompact space. The topological algebra B is a subalgebra of the topological algebra $(F(X, K)\times A, \mathcal{J}_d\times\mathcal{J})$. This completes the proof.

9.7. Remark. Let $j\in\{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$, $i\in\{3, 3\frac{1}{2}\}$ and V is a complete non-trivial T_j-variety of topological E-algebras. We put $K_i(V)=VK_i(E)$. Then $K_i(V)$ is a complete non-trivial T_i-variety of topological E-algebras. Therefore from Theorem 9.5 we have

9.8. Corollary. *Every first countable regular algebra $A\in V$ is a factor-algebra of a F_σ-metrizable first countable zero-dimensional paracompact algebra $B\in V$ and $w(A)=w(B)$. If the space A is metrizable, then B is metrizable as well.*

10. Permutability of congruence

The theorem of A. I. Malčev [16] states that a complete T_i-variety V of topological E-algebras has permutable congruence iff there exists a polynomial $p(x, y, z)$ such that the equations

$$x=p(y, y, x),\quad x=p(x, y, y)$$

hold identically in V. By this fact A. I. Malčev [16] has proved the following statements. If V is a congruence-permutable complete T_i-variety of topological E-algebras then:

1. Every factor-homomorphism in V is an open map.
2. If $A\in V$ is a T_0-space then A is a T_2-space.
3. If $A\in V$ and q is any congruence on A then the quotient space A/q is a topological E-algebra.

10.1. Definition. *A class K of E-algebras is called a class of E-quasy-groups if for some polynomials p, l, r of type 2 we have*

$$p(x, l(x, y))=y,\quad l(x, p(x, y))=y;$$
$$p(r(y, x), x)=y,\quad r(p(y, x), x)=y;$$
$$r(x, l(y, x))=y,\quad l(r(x, y), x)=y.$$

A. I. Malčev [16] proved that every class of E-quasi-groups is congruence-permutable, and every T_0-topological quasi-group is a T_3-space.

10.2. Corollary. *Let V be a non-trivial complete T_i-variety of topological E-quasi-groups where $i\in\{-1, 0, 1, 2, 3, 3\frac{1}{2}\}$ and the set E is countable. Then:*

a. *The space X is a F_σ-metrizable or paracompact σ-space iff $F(X, V)$ is of the same type.*

b. *Let X be a zero-dimensional metrizable space. Then $\dim F(X, V)=0$.*

10.2. Theorem. *Every first countable T_0-topological quasi-group is a Moore space.*

Proof. Let X be a T_0-topological first countable quasi-group with binary operations p, l, r. We fix an element $1 \in X$ and put $P(x, y) = r(p(x, l(1,1)), l(y, 1))$, $L(y, x) = r(1, l(x, p(y, l(1, 1))))$, $R(x, y) = r(p(x, l(y, 1)), l(1,1))$. Then X is a topological loop relatively to operations P, L, R and $P(x, 1) = P(1, x) = x$ for every $x \in X$ (see [1], [16]). We fix a countable base $\{U_n : n \in N\}$ at the point $1 \in X$. For every operation $e : X \times X \to X$ we put $e(A, B) = e(A \times B)$, where $A \subset X$ and $B \subset X$. Then the families $\gamma_n = \{P(x, U_n) : x \in X\}$ are open covers of the space X.

Let $x_0, z_n \in P(x_n, U_n)$, where $n \in N$. Hence $L(x_n, x_0) \in L(x_n, P(x_n, U_n)) = U_n$, $\lim L(x_n, x_0) = 1$ and $\lim x_n = \lim R(x_0, L(x_n, x_0)) = R(x_0, 1) = P(R(x_0, 1), 1) = x_0$. The inclusion $z_n \in P(x_n, U_n)$ implies that $L(x_n, z_n) \in U_n$, $\lim L(x_n, z_n) = 1$, $\lim z_n = \lim P(x_n, z_n) = P(x_0, 1) = x_0$. Then for every point $x \in X$, any neighbourhood U of x we have $\gamma_n(x) = \cup\{P(y, U_n) : x \in P(y, U_n)\} \subset U$ for some $n \in N$. Therefore $\{\gamma_n : n \in N\}$ is a development for X.

10.3. Corollary. *A topological quasi-group is metrizable if and only if it is first countable and collectionwise normal.*

10.4. Corollary. *Let V be a complete T_i-variety of topological E-quasi-groups, where $i \in \{-1, 0, 3\frac{1}{2}\}$ and the set E is countable. Every first countable T_0-quasi-group $A \in V$ is a factor-quasi-group of a metrizable zero-dimensional quasi-group $B \in V$ and $w(A) = w(B)$.*

11. Application to equivalence relations

In this section we discuss certain results related to the notion of K-equivalence. We fix a countable signature E, an element $i \in \{3, 3\frac{1}{2}\}$ and a complete non-trivial T_i-variety K of topological E-algebras.

11.1. Definition. *We say that X is K-equivalent to Y if $F(X, K)$ and $F(Y, K)$ are topologically isomorphic E-algebras.*

The results of the paragraph 9, together with the Theorem 6.9 and with the results of A. V. Arhangel'skiĭ [3, 4], yield.

11.2. Corollary. *Let $F(X, K)$ and $F(Y, K)$ be topologically isomorphic, where X and Y are Tychonoff spaces. Then:*

1. *If X is pseudocompact, then Y is pseudocompact;*
2. *If X is σ-compact, then Y is σ-compact;*
3. *If X is compact, then Y is compact;*
4. *If X is a cosmic space, then Y is a cosmic space;*
5. *If X is a $\,_0$-space, then Y is a $\,_0$-space;*
6. *If X is a left space, then Y is a left space;*
7. *If X is a Corson compact, then Y is a Corson compact;*
8. *If X is an Eberlein compact, then Y is an Eberlein compact;*
9. *If X is a sequential or bisequential compact, then Y is of the same type.*

12. Other applications

In this section any pseudometric is considered normed. The metric product of pseudometric spaces $\{(X_a, d_a) : a \in A\}$ is called the set $X = \Pi\{X_a : a \in A\}$ with the pseudometric $d(\{X_a : a \in A\}, y_a : a \in A\}) = \sup\{d(x_a, y_a) : a \in A\}$.

We fix a signature E with the discrete metric $b(x, y)=1$ for $x \neq y$. An E-algebra G together with a given uniformity on it is called a uniform E-algebra if all the maps P_{nG} are uniformly continuous. Cartesian product of the uniform E-algebras is a uniform E-algebra.

A metric E-algebra is a pair (G, d) consisting of E-algebra G and a stable metric d. The metric product of the metric E-algebras is a metric E-algebra. Every metric E-algebra is a uniform E-algebra.

12.1. Definition. *A class K of metric algebras is called a complete M-variety if it is closed under the formation of subalgebras, metric products and homomorphic images.*

12.2. Definition. *A class K of uniform E-algebras is called a complete U-variety if it is closed under the formation of subalgebras, cartesian products and homomorphic images.*

12.3. Definition. *Let K be a non-trivial complete M-variety of metric E-algebras and (X, d) be a metric space. A system $(F(X, K), \bar{d}, i_x)$, where $(F(X, K), \bar{d}) \in K$ and $i_x : X \to F(X, K)$ is an isometry, is called a free metric E-algebra of a metric space (X, d) in the class K if $i_x(X)$ algebraically generates $F(X, K)$ and for any map $f : X \to G$ with $(G, p) \in K$ and $p(f(x), f(y)) \leqq d(x, y)$ for $x, y \in X$ there exists a homomorphism $\hat{f} : F(X, K) \to G$ with $f = \hat{f} \circ i_x$ and $p(\hat{f}(x), \hat{f}(y)) \leqq \bar{d}(x, y)$ for every $x, y \in F(X, K)$.*

Theorems 6.9 and 7.7 yield

12.4. Corollary. *For every non-trivial complete M-variety K of metric E-algebras and any metric space (X, d) we have:*

1. *The free algebra $(F(X, K), \bar{d}, i_x)$ exists and is unique.*
2. *The algebra $F(X, K)$ is algebraically free in K.*
3. *The topology $\mathscr{I}_{\bar{d}}$ is an admissible topology on $F(X, K)$.*
4. *If the set E is countable, then* $\dim F(x, K) \leqq \sup \dim X^n : n \in N$.

12.5. Corollary. *Let K be a non-trivial complete M-variety of metric E-algebras and the set E be countable. Every algebra $A \in K$ is a factor-algebra of a zero-dimensional algebra $B \in K$ and $w(A) = w(B)$.*

12.6. Definition. *Let K be a non-trivial complete U-variety of uniform E-algebras and X be a uniform space. A Couple $(F(X, K), i_x)$, where $F(X, K) \in K$ and $i_x : X \to F(X, K)$ is a uniform embedding of the space X in $F(X, K)$, is called a free uniform E-algebra of the uniform space X in the class K, if $i_x(X)$ algebraically generates $F(X, K)$ and for any uniformly continuous map $f : X \to G$ with $G \in K$ there exists a uniformly continuous homomorphism $\hat{f} : F(X, K) \to G$ with $f = \hat{f} \circ i_x$.*

Theorems 6.9 and 7.7 yield

12.7. Corollary. *For every non-trivial complete U-variety K of uniform E-algebras and any uniform space X we have:*

1. *The free algebra $(F(X, K), i_x)$ exists and is unique.*
2. *The algebra $F(X, K)$ is algebraically free in K.*
3. *The topology on $F(X, K)$ is admissible.*
4. *If the set E is countable and X is a paracompact σ-space, then*

a. *$F(X, K)$ is a paracompact σ-space:*

b. *if X is F_σ-metrizable and* $\dim X = 0$, *then* $\dim F(X, K) = 0$.

12.8. Corollary. *Let K be a non-trivial complete U-variety of uniform E-algebras and the set E be countable. Every first countable algebra $A \in K$ is a factor-algebra of a F_σ-metrizable first countable zero-dimensional paracompact algebra $B \in K$ and $w(A)=w(B)$, $uw(A)=uw(B)$. If the space A is metrizable or the uniformity on A is induced by a metric on X, then B is of the same type.*

References

1. A. A. Albert. Quasi-groups 1. *Trans. Amer. Math. Soc.*, **54**, 1943, 507-519.
2. A. V. Arhangel'skiĭ. An addition theorem for the weight of sets lying in bicompacta. *DAN SSSR*, **126**, 1959, 239-241 (in Russian).
3. A. V. Arhangel'skiĭ. On correlation between invariants of topological groups and it subspaces. *Uspehi Mat. Nauk*, **35**, No 3, 1980, 3-22 (in Russian).
4. A. V. Arhangel'skiĭ. Classes of topological groups. *Uspehi Mat. Nauk*, **36**, No 3, 1981, 127-146 (in Russian).
5. M. M. Čoban. On the theory of topological systems. *Trudy Mosk. Matem. Obshch.*, **48**, 1985, 106-149 (in Russian).
6. M. M. Čoban. On some questions of the theory of topological groups. — In: Obščaja algebra i diskretnaja geometrija. Kishinev: Shtiintsa, 1980, 120-135 (in Russian).
7. M. M. Čoban. On the mappings of topological groups. *DAN SSSR*, **226**, No 4, 1976, 749-753 (in Russian).
8. M. M. Čoban. Algebras and some questions of the theory of maps. Proceed. Fifth Prague Topol. Symposium. 1981. Copyright Helderman verlag, Berlin, 1983, 86-97.
9. M. M. Čoban. On the theory of free topological groups. *Topology theory and applications*. Colloquia Math. Soc. J. Bolyai, **41**, 1985, 159-175.
10. P. M. Cohn. Universal algebras. New York, Evanston, London, 1965.
11. S. S. Dumitrascu, M. M. Čoban. On free algebras with continuous signature. — In: Algebraicheskijwe i topologicheskije systemy. Kishinev: Štiintsa, 1982, 27-53 (in Russian).
12. B. K. Belnov. On the zero-dimensional topological groups. *DAN SSSR*, **226**, No 4, 1976, 749-753 (in Russian).
13. R. Engelking. General topology. Warszawa: PWN, 1977.
14. M. I. Graev. Free topological groups. *Izv. AN SSSR*, **12**, 1948, 279-324 (in Russian). English translation in *Amer. Math. Soc. Transl.*, **8**, 1962, 305-364.
15. A. I. Malčev. Algebraic systems. Moscow: Nauka, 1970 (in Russian).
16. A. I. Malčev. On the general theory of algebraic systems. *Mat. Sb.*, **35**, 1954, 3-20 (in Russian). English translation in *Amer. Math. Soc. Transl.*, **27**, 1963, 125-142.
17. A. I. Malčev. Free topological algebras. *Izv. AN SSSR*, **21**, 1957, 171-198 (in Russian).
18. A. A. Markov. On free topological groups. *Izv. AN SSSR*, **9**, 1945, 3-64 (in Russian). English translation in *Amer. Math. Soc. Transl.*, **8**, 1962, 195-272.
19. S. A. Morris, P. Nicolas. Locally invariant topologies of free groups. *Pacific I. Math.*, **103**, No 2, 1962, 523-577.
20. E. Michael. $_0$-spaces. *J. Math. Mech.*, **15**, 1966, 983-1002.
21. A. Okuyama. A survey on the theory of σ-spaces. *General Topol. Appl.*, **1**, 1971, 57-63.
22. V. I. Ponomarev. Axioms of countability and continuous mappings. *Bull. Acad. Pol. Sci. Ser. Math.*, **8**, 1960, 127-133 (in Russian).
23. S. Swierczkowski. Topologies in free algebras. *Proc. London Math. Soc.* (3) **14**, No 55, 1964, 566-576.
24. W. Taylor. Varieties of topological algebras. *J. Austral. Math. Soc., ser.* **A, 23**, No 2, 1977, 207-241.
25. W. Taylor. Varieties obeying homotopy laws. *Canad. J. Math.*, **29**, No 3, 1977, 498-527.
26. S. I. Nedev, M. M. Čoban. General conception of metrization of topological spaces. *Annuaire de l'universite de Sofia, faculte des math.*, **65**, 1973, 11-165 (in Russian).

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